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LONG-TIME EXISTENCE OF QUASILINEAR WAVE EQUATIONS EXTERIOR TO STAR-SHAPED OBSTACLES VIA ENERGY METHODS*

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Abstract. We establish long-time existence results for quasilinear wave equations in the exterior of star-shaped obstacles. To do so, we prove an analogue of the mixed-norm estimates of Keel, Smith, and Sogge for the perturbed wave equation. The arguments that are presented rely only upon the invariance of the wave operator under translations and spatial rotations.

Key words. systems of nonlinear wave equations, exterior domains, almost global existence, star-shaped

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1. Introduction.

The purpose of this article is to establish long-time existence results for quasilinear wave equations in the exterior of a star-shaped obstacle. The proofs that are presented rely upon the classical invariance of the wave operator under translations and spatial rotations. These techniques use only energy methods, and thus we are optimistic about their potential use in other applications. A key step in completing the proof is to establish a weighted $L^2_tL^2_x$-estimate for the perturbed equation that is analogous to the one of Keel, Smith, and Sogge [8] for the free wave equation.

Let us more explicitly describe the initial value boundary value problem that we will study. We begin by fixing an obstacle $\mathcal{K}\subset\mathbb{R}^n$ that is compact, has smooth boundary, and is star-shaped with respect to the origin. The latter condition means that there is a smooth positive function ψ on S^{n-1} so that

$$\mathcal{K}=\{(r,\omega)\,:\,\psi(\omega)-r\ge 0\}.$$

Here, we have expanded x in polar coordinates as $x=r\omega$, $(r,\omega)\in[0,\infty)\times S^{n-1}$.

For such a fixed $\mathcal{K}$, we examine the quasilinear wave equation

$$\begin{cases}\Box u=Q(du,d^2u), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n\backslash\mathcal{K},\\ u(t,\,\cdot\,)|_{\partial\mathcal{K}}=0, & \\ u(0,\,\cdot\,)=f, \quad \partial_t u(0,\,\cdot\,)=g. & \end{cases}\tag{1.1}$$

Here and throughout, $\Box=(\partial_t^2-\Delta)$ denotes the standard d'Alembertian.

The nonlinearity $Q(du,d^2u)$ in (1.1) is quadratic in its arguments and is linear in d^2u. We can expand

$$Q(du,d^2u)=B(du)+\sum_{0\le\alpha,\beta,\gamma\le n}B^{\alpha\beta}_{\gamma}\partial_\gamma u\partial_\alpha\partial_\beta u,\tag{1.2}$$

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where $B(du)$ is a quadratic form and the $B_\gamma^{\alpha\beta}$ are real constants. We assume the symmetry condition

$$B_\gamma^{\alpha\beta} = B_\gamma^{\beta\alpha}. \tag{1.3}$$

By scaling, we note that it suffices to choose $\mathcal{K} \subset \{|x| < 1\}$, and we will make this assumption throughout.

In order to solve (1.1), the data must be assumed to satisfy the relevant compatibility conditions. Briefly, this means that if we set $J_k u = \{\partial_x^\alpha u : 0 \le |\alpha| \le k\}$ and if u is a formal H^m solution for some fixed m, then we can write $\partial_t^k u(0, \cdot\,) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$, for compatibility functions ψ_k which depend on Q, $J_k f$, and $J_{k-1} g$. The compatibility condition for $(f, g) \in H^m \times H^{m-1}$ states that ψ_k must vanish on $\partial\mathcal{K}$ when $0 \le k \le m-1$. Additionally, $(f, g) \in C^\infty$ are said to satisfy the compatibility condition to infinite order if this condition holds for all m. For a more detailed exposition on compatibility conditions, see, e.g., [7].

In describing the main results, we will use the notation $\{\Omega\} = \{x_i\partial_j - x_j\partial_i : 1 \le i < j \le n\}$ to denote the generators of the spatial rotations. We will also use $\{Z\} = \{\partial_k, \Omega : 0 \le k \le n\}$ to denote the generators of translations and spatial rotations.

Our main results are as follows. The first states that small-data solutions to (1.1) exist almost globally if $n = 3$.

THEOREM 1.1. *Assume that the star-shaped obstacle $\mathcal{K} \subset \mathbb{R}^3$ and the nonlinearity $Q(du, d^2u)$ are as above. Suppose that the initial data $(f, g) \in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfy the compatibility condition to infinite order. Then, there are constants $\kappa, \varepsilon_0 > 0$ and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$ and data satisfying*

$$\sum_{|\mu|\le N} \|Z^\mu \nabla_x f\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\mu|\le N} \|Z^\mu g\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon, \tag{1.4}$$

(1.1) *has a unique solution $u \in C^\infty([0, T_\varepsilon] \times \mathbb{R}^3\backslash\mathcal{K})$ with*

$$T_\varepsilon = \exp(\kappa/\varepsilon). \tag{1.5}$$

This bound on the lifespan of solutions in $n = 3$ is sharp, as is illustrated by finite propagation speed and the counterexamples of John [5] and Sideris [21] in the boundaryless case.

The second main result states that small-data solutions exist globally in higher dimensions.

THEOREM 1.2. *Suppose $n \ge 4$. Assume that the star-shaped obstacle $\mathcal{K} \subset \mathbb{R}^n$ and the nonlinearity $Q(du, d^2u)$ are as above. Suppose that the initial data $(f, g) \in C^\infty(\mathbb{R}^n\backslash\mathcal{K})$ satisfy the compatibility condition to infinite order. Then, there are a constant $\varepsilon_0 > 0$ and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$ and data satisfying*

$$\sum_{|\mu|\le N} \|Z^\mu \nabla_x f\|_{L^2(\mathbb{R}^n\backslash\mathcal{K})} + \sum_{|\mu|\le N} \|Z^\mu g\|_{L^2(\mathbb{R}^n\backslash\mathcal{K})} \le \varepsilon, \tag{1.6}$$

(1.1) *has a unique solution $u \in C^\infty([0, \infty) \times \mathbb{R}^n\backslash\mathcal{K})$.*

While we have stated only the theorems for scalar wave equations, as the proofs rely only upon energy methods, straightforward modifications would yield the results for multiple speed systems of wave equations. In order not to further complicate the notation, we will prove only the scalar case. A more detailed exposition concerning the multiple speed case will be available in a forthcoming paper on null-form wave equations.

Theorem 1.1 was first proved by Keel, Smith, and Sogge [9]. It is an analogue of the results concerning boundaryless wave equations of John and Klainerman [6] and Klainerman and Sideris [11]. Theorem 1.2 was previously shown by the authors [17]. This generalizes the work on wave equations in higher dimensions previously completed by Metcalfe [13], Shibata and Tsutsumi [20], and Hayashi [3]. It is also worth pointing out the following works for related problems involving null-form nonlinearities: Keel, Smith, and Sogge [7]; Metcalfe and Sogge [16]; and Metcalfe, Nakamura, and Sogge [14, 15]. The techniques in this paper appear to allow for some simplifications of these proofs, and this will be explored in a subsequent paper. The arguments, however, are more involved as they require the use of the scaling vector field and decay estimates of Klainerman and Sideris [11].

The techniques used to prove Theorem 1.1 represent an improvement over those in [9] in a number of ways. Most importantly, the proofs in this article make no reference to the fundamental solution of the wave equation or to the sharp Huygens' principle. Thus, it is believed that these techniques will be more suitable for other applications. For example, one might compare the methods of Sideris and Tu [24] to those used in Sideris [22, 23]. Additionally, we are not required to use the scaling vector field $L = t\partial_t + r\partial_r$. On a lesser note, we remark that the proofs herein seem to require less regularity of the initial data and less regularity of the boundary of the obstacle. As neither proof takes care to minimize such regularity, there is much possibility for further improvement in this direction. The proof of Theorem 1.2 improves upon the techniques of previous works in similar ways.

It is interesting to note that our arguments never make explicit use of the well-known decay of local energy. See, e.g., Lax, Morawetz, and Phillips [12]. We do, however, rely upon a geometrical condition that is sufficient to ensure such estimates. This condition is used in ways that are reminiscent of those of Morawetz [18] in proving said decay estimates.

A key estimate which is common to many of the previous studies of wave equations in exterior domains was established by Keel, Smith, and Sogge [8] and states that for $n \ge 3$

$$(\log(2+T))^{-1/2}\Big(\|\langle x\rangle^{-1/2}\nabla_{t,x}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} + \|\langle x\rangle^{-3/2}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)}\Big) \lesssim \|\nabla_{t,x}\phi(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\,ds. \tag{1.7}$$

The proof is easily modified to yield the second estimate

$$\|\langle x\rangle^{-1/2-}\nabla_{t,x}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} + \|\langle x\rangle^{-3/2-}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} \lesssim \|\nabla_{t,x}\phi(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\,ds. \tag{1.8}$$

Here, we are using the notation $\langle x\rangle = \langle r\rangle = (1+|x|^2)^{1/2}$. We are also using the notation $\langle x\rangle^{-1/2-}$ and $\langle x\rangle^{-3/2-}$ to indicate that (1.8) holds with the weights replaced, respectively, by $\langle x\rangle^{-1/2-\delta}$ and $\langle x\rangle^{-3/2-\delta}$ for any $\delta > 0$. Moreover, we are using $A \lesssim B$ to indicate $A \le CB$ for some positive, unspecified constant C.

These estimates are related to an earlier one of Strauss [27] and were used by Keel, Smith, and Sogge [8] to give a proof of almost global existence to semilinear wave equations in exterior domains. Using this estimate, long-time existence was established using the $O(1/|x|)$ decay of the wave equation rather than the more standard $O(1/t)$ decay which is much more difficult to prove when there is a boundary. Metcalfe [13] completed the analogous result for higher dimensions using an estimate of the form (1.8) and arguments that are reminiscent of [8].

It should be noted that estimates similar to (1.7) and (1.8) which hold in all dimensions have been shown by Hidano and Yokoyama [4]. The above estimates should also be compared to the Morawetz identities (see, e.g., [19])

$$\||x|^{-1/2}\not\nabla\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} + \||x|^{-3/2}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} \lesssim \|\nabla_{t,x}\phi(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\,ds, \quad n\ge 4,$$

and

$$\||x|^{-1/2}\not\nabla\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^3)} + \|\phi(\,\cdot\,,0)\|_{L^2_t([0,T])} \lesssim \|\nabla_{t,x}\phi(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\,ds, \quad n=3,$$

which correspond to choosing $f(r)\equiv 1$ in the proof of Lemma 4.1. Here $\not\nabla$ denotes the angular portion of ∇_x.

The estimates (1.7) and (1.8) are proved by scaling a version of (1.8) where the norms in the left side are taken over $[0,T]\times\{|x|<1\}$. These local versions are established either, in odd dimensions, by noticing that the backward light cones $s+|x|\in(j-1,j]$, $j=1,2,3,\dots$, have finite overlap or by an argument using Plancherel's identity (see Smith and Sogge [25]). Then, using techniques that resemble those of [25], one can show that an estimate for the Dirichlet-wave equation follows from those for the free equation.

The estimates (1.7) and (1.8) are, however, insufficient to give a proof of long-time existence for quasilinear equations as there is a loss of regularity in the right side. In order to get around this, previous works have had to rely on pointwise estimates that involve direct estimation of the fundamental solution of the free wave equation.

Recently, Rodnianski [26, Appendix] has given a new proof of an estimate related to (1.8). This new proof relies only upon energy methods. A main topic of this paper is the further study of this argument. In particular, we show that Rodnianski's argument can be used to prove (1.7) and (1.8). Moreover, we show that this argument can be used to directly prove an estimate for the Dirichlet-wave equation if the obstacle is assumed to be star-shaped. Thus, we will not rely on the cutoff methods used previously. Last, this new geometric argument, unlike the previously established proofs, lends itself well to establishing similar weighted estimates for perturbed equations. With such estimates for the perturbed equation, one can prove Theorems 1.1 and 1.2 using the arguments of [8].

The mixed-norm estimates for the perturbed equation in Theorem 5.1 give a partial answer to questions raised in Alinhac [1] concerning the adaptability of the Keel–Smith–Sogge estimates to more general settings. During final preparations of this article, it was learned that Alinhac [2] had independently obtained a Keel–Smith–Sogge-type estimate for the perturbed wave equation using different (although related) techniques. This argument, however, requires assumptions on the perturbation that are less favorable in the current setting. In particular, it is required that the perturbation decay in t. When there is a boundary, such decay is quite difficult to prove, and we are using the mixed-norm estimates in place of such decay. Thus, it is essential that we require the perturbation terms to have decay only in $|x| = r$.

Before proceeding, we fix some notation. Throughout the paper, we will use the Einstein convention where repeated indices are summed. We will use Greek indices $\alpha, \beta, \gamma, \delta$ when the indices are to run from $0, \dots, n$. We will use Latin indices a, b when the implicit summations run from $1, \dots, n$. We will let $g_{\alpha\beta} = \operatorname{diag}(-1, 1, \dots, 1)$ be the Minkowski metric, and $\langle \cdot\,, \cdot \rangle$ will occasionally be used to denote the Euclidean inner product on $\mathbb{R}^n$. Unless explicitly stated to the contrary, L^2 norms are taken over $\mathbb{R}^n \backslash \mathcal{K}$. We will let $S_T = [0, T] \times \mathbb{R}^n \backslash \mathcal{K}$ denote a time strip of height T. We will use the notation $t = x_0$, $\partial_t = \partial_0$ interchangeably. And, when convenient, we will use $' = \partial = \nabla_{t,x} = (\partial_t, \nabla_x)$ to denote the full space-time gradient. We will use D to denote the Levi–Civita connection of $g_{\alpha\beta}$, but as this metric is flat, we have the correspondence $D^\alpha = \partial^\alpha$.

This paper is organized as follows. In the next section, we will give the weighted Sobolev inequality from which we easily obtain the required $O(1/|x|^{(n-1)/2})$ decay for solutions to the wave equation. In the third section, we prove the basic energy estimates that will be used in the proofs of long-time existence. In the fourth section, we give the new geometrical proof of the mixed-norm estimates of Keel, Smith, and Sogge. This argument follows that of Rodnianski [26, Appendix] quite closely. In the following section, we show that the energy methods used to prove the mixed-norm estimates are stable under small perturbations. In the final two sections, we give the proofs of Theorems 1.1 and 1.2, respectively.

2. Sobolev estimates.

In this section, we give the now standard weighted Sobolev estimate from which one can obtain the necessary $O(1/|x|^{(n-1)/2})$ decay in order to show our long-time existence results. See [10].

LEMMA 2.1. *Suppose that $h \in C^\infty(\mathbb{R}^n)$. Then, for $R \geq 1$,*

$$\|h\|_{L^\infty(R/2<|x|<R)} \lesssim R^{-(n-1)/2} \sum_{|\mu|+|\nu| \leq \frac{n+2}{2}} \|\Omega^\mu \partial_x^\nu h\|_{L^2(R/4<|x|<2R)}. \tag{2.1}$$

3. Energy estimates.

In this section, we will collect the energy estimates that we will require. These results are rather standard. We will be concerned with solutions $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K})$ of the Dirichlet-wave equation

$$\begin{cases} \Box_h \phi = F, \\ \phi|_{\partial\mathcal{K}} = 0, \end{cases} \tag{3.1}$$

where

$$\Box_h \phi = -\partial^\alpha \partial_\alpha \phi + h^{\alpha\beta} \partial_\alpha \partial_\beta \phi = (\partial_t^2 - \Delta)\phi + \sum_{\alpha,\beta=0}^n h^{\alpha\beta}(t, x) \partial_\alpha \partial_\beta \phi.$$

We shall assume that the $h^{\alpha\beta}$ satisfy the symmetry conditions

$$h^{\alpha\beta} = h^{\beta\alpha} \tag{3.2}$$

as well as the size conditions

$$\sum_{\alpha,\beta=0}^n |h^{\alpha\beta}(t, x)| \leq \delta \ll 1. \tag{3.3}$$

We define the energy form associated with $\Box_h$

$$e_0 = e_0(\phi) = (\partial_0\phi)^2 + \frac{1}{2}\partial^\gamma\phi\partial_\gamma\phi - g_{0\gamma}h^{\gamma\delta}\partial_\delta\phi\partial_0\phi - \frac{1}{2}h^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi. \tag{3.4}$$

Our most basic estimate involves

$$E_M(t) = E_M(\phi)(t) = \int_{\mathbb{R}^n\backslash\mathcal{K}} \sum_{j=0}^{M} e_0(\partial_t^j\phi)(t,x)\,dx.$$

Lemma 3.1. *Fix $M = 0, 1, 2, \dots$ and assume that the perturbation terms $h^{\alpha\beta}$ are as above. Suppose also that $\phi \in C^\infty$ solves* (3.1) *and that for every t, $\phi(t,x) = 0$ for large $|x|$. Then,*

$$\begin{aligned} E_M(T) \lesssim E_M(0) &+ \sum_{j,k=0}^{M}\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} |(\partial_0\partial_t^k\phi)(\Box_h\partial_t^j\phi)|\,dx\,dt \\ &+ \sum_{j,k=0}^{M}\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(|(\partial_\alpha h^{\alpha\beta})(\partial_0\partial_t^j\phi)(\partial_\beta\partial_t^k\phi)| + |(\partial_0 h^{\alpha\beta})(\partial_\alpha\partial_t^j\phi)(\partial_\beta\partial_t^k\phi)|\Big)\,dx\,dt. \end{aligned} \tag{3.5}$$

We first note that since $\partial_t^j\phi$ satisfies the Dirichlet boundary conditions for $1 \le j \le M$ it suffices to prove the result for $M = 0$. To proceed with the proof, we must define the other components of the energy-momentum vector. For $k = 1, \dots, n$, we set

$$e_k = e_k(\phi) = \partial_k\phi\partial_0\phi - g_{k\gamma}h^{\gamma\delta}\partial_\delta\phi\partial_0\phi. \tag{3.6}$$

Calculating the divergence of this energy-momentum vector, we see that

$$D^\alpha e_\alpha = -\partial_0\phi\Box_h\phi - (\partial_\alpha h^{\alpha\delta})\partial_\delta\phi\partial_0\phi + \frac{1}{2}(\partial_0 h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi. \tag{3.7}$$

If we integrate in the spatial components and apply the divergence theorem, it follows that

$$\begin{aligned} \partial_t\int_{\mathbb{R}^n\backslash\mathcal{K}} e_0\,dx + \int_{\partial\mathcal{K}} e_a n^a\,d\sigma = \int_{\mathbb{R}^n\backslash\mathcal{K}} &\partial_0\phi\Box_h\phi\,dx \\ &+ \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big((\partial_\alpha h^{\alpha\delta})\partial_\delta\phi\partial_0\phi - \frac{1}{2}(\partial_0 h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi\Big)\,dx, \end{aligned} \tag{3.8}$$

where $\vec{n}$ is the outward unit normal to $\mathcal{K}$ and $d\sigma$ is the surface measure on $\partial\mathcal{K}$.

Since ∂_t preserves the Dirichlet boundary condition, we have that $\partial_t\phi$ vanishes on $\partial\mathcal{K}$ and that the integrand of the second term in the left side of (3.8) vanishes. If we integrate the remaining terms over a time interval $[0,T]$, (3.5) follows easily. □

Next, we will need energy estimates that also involve spatial derivatives.

Lemma 3.2. *Suppose that the* $h^{\alpha\beta}$ *are as above with* δ *chosen sufficiently small. Then, if* ϕ *solves* (3.1) *and if* $N = 0, 1, 2, \dots$ *is fixed,*

$$\begin{aligned}\sum_{|\mu|\le N} \|\partial_{t,x}^{\mu}\phi'(T,\,\cdot\,)\|_2 &\lesssim \sum_{j\le N}\|\partial_t^j\phi'(0,\,\cdot\,)\|_2 \\ &\quad+\sum_{j,k=0}^{N}\Bigl(\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} |(\partial_0\partial_t^k\phi)(\Box_h\partial_t^j\phi)|\,dx\,dt\Bigr)^{1/2} \\ +\sum_{j,k=0}^{N}\Bigl[\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}}\Bigl(|(\partial_\alpha h^{\alpha\beta})(\partial_0\partial_t^j\phi)(\partial_\beta\partial_t^k\phi)| &+ |(\partial_0 h^{\alpha\beta})(\partial_\alpha\partial_t^j\phi)(\partial_\beta\partial_t^k\phi)|\Bigr)\,dx\,dt\Bigr]^{1/2} \\ &\quad+\sum_{|\mu|\le N-1}\|\Box\partial_{t,x}^{\mu}\phi(T,\,\cdot\,)\|_2.\end{aligned}\tag{3.9}$$

Since

$$\frac{1}{2}|\phi'(t,x)|^2 \le e_0(t,x) \le 2|\phi'(t,x)|^2$$

for δ in (3.3) sufficiently small, this follows from (3.9) and a standard elliptic regularity argument. The interested reader can see, e.g., Lemma 2.3 of [16] or Theorem 5.2 of [9]. □

Finally, we will need energy estimates that involve the generators of spatial rotations as well as derivatives.

Lemma 3.3. *Fix* $N = 0, 1, 2, \dots$ *and set*

$$Y_N(t) = \sum_{|\mu|\le N}\int e_0(Z^\mu\phi)(t,x)\,dx.$$

Suppose that (3.3) *holds for* δ *sufficiently small. Then*

$$\begin{aligned}\partial_t Y_N(t) \lesssim &\sum_{|\mu|,|\nu|\le N}\int_{\mathbb{R}^n\backslash\mathcal{K}}|(\partial_0 Z^\mu\phi)(\Box_h Z^\nu\phi)|\,dx \\ &+\sum_{|\mu|,|\nu|\le N}\int_{\mathbb{R}^n\backslash\mathcal{K}}\Bigl(|(\partial_\gamma h^{\gamma\delta})(\partial_0 Z^\mu\phi)(\partial_\delta Z^\nu\phi)| + |(\partial_0 h^{\gamma\delta})(\partial_\gamma Z^\mu\phi)(\partial_\delta Z^\nu\phi)|\Bigr)\,dx \\ &+\sum_{|\mu|\le N+1}\|\partial^\mu u'(s,\,\cdot\,)\|^2_{L^2(|x|<1)}.\end{aligned}\tag{3.10}$$

In order to prove (3.10), we argue as in the proof of Lemma 3.1 and find that

$$\begin{aligned}\partial_t Y_N \lesssim &\sum_{|\mu|,|\nu|\le N}\int_{\mathbb{R}^n\backslash\mathcal{K}}|(\partial_0 Z^\mu\phi)(\Box_h Z^\nu\phi)|\,dx \\ &+\sum_{|\mu|,|\nu|\le N}\int_{\mathbb{R}^n\backslash\mathcal{K}}\Bigl(|(\partial_\gamma h^{\gamma\delta})(\partial_0 Z^\mu\phi)(\partial_\delta Z^\nu\phi)| + |(\partial_0 h^{\gamma\delta})(\partial_\gamma Z^\mu\phi)(\partial_\delta Z^\nu\phi)|\Bigr)\,dx \\ &+\int_{\partial\mathcal{K}}|e_k^N n^k|\,d\sigma,\end{aligned}$$

where $\vec{n}$ is the outward normal at a given point of $\partial\mathcal{K}$ and the $e_k^N = \sum_{|\mu|\le N} e_k^N(Z^\mu\phi)(t,x)$ are as in (3.6). Since $\mathcal{K}\subset\{|x|<1\}$ and since

$$\sum_{|\mu|\le N} |Z^\mu\phi(t,x)| \lesssim \sum_{|\mu|\le N} |\partial^\mu\phi(t,x)|, \quad x\in\partial\mathcal{K},$$

we have

$$\int_{\partial\mathcal{K}} |e_k^N n^k|\, d\sigma \lesssim \int_{\{x\in\mathbb{R}^n\setminus\mathcal{K}\,:\,|x|<1\}} \sum_{|\mu|\le N+1} |\partial^\mu\phi'(t,x)|^2\, dx,$$

which completes the proof. $\square$

4. Geometric approach to $L_t^2L_x^2$-estimates. In this section, we show that the estimates (1.7) and (1.8) hold in the exterior of a star-shaped obstacle. In the next section, we will show that these estimates also hold for the perturbed wave equation. For clarity of exposition, we begin here with the proofs for the standard d'Alembertian. These estimates will be shown directly using energy methods and result from straightforward modifications of Rodnianski's argument [26, Appendix].

LEMMA 4.1. *Suppose that $\mathcal{K}$ is as above and $n\ge 3$. Suppose also that $\phi\in C^\infty$ satisfies $\phi|_{\partial\mathcal{K}}=0$ and that ϕ vanishes for large $|x|$ for every t. Then, we have*

$$(4.1)\quad (\log(2+T))^{-1/2}\Big(\|\langle x\rangle^{-1/2}\phi'\|_{L_t^2L_x^2(S_T)} + \|\langle x\rangle^{-3/2}\phi\|_{L_t^2L_x^2(S_T)}\Big) \lesssim \|\phi'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\, ds$$

and

$$(4.2)\quad \|\langle x\rangle^{-1/2-}\phi'\|_{L_t^2L_x^2(S_T)} + \|\langle x\rangle^{-3/2-}\phi\|_{L_t^2L_x^2(S_T)} \lesssim \|\phi'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box\phi(s,\,\cdot\,)\|_2\, ds$$

for any $T>0$. The implicit constants in (4.1) *and* (4.2) *are independent of $\mathcal{K}$.*

By Duhamel's principle, we shall need only the homogeneous case. So, let ϕ be a solution to

$$(4.3)\quad \begin{cases} \Box\phi = (\partial_t^2-\Delta)\phi = 0, \\ \phi|_{\partial\mathcal{K}} = 0. \end{cases}$$

We let $Q_{\alpha\beta}$ denote its energy-momentum tensor

$$(4.4)\quad Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi.$$

It is well known that $Q_{\alpha\beta}$ is divergence free. That is,

$$D^\alpha Q_{\alpha\beta}[\phi] = 0.$$

In order to get the weighted estimates, we define the momentum density

$$(4.5)\quad P_\alpha[\phi,X] = Q_{\alpha\beta}[\phi]X^\beta$$

by contracting $Q_{\alpha\beta}[\phi]$ with the radial vector field

$$X = f(r)\partial_r \tag{4.6}$$

(and thus, $X^a = \frac{f(r)}{r}x^a$ for $a = 1, \dots, n$ and $X^0 = 0$). One can check that this satisfies

$$D^\alpha P_\alpha[\phi, X] = \frac{1}{2} Q_{\alpha\beta}[\phi]\pi^{\alpha\beta},$$

where

$$\pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha$$

is the deformation tensor of X.

A direct calculation then yields that

$$D^\alpha P_\alpha[\phi, X] = f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r}|\not\nabla\phi|^2 - \frac{1}{2}\mathrm{tr}\pi\partial^\gamma\phi\partial_\gamma\phi, \tag{4.7}$$

where, as you can check,

$$\mathrm{tr}\pi = f'(r) + (n-1)\frac{f(r)}{r}. \tag{4.8}$$

Here, $\not\nabla\phi$ denotes the angular portion of the spatial gradient $\nabla_x\phi$. At this point, we define the modified momentum density

$$\tilde{P}_\alpha[\phi, X] = P_\alpha[\phi, X] + \frac{1}{2}\mathrm{tr}\pi\phi\partial_\alpha\phi - \frac{1}{4}\partial_\alpha(\mathrm{tr}\pi)|\phi|^2, \tag{4.9}$$

which satisfies

$$D^\alpha \tilde{P}_\alpha[\phi, X] = f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r}|\not\nabla\phi|^2 - \frac{1}{4}\Delta(\mathrm{tr}\pi)|\phi|^2. \tag{4.10}$$

If we integrate this identity over a time strip $[0,T] \times \mathbb{R}^n\backslash\mathcal{K}$ and apply the divergence theorem, we see that

$$\begin{aligned} \int_{\mathbb{R}^n\backslash\mathcal{K}} \tilde{P}_0[\phi, X](0)\,dx - \int_{\mathbb{R}^n\backslash\mathcal{K}} \tilde{P}_0[\phi, X](T)\,dx - \int_0^T \int_{\partial\mathcal{K}} \tilde{P}_a[\phi, X](t)n^a\,d\sigma\,dt \\ = \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r}|\not\nabla\phi|^2 - \frac{1}{4}\Delta(\mathrm{tr}\pi)|\phi|^2 \Big)\,dx\,dt, \end{aligned} \tag{4.11}$$

where $\vec{n}$ is the outward unit normal to $\mathcal{K}$ and $d\sigma$ is the surface measure on $\partial\mathcal{K}$. Here

$$\int_{\mathbb{R}^n\backslash\mathcal{K}} \tilde{P}_0[\phi, X](0)\,dx = \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(X^a\partial_t\phi(0)\partial_a\phi(0) + \frac{1}{2}\mathrm{tr}\pi\phi(0)\partial_t\phi(0) \Big)\,dx.$$

There is an identical expression for the time T piece on the left side of (4.11), since $\mathrm{tr}\pi$ is independent of t. If one chooses $f(r)$ so that $|f(r)| \lesssim 1$ and $|f'(r)| \lesssim \frac{1}{r}$, it follows from (4.8) and the Schwarz inequality that

$$\begin{aligned} \left| \int_{\mathbb{R}^n\backslash\mathcal{K}} \tilde{P}_0[\phi, X](0)\,dx \right| &\lesssim \Big(\|r^{-1}\phi(0)\|_2 + \|\nabla_{t,x}\phi(0)\|_2 \Big) \|\nabla_{t,x}\phi(0)\|_2 \\ &\lesssim \|\nabla_{t,x}\phi(0)\|_2^2. \end{aligned}$$

The last inequality above follows from the Hardy inequality. Analogous estimates hold for the $\tilde{P}_0[\phi, X](T)$ term. Thus, by conservation of energy, this term is also controlled by $\|\nabla_{t,x}\phi(0)\|_2^2$.

Since $\phi|_{\partial\mathcal{K}} = 0$ and since ∂_t preserves the Dirichlet boundary condition, for the remaining boundary term, we have

$$-\int_0^T \int_{\partial\mathcal{K}} \tilde{P}_a[\phi, X](t) \cdot n^a \, d\sigma \, dt = -\int_0^T \int_{\partial\mathcal{K}} \frac{f(r)}{r} \Big(\partial_{\vec{n}}\phi \partial_\beta \phi x^\beta - \frac{1}{2} |\nabla\phi|^2 \langle x, \vec{n}\rangle \Big) \, d\sigma \, dt.$$

Here, $\partial_{\vec{n}}\phi = \langle \vec{n}, \nabla_x\rangle\phi$ denotes differentiation with respect to the outward normal to $\mathcal{K}$. Since $\phi = 0$ on $\partial\mathcal{K}$, we have that $\partial_\beta\phi = \partial_{\vec{n}}\phi \, n_\beta$. And, thus, we see that

$$-\int_0^T \int_{\partial\mathcal{K}} \tilde{P}_a[\phi, X](t) \cdot n^a \, d\sigma \, dt = -\frac{1}{2}\int_0^T \int_{\partial\mathcal{K}} \frac{f(r)}{r} (\partial_{\vec{n}}\phi)^2 \langle x, \vec{n}\rangle \, d\sigma \, dt.$$

This term is then easily seen to be negative as $\langle x, \vec{n}\rangle > 0$ for star-shaped $\mathcal{K}$.

Combining (4.11) and these estimates for the boundary terms, we see that

$$\int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r} |\nabla\!\!\!/\phi|^2 - \frac{1}{4}\Delta(\mathrm{tr}\pi)|\phi|^2 \Big) \, dx \, dt \lesssim \|\nabla_{t,x}\phi(0)\|_2^2. \tag{4.12}$$

At this point, we choose the weight function

$$f(r) = \frac{r}{\rho + r} \tag{4.13}$$

for some $\rho > 0$. It is easy to check that

$$\Delta(\mathrm{tr}\pi) = -\frac{1}{r(\rho+r)^4}\Big((n-1)(n-3)r^2 + 2(n^2-2n-2)r\rho + (n+1)(n-1)\rho^2 \Big) < 0 \tag{4.14}$$

for $n \geq 3$. Indeed, each term above is nonpositive. This, therefore, gives the a priori estimate

$$\int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(\frac{\rho}{(\rho+r)^2}|\partial_r\phi|^2 + \frac{1}{r+\rho}|\nabla\!\!\!/\phi|^2 + \frac{\rho}{(\rho+r)^4}|\phi|^2 \Big) \, dx \, dt \lesssim \|\nabla_{t,x}\phi(0)\|_2^2. \tag{4.15}$$

The implicit constant is independent of ρ.

By choosing $\rho = 1$, this yields the estimate

$$\int_0^T \int_{|x|\leq 1} \Big(|\nabla_x\phi|^2 + |\phi|^2 \Big) \, dx \, dt \lesssim \|\nabla_{t,x}\phi(0)\|_2^2. \tag{4.16}$$

Similarly, if we choose $\rho = 2^k$ for an integer $k \geq 0$, we get

$$\int_0^T \int_{2^{k-1}\leq|x|\leq 2^k} \Big(\frac{|\nabla_x\phi|^2}{r} + \frac{|\phi|^2}{r^3} \Big) \, dx \, dt \lesssim \|\nabla_{t,x}\phi(0)\|_2^2. \tag{4.17}$$

If we combine (4.16) and (4.17) and sum over $k \geq 1$, we see immediately that

$$\|\langle r\rangle^{-1/2-}\nabla_x\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n\backslash\mathcal{K})} + \|\langle x\rangle^{-3/2-}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n\backslash\mathcal{K})} \lesssim \|\nabla_{t,x}\phi(0)\|_2. \tag{4.18}$$

The same argument also yields

$$(4.19)\quad (\log(2+T))^{-1/2}\Big(\|\langle r\rangle^{-1/2}\nabla_x\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n\backslash\mathcal{K})} + \|\langle x\rangle^{-3/2}\phi\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n\backslash\mathcal{K})}\Big)\lesssim\|\nabla_{t,x}\phi(0)\|_2.$$

Indeed, (4.19) follows trivially from the energy inequality and a Hardy inequality if the norms on the left side are taken over $|x|\ge T$. Thus, we need only sum over the $O(\log(2+T))$ choices of k with $2^{k-1}\lesssim T$.

It remains to see that a similar bound holds for $\partial_t\phi$. To do so, we define another modified momentum density

$$(4.20)\qquad \bar P_\alpha[\phi,X]=P_\alpha[\phi,X]+\frac{n-1}{2}\frac{f(r)}{r}\phi\partial_\alpha\phi-\frac{n-1}{4}\partial_\alpha\Big(\frac{f(r)}{r}\Big)|\phi|^2.$$

Calculating the divergence, we have

$$(4.21)\qquad \begin{aligned} D^\alpha\bar P_\alpha[\phi,X]&=f'(r)(\partial_r\phi)^2+\frac{f(r)}{r}|\not\nabla\phi|^2-\frac12 f'(r)\partial^\gamma\phi\partial_\gamma\phi\\ &\quad-\frac{n-1}{4}\Delta\Big(\frac{f(r)}{r}\Big)|\phi|^2.\end{aligned}$$

Thus, if we integrate both sides of (4.21) over a time strip, apply the divergence theorem, and use similar arguments for controlling the boundary terms, it follows that

$$(4.22)\quad \begin{aligned}\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} f'(r)(\partial_r\phi)^2+\frac{f(r)}{r}|\not\nabla\phi|^2-\frac12 f'(r)\partial^\gamma\phi\partial_\gamma\phi-\frac{n-1}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2\\ \lesssim\|\nabla_x\phi(0)\|_2^2.\end{aligned}$$

For f as in (4.13), we have $f'(r)<f(r)/r$ and

$$(4.23)\qquad \Delta\left(\frac{f(r)}{r}\right)=\frac{-(n-3)r-(n-1)\rho}{r(r+\rho)^3}<0$$

for $n\ge 3$. Since $\partial^\gamma\phi\partial_\gamma\phi=-(\partial_t\phi)^2+(\partial_r\phi)^2+|\not\nabla\phi|^2$, we see from (4.22) that

$$\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}}\frac{\rho}{(r+\rho)^2}(\partial_t\phi)^2\,dx\,dt\lesssim\|\nabla_{t,x}\phi(0)\|_2^2.$$

By choosing $\rho=1$ and $\rho=2^k$ for $k\ge 1$, we get the estimates analogous to (4.16) and (4.17) for $\partial_t\phi$. Summing over k as above and combining this with (4.18) and (4.19) immediately yield (4.1) and (4.2). □

It is worth noting that the proof shows that there is additional decay for certain derivatives of the solution ϕ, namely, for $\not\nabla\phi$. Indeed, from (4.15), one can see that the $\log(2+T)$ factor in (4.1) is not necessary for the angular derivatives. This extra decay is something that has been exploited in other estimates and applications. See, e.g., Alinhac [1].

5. $L^2_tL^2_x$-estimate for the perturbed equation. The goal of this section is to show that the methods of the previous section can be adapted to give similar bounds for perturbed wave equations. This is the main new estimate of this article.

Theorem 5.1. *Suppose $\mathcal{K}$ is as above and that $n \geq 3$. Let $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K})$ be a solution of* (3.1)*. Suppose that $h^{\alpha\beta}$ satisfies* (3.2) *and* (3.3) *for a small choice of δ. Then, if $\Box_h \phi = F$, we have*

$$\begin{aligned}(5.1)\quad &\|\langle x\rangle^{-1/2-}\phi'\|_{L^2_tL^2_x(S_T)} + (\log(2+T))^{-1/2}\|\langle x\rangle^{-1/2}\phi'\|_{L^2_tL^2_x(S_T)} \\ &\quad + \|\langle x\rangle^{-3/2-}\phi\|_{L^2_tL^2_x(S_T)} + (\log(2+T))^{-1/2}\|\langle x\rangle^{-3/2}\phi\|_{L^2_tL^2_x(S_T)} \\ &\quad \lesssim \|\phi'(0,\,\cdot\,)\|_2 + \left(\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)|F|\,dx\,dt\right)^{1/2} \\ &\qquad + \left[\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\partial h| + \frac{|h|}{r}\right)|\nabla_{t,x}\phi|\left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)dx\,dt\right]^{1/2}\end{aligned}$$

for any $T \geq 0$.

Here, in an abuse of notation, we are setting

$$|h| = \sum_{\alpha,\beta=0}^n |h^{\alpha\beta}(t,x)|, \quad |\partial h| = \sum_{\alpha,\beta,\gamma=0}^n |\partial_\gamma h^{\alpha\beta}(t,x)|.$$

Although we shall not use this, we point out that this estimate holds when the regularity of the boundary of $\mathcal{K}$ is merely C^1. Also, the implicit constants in (5.1) are independent of the choice of star-shaped obstacle $\mathcal{K}$.

To prove (5.1), we set

$$(5.2)\qquad Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi - g_{\alpha\gamma}h^{\gamma\delta}\partial_\delta\phi\partial_\beta\phi + \frac{1}{2}g_{\alpha\beta}h^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi.$$

Then it is straightforward to check that

$$D^\alpha Q_{\alpha\beta}[\phi] = -(\partial_\beta\phi)F - (\partial_\gamma h^{\gamma\delta})\partial_\delta\phi\partial_\beta\phi + \frac{1}{2}(\partial_\beta h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi.$$

As above, we contract this with a radial vector field $X = f(r)\partial_r$, define $P_\alpha[\phi, X] = Q_{\alpha\beta}[\phi]X^\beta$, and compute that

$$\begin{aligned}D^\alpha P_\alpha[\phi,X] &= f'(r)(\partial_r\phi)^2 + \frac{f(r)}{r}|\nabla\!\!\!/\,\phi|^2 - \frac{1}{2}\mathrm{tr}\pi\partial^\gamma\phi\partial_\gamma\phi - F(\partial_r\phi)f(r) \\ &\quad - (\partial_\gamma h^{\gamma\delta})\partial_\delta\phi\partial_r\phi f(r) + \frac{1}{2}(\partial_r h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi f(r) - x_a h^{a\delta}\partial_\delta\phi\partial_r\phi\frac{f'(r)}{r} \\ &\quad + x_a h^{a\delta}\partial_\delta\phi\partial_r\phi\frac{f(r)}{r^2} - h^{a\delta}\partial_\delta\phi\partial_a\phi\frac{f(r)}{r} + \frac{1}{2}(\mathrm{tr}\pi)h^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi,\end{aligned}$$

where $\mathrm{tr}\pi$ is given as in (4.8).

By modifying this momentum density as in (4.20) and setting

$$\begin{aligned}(5.3)\quad \bar{P}_\alpha[\phi,X] &= P_\alpha[\phi,X] + \frac{n-1}{2}\Big(\frac{f(r)}{r}\Big)\phi\partial_\alpha\phi \\ &\quad - \frac{n-1}{4}\partial_\alpha\left(\frac{f(r)}{r}\right)|\phi|^2 - \frac{n-1}{2}\left(\frac{f(r)}{r}\right)g_{\alpha\gamma}h^{\gamma\beta}\phi\partial_\beta\phi,\end{aligned}$$

it follows that

$$
\begin{aligned}
(5.4)\qquad D^\alpha \bar{P}_\alpha[\phi, X] = {} & f'(r)(\partial_r\phi)^2 + \frac{f(r)}{r}|\not\nabla\phi|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi \\
& - \frac{n-1}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2 - F(\partial_r\phi)f(r) - \frac{n-1}{2}F\frac{\phi}{r}f(r) \\
& - (\partial_\gamma h^{\gamma\delta})\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right)f(r) + \frac{1}{2}(\partial_r h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi f(r) \\
& - x_a h^{a\delta}\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right)\frac{f'(r)}{r} \\
& + x_a h^{a\delta}\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right)\frac{f(r)}{r^2} \\
& - h^{a\delta}\partial_\delta\phi\partial_a\phi\frac{f(r)}{r} + \frac{1}{2}f'(r)h^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi.
\end{aligned}
$$

Integrating both sides of (5.4) in a time strip S_T yields

$$
\begin{aligned}
(5.5)\qquad \int_{\mathbb{R}^n\backslash\mathcal{K}} \bar{P}_0[\phi, X](0)\,dx - \int_{\mathbb{R}^n\backslash\mathcal{K}} \bar{P}_0[\phi, X](T)\,dx - \int_0^T\int_{\partial\mathcal{K}} \bar{P}_a[\phi, X]n^a\,d\sigma\,dt \\
= \int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} D^\alpha\bar{P}_\alpha[\phi, X]\,dx\,dt.
\end{aligned}
$$

For f as given by (4.13), we have $|f(r)| \lesssim 1$ and $|f'(r)| \lesssim \frac{1}{r}$. Thus, we see that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n\backslash\mathcal{K}} \bar{P}_0[\phi, X](0)\,dx\right| = {} & \Big|\int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(\partial_t\phi(0)\partial_r\phi(0)f(r) - g_{0\gamma}h^{\gamma\delta}(0)\partial_\delta\phi(0)\partial_r\phi(0)f(r) \\
& + \frac{n-1}{2}\frac{f(r)}{r}\phi(0)\partial_t\phi(0) - \frac{n-1}{2}\frac{f(r)}{r}g_{0\gamma}h^{\gamma\beta}\phi(0)\partial_\beta\phi(0)\Big)\,dx\Big| \\
\lesssim {} & \|\nabla_{t,x}\phi(0)\|_2^2.
\end{aligned}
$$

For the last inequality, we are, as above, using the Schwarz inequality and the Hardy inequality. We are also using the bound (3.3).

A similar estimate holds for the $\bar{P}_0[\phi, X](T)$ term. And, thus, by the energy inequality (3.5),

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n\backslash\mathcal{K}} \bar{P}_0[\phi, X](T)\,dx\right| \lesssim {} & \|\nabla_{t,x}\phi(T)\|_2^2 \\
\lesssim {} & \|\nabla_{t,x}\phi(0)\|_2^2 + \int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} |\partial_t\phi||F|\,dx\,dt \\
& + \int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(|(\partial_\alpha h^{\alpha\beta})\partial_t\phi\partial_\beta\phi| + |(\partial_t h^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi|\Big)\,dx\,dt.
\end{aligned}
$$

For the remaining boundary terms, we use the fact that the Dirichlet boundary conditions permit us to write $\partial_a\phi = \partial_{\vec{n}}\phi\, n_a$ on $\partial\mathcal{K}$. Thus, if δ in (3.3) is small enough, we have

$$\begin{aligned}
-\int_0^T \int_{\partial\mathcal{K}} \bar{P}_a[\phi, X] n^a \, d\sigma \, dt
&= -\frac{1}{2}\int_0^T \int_{\partial\mathcal{K}} \frac{f(r)}{r}\Big((\partial_{\vec{n}}\phi)^2\langle x, \vec{n}\rangle - (\partial_{\vec{n}}\phi)^2 (h^{ab} n_a n_b)\langle x, \vec{n}\rangle\Big)\, d\sigma\, dt \\
&\le -\frac{1}{4}\int_0^T \int_{\partial\mathcal{K}} \frac{f(r)}{r}(\partial_{\vec{n}}\phi)^2\langle x, \vec{n}\rangle\, d\sigma\, dt \le 0.
\end{aligned}$$

For the first inequality, we are using (3.3). For the second inequality, we use the fact that $\langle x, \vec{n}\rangle > 0$ for star-shaped $\mathcal{K}$.

Using these bounds in (5.5), fixing f as in (4.13), and applying (5.4), it follows that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} & f'(r)(\partial_r\phi)^2 + \frac{f(r)}{r}|\not\nabla\phi|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi - \frac{n-1}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2 \\
& - (\partial_\gamma h^{\gamma\delta})\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right) f(r) + \frac{1}{2}(\partial_r h^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi f(r) \\
& - x_a h^{a\delta}\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right)\frac{f'(r)}{r} + x_a h^{a\delta}\partial_\delta\phi\left(\partial_r\phi + \frac{n-1}{2}\frac{\phi}{r}\right)\frac{f(r)}{r^2} \\
& - h^{a\delta}\partial_\delta\phi\partial_a\phi\frac{f(r)}{r} + \frac{1}{2}f'(r)h^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi\, dx\, dt \\
& \lesssim \|\nabla_{t,x}\phi(0)\|_2^2 + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)|F|\, dx\, dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(|(\partial_\alpha h^{\alpha\beta})\partial_t\phi\partial_\beta\phi| + |(\partial_t h^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi|\Big)\, dx\, dt.
\end{aligned}$$

Using (4.23), this yields

$$\begin{aligned}
\text{(5.6)} \quad \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} & \frac{\rho}{(r+\rho)^2}(\partial_r\phi)^2 + \frac{1}{r+\rho}|\not\nabla\phi|^2 + \frac{\rho}{(r+\rho)^2}(\partial_t\phi)^2 \\
& + \frac{\rho}{r(r+\rho)^3}|\phi|^2\, dx\, dt \\
& \lesssim \|\nabla_{t,x}\phi(0)\|_2^2 + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\Big)|F|\, dx\, dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \Big(|\partial h| + \frac{|h|}{r}\Big)|\nabla_{t,x}\phi|\Big(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\Big)\, dx\, dt,
\end{aligned}$$

since $f'(r) < f(r)/r$. Thus, it follows that

$$\begin{aligned}
\text{(5.7)} \quad \int_0^T \int_{|x|\le 1} & \Big(|\nabla_x\phi|^2 + |\partial_t\phi|^2 + |\phi|^2\Big)\, dx\, dt \\
& \lesssim \|\nabla_{t,x}\phi(0)\|_2^2 + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)|F|\, dx\, dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\partial h| + \frac{|h|}{r}\right)|\nabla_{t,x}\phi|\left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)\, dx\, dt
\end{aligned}$$

for the choice $\rho = 1$ (since $0 \in \mathcal{K}$ and thus $1/r$ is bounded on the complement of $\mathcal{K}$) and

$$\begin{aligned}\int_0^T \int_{2^{k-1}\le|x|\le 2^k} \left(\frac{|\nabla_x\phi|^2}{r} + \frac{|\partial_t\phi|^2}{r} + \frac{|\phi|^2}{r^3}\right) dx\,dt \\ \lesssim \|\nabla_{t,x}\phi(0)\|_2^2 + \int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\nabla_{t,x}\phi| + \frac{|\phi|}{r}\right)|F|\,dx\,dt \\ + \int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}}\left(|\partial h| + \frac{|h|}{r}\right)|\nabla_{t,x}\phi|\left(|\nabla_{t,x}\phi|+\frac{|\phi|}{r}\right)dx\,dt\end{aligned} \tag{5.8}$$

for $\rho = 2^k$. If we sum over $k \ge 1$ as above, (5.1) follows, which completes the proof. □

If we use an elliptic regularity argument as above, the following lemma holds.

Lemma 5.2. *Suppose that $\mathcal{K}$ is as above and $n \ge 3$. Let $\phi \in C^\infty(\mathbb{R}_+\times\mathbb{R}^n\backslash\mathcal{K})$ be a solution of* (3.1). *Suppose that $h^{\alpha\beta}$ satisfies* (3.2) *and* (3.3) *for a small choice of δ. Then, we have*

$$\begin{aligned}(\log(2+T))^{-1/2}\sum_{|\mu|\le N}\|\langle x\rangle^{-1/2}\partial^\mu_{t,x}\phi'\|_{L^2_tL^2_x(S_T)} \\ + \sum_{|\mu|\le N}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\phi'\|_{L^2_tL^2_x(S_T)} \\ \lesssim \sum_{j\le N}\|\partial_t^j\phi'(0,\,\cdot\,)\|_2 + \sum_{j,k\le N}\left(\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}}\left(|\nabla_{t,x}\partial_t^j\phi| + \frac{|\partial_t^j\phi|}{r}\right)|\Box_h\partial_t^k\phi|\,dx\,dt\right)^{1/2} \\ + \sum_{j,k\le N}\left[\int_0^T\int_{\mathbb{R}^n\backslash\mathcal{K}}\left(|\partial h|+\frac{|h|}{r}\right)|\nabla_{t,x}\partial_t^j\phi|\left(|\nabla_{t,x}\partial_t^k\phi| + \frac{|\partial_t^k\phi|}{r}\right)dx\,dt\right]^{1/2} \\ + \sum_{|\mu|\le N-1}\|\Box\partial^\mu_{t,x}\phi\|_{L^2_tL^2_x(S_T)}\end{aligned} \tag{5.9}$$

for any $T > 0$ and $N = 0, 1, 2, \dots$.

Indeed, in order to obtain (5.9), we argue inductively where (5.1) is the base case ($N = 0$). We then notice that

$$\begin{aligned}\sum_{|\mu|\le N}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\phi'\|_{L^2_tL^2_x(S_T)} \le \sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\partial^2_x\phi\|_{L^2_tL^2_x(S_T)} \\ + \sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}(\partial_t\phi)'\|_{L^2_tL^2_x(S_T)} + \|\langle x\rangle^{-1/2-}\phi'\|_{L^2_tL^2_x(S_T)}.\end{aligned} \tag{5.10}$$

The estimate for the last term on the right follows trivially from (5.1). Since ∂_t preserves the Dirichlet boundary condition, we can use the inductive hypothesis to bound the second term on the right.

In order to bound the first term on the right side of (5.10), we will use elliptic regularity. To see this, we fix a smooth cutoff function β with $\beta \equiv 1$ for $1/2 < |x| < 1$ and $\beta \equiv 0$ outside of $1/4 \le |x| \le 2$. Applying elliptic regularity to $\beta(x/R)\phi(t,x)$, we see that

$$\sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\partial_x^2\phi(t,\,\cdot\,)\|_{L^2_x(\{|x|\in[R/2,R]\})} \tag{5.11}$$

$$\lesssim \sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\Delta\phi(t,\,\cdot\,)\|_{L^2_x(\{|x|\in[R/4,2R]\})}$$

$$+\sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\phi'(t,\,\cdot\,)\|_{L^2_x(\{|x|\in[R/4,2R]\})}+\|\langle x\rangle^{-1}\phi(t,\,\cdot\,)\|_{L^2_x(\{|x|\in[R/4,2R]\})}$$

for $R\ge 2$. Similarly

$$\sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\partial_x^2\phi(t,\,\cdot\,)\|_{L^2_x(\{|x|\le 1\})}\lesssim \sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\Delta\phi(t,\,\cdot\,)\|_{L^2_x(\{|x|\le 2\})}$$

$$+\sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\phi'(t,\,\cdot\,)\|_{L^2_x(\{|x|\le 2\})},$$

where we have used the fact that the Dirichlet boundary conditions allow us to control ϕ locally by ϕ'. By multiplying both sides of (5.11) by $R^{-1/2-}$, summing over $R=2^k$, $k=1,2,\dots$, and integrating in time, we see that

$$\sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\partial_x^2\phi\|_{L^2_tL^2_x(S_T)}\lesssim \sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\Delta\phi\|_{L^2_tL^2_x(S_T)} \tag{5.12}$$

$$+\sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\phi'\|_{L^2_tL^2_x(S_T)}+\|\langle x\rangle^{-3/2-}\phi\|_{L^2_tL^2_x(S_T)}.$$

The estimates for the last two terms can, again, be obtained by the inductive hypothesis and (5.1), respectively.

For the first term in the right side of (5.12), we simply notice that

$$\sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\Delta\phi\|_{L^2_tL^2_x(S_T)}\lesssim \sum_{|\mu|\le N-1}\|\langle x\rangle^{-1/2-}\partial^\mu_{t,x}\partial_t^2\phi\|_{L^2_tL^2_x(S_T)}$$

$$+\sum_{|\mu|\le N-1}\|\partial^\mu_{t,x}\Box\phi\|_{L^2_tL^2_x(S_T)}.$$

As ∂_t preserves the boundary condition, we may use the inductive hypothesis to see that the first term on the right is bounded by the right side of (5.9). As a similar argument may be used to obtain the estimate for the first term on the left of (5.9), the proof of Lemma 5.2 is complete. $\square$

Similarly, if as above (Lemma 3.3), we repeat the argument with ϕ replaced by $Z^\mu\phi$ for some multi-index μ, we see that the following holds.

LEMMA 5.3. *Suppose that* $\mathcal{K}$ *is as above and* $n \geq 3$. *Let* $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K})$ *be a solution of* (3.1). *Suppose that* $h^{\alpha\beta}$ *satisfies* (3.2) *and* (3.3) *for a small choice of* δ. *Then, we have*

$$\begin{aligned}&(\log(2+T))^{-1/2} \sum_{|\mu|\leq N} \|\langle x\rangle^{-1/2} Z^\mu \phi'\|_{L^2_t L^2_x(S_T)} \\ &\qquad + \sum_{|\mu|\leq N} \|\langle x\rangle^{-1/2-} Z^\mu \phi'\|_{L^2_t L^2_x(S_T)} \\ &\lesssim \sum_{|\mu|\leq N} \|Z^\mu \phi'(0,\,\cdot\,)\|_2 + \sum_{|\mu|,|\nu|\leq N} \left(\int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\nabla_{t,x} Z^\mu \phi| + \frac{|Z^\mu \phi|}{r} \right) |\Box_h Z^\nu \phi| \, dx\, dt \right)^{1/2} \\ &\quad + \sum_{|\mu|,|\nu|\leq N} \left[\int_0^T \int_{\mathbb{R}^n\backslash\mathcal{K}} \left(|\partial h| + \frac{|h|}{r} \right) |\nabla_{t,x} Z^\mu \phi| \left(|\nabla_{t,x} Z^\nu \phi| + \frac{|Z^\nu \phi|}{r} \right) dx\, dt \right]^{1/2} \\ &\qquad + \sum_{|\mu|\leq N+1} \|\partial_x^\mu \phi'\|_{L^2_t L^2_x([0,T]\times\{|x|<1\})}\end{aligned} \tag{5.13}$$

for any $T > 0$ *and* $N = 0, 1, 2, \ldots$.

In particular, notice that the last term of (5.13) can be controlled using (5.9). The same is true for the boundary term of (3.10).

6. Almost global existence for $n = 3$. In this section, we shall prove Theorem 1.1. Since (5.9) and (5.13) have been established, this proof will resemble that of the semilinear case [8], which is much easier than the subsequent proofs for the quasilinear case. We shall use an iteration argument to solve (1.1) and to show that the solution satisfies

$$\begin{aligned}&\sum_{|\mu|\leq 15} \Big(\|\partial^\mu u'(t,\,\cdot\,)\|_2 + (\log(2+t))^{-1/2} \|\langle x\rangle^{-1/2} \partial^\mu u'\|_{L^2_t L^2_x(S_t)} \Big) \\ &\qquad + \sum_{|\mu|\leq 14} \Big(\|Z^\mu u'(t,\,\cdot\,)\|_2 + (\log(2+t))^{-1/2} \|\langle x\rangle^{-1/2} Z^\mu u'\|_{L^2_t L^2_x(S_t)} \Big) \leq C\varepsilon\end{aligned} \tag{6.1}$$

for $0 \leq t \leq T_\varepsilon$ and for uniform constant C.

Here, we let $u_{-1} \equiv 0$ and then recursively define u_k, $k = 0, 1, 2, \ldots$, to solve

$$\begin{cases} \Box u_k(t,x) = Q(du_{k-1}, d^2 u_k), \quad (t,x) \in [0,T] \times \mathbb{R}^3\backslash\mathcal{K}, \\ u_k|_{\partial\mathcal{K}} = 0, \\ u_k(0,\,\cdot\,) = f, \quad \partial_t u_k(0,\,\cdot\,) = g. \end{cases} \tag{6.2}$$

We set

$$\begin{aligned}M_k(T) = \sup_{0\leq t\leq T} \Bigg[&\sum_{|\mu|\leq 15} \Big(\|\partial^\mu u_k'(t,\,\cdot\,)\|_2 + (\log(2+t))^{-1/2} \|\langle x\rangle^{-1/2} \partial^\mu u_k'\|_{L^2_t L^2_x(S_t)} \Big) \\ &+ \sum_{|\mu|\leq 14} \Big(\|Z^\mu u_k'(t,\,\cdot\,)\|_2 + (\log(2+t))^{-1/2} \|\langle x\rangle^{-1/2} Z^\mu u_k'\|_{L^2_t L^2_x(S_t)} \Big) \Bigg].\end{aligned} \tag{6.3}$$

Clearly, by (1.3), the standard energy inequality, and (5.9) and (5.13) (with $h^{\alpha\beta} \equiv 0$), there is a uniform constant C_0 so that

$$M_0(T) \le C_0\varepsilon$$

for any T. Here, C_0 can be chosen to be larger than the implicit constants of (3.9), (3.10), (5.9), and (5.13). For $\varepsilon < \varepsilon_0$ sufficiently small and for κ in (1.5) small, we will show inductively that for $k = 1, 2, 3, \dots$

$$M_k(T_\varepsilon) \le 10C_0\varepsilon. \tag{6.4}$$

By (1.4), (3.9), (3.10), (5.9), and (5.13), we have

(6.5)

$$\begin{aligned}
M_k(T_\varepsilon) \le 4C_0\varepsilon &+ \sum_{|\mu|,|\nu|\le 15} \Big(\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\nabla_{t,x}\partial^\mu u_k| + \frac{|\partial^\mu u_k|}{r}\Big)|\partial^\nu \Box_h u_k|\,dx\,dt\Big)^{1/2} \\
&+ \sum_{|\mu|,|\nu|\le 15} \Big(\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\nabla_{t,x}\partial^\mu u_k| + \frac{|\partial^\mu u_k|}{r}\Big)|[\partial^\nu, \Box_h]u_k|\,dx\,dt\Big)^{1/2} \\
&+ \sum_{|\mu|,|\nu|\le 15} \Big[\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\partial h| + \frac{|h|}{r}\Big)|\nabla_{t,x}\partial^\mu u_k|\Big(|\nabla_{t,x}\partial^\nu u_k| + \frac{|\partial^\nu u_k|}{r}\Big)\,dx\,dt\Big]^{1/2} \\
&+ \sum_{|\mu|,|\nu|\le 14} \Big(\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\nabla_{t,x}Z^\mu u_k| + \frac{|Z^\mu u_k|}{r}\Big)|Z^\nu \Box_h u_k|\,dx\,dt\Big)^{1/2} \\
&+ \sum_{|\mu|,|\nu|\le 14} \Big(\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\nabla_{t,x}Z^\mu u_k| + \frac{|Z^\mu u_k|}{r}\Big)|[Z^\nu, \Box_h]u_k|\,dx\,dt\Big)^{1/2} \\
&+ \sum_{|\mu|,|\nu|\le 14} \Big[\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \Big(|\partial h| + \frac{|h|}{r}\Big)|\nabla_{t,x}Z^\mu u_k|\Big(|\nabla_{t,x}Z^\mu u_k| + \frac{|Z^\mu u_k|}{r}\Big)\,dx\,dt\Big]^{1/2} \\
&+ \sup_{0\le t\le T_\varepsilon}\Big[\sum_{|\mu|\le 14}\|\partial^\mu_{t,x}\Box u_k(t,\,\cdot\,)\|_2\Big] + \sum_{|\mu|\le 14}\|\partial^\mu_{t,x}\Box u_k\|_{L^2_t L^2_x(S_{T_\varepsilon})}.
\end{aligned}$$

Here, we set $h^{\gamma\delta} = -\sum_{0\le\beta\le 3} B^{\gamma\delta}_\beta \partial_\beta u_{k-1}$. We note that

$$\begin{aligned}
&\sum_{|\mu|\le 15}\Big(|\partial^\mu \Box_h u_k| + \sum_{|\mu|\le 15}|[\partial^\mu, \Box_h]u_k|\Big) \\
&\qquad\lesssim \sum_{|\mu|\le 7}|\partial^\mu u'_{k-1}|\sum_{|\nu|\le 15}|\partial^\nu u'_k| + \sum_{|\mu|\le 8}|\partial^\mu u'_k|\sum_{|\nu|\le 15}|\partial^\nu u'_{k-1}| \\
&\qquad\qquad + \sum_{|\mu|\le 7}|\partial^\mu u'_{k-1}|\sum_{|\nu|\le 15}|\partial^\nu u'_{k-1}|.
\end{aligned}$$

Similarly,

$$\sum_{|\mu|\le 14}\Big(|Z^\mu \Box_h u_k| + \sum_{|\mu|\le 14} |[Z^\mu, \Box_h]u_k|\Big)$$

$$\lesssim \sum_{|\mu|\le 7} |Z^\mu u'_{k-1}| \sum_{|\nu|\le 14} |Z^\nu u'_k| + \sum_{|\mu|\le 8} |Z^\mu u'_k| \sum_{|\nu|\le 14} |Z^\nu u'_{k-1}|$$

$$+ \sum_{|\mu|\le 7} |Z^\mu u'_{k-1}| \sum_{|\nu|\le 14} |Z^\nu u'_{k-1}|.$$

If we use this, (2.1), the Schwarz inequality, and a Hardy inequality, it follows that the second, third, and fourth terms in (6.5) are controlled by

$$\sum_{|\mu|\le 9} \|\langle x\rangle^{-1/2} Z^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})} \sum_{|\mu|\le 15} \|\langle x\rangle^{-1/2} \partial^\mu u'_k\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})} \tag{6.6}$$

$$\times \sup_{0\le t\le T_\varepsilon} \sum_{|\mu|\le 15} \|\partial^\mu u'_k(t,\,\cdot\,)\|_2^{1/2}$$

$$+ \left(\sum_{|\mu|\le 10} \|\langle x\rangle^{-1/2} Z^\mu u'_k\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})} + \sum_{|\mu|\le 9} \|\langle x\rangle^{-1/2} Z^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}\right)$$

$$\times \sum_{|\mu|\le 15} \|\langle x\rangle^{-1/2} \partial^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})} \left(\sup_{0\le t\le T_\varepsilon} \sum_{|\mu|\le 15} \|\partial^\mu u'_k(t,\,\cdot\,)\|_2^{1/2}\right).$$

In particular, for the fourth term in (6.5), notice that for the given choice of h, we have

$$\left(|\partial h| + \frac{|h|}{r}\right) \lesssim \sum_{|\mu|\le 1} |\partial^\mu u'_{k-1}|.$$

Here, we use that $\frac{1}{r}$ is bounded on $\mathbb{R}^3\backslash\mathcal{K}$ since $0\in\mathcal{K}$. Thus, by (2.1) and the Schwarz inequality, we have

$$\sum_{|\mu|,|\nu|\le 15} \int_{2^{j-1}}^{2^j} \left(|\partial h| + \frac{|h|}{r}\right) |\nabla_{t,x}\partial^\mu u_k(t,x)| \left(|\nabla_{t,x}\partial^\nu u_k(t,x)| + \frac{|\partial^\nu u_k(t,x)|}{r}\right)\,dx$$

$$\lesssim \sum_{|\mu|\le 3} \|\langle x\rangle^{-1/2} Z^\mu u'_{k-1}(t,\,\cdot\,)\|_{L^2(2^{j-2}\le|x|\le 2^{j+1})} \sum_{|\mu|\le 15} \|\langle x\rangle^{-1/2}\partial^\mu u'_k(t,\,\cdot\,)\|_{L^2(2^{j-1}\le|x|\le 2^j)}$$

$$\times \left(\sum_{|\mu|\le 15} \|\partial^\nu u'_k(t,\,\cdot\,)\|_2 + \left\|\frac{1}{r}u_k(t,\,\cdot\,)\right\|_2\right).$$

A similar bound holds over $|x|\in[0,1]$. If we sum over j and integrate in t, it follows that the fourth term in (6.5) is indeed controlled by the first term in (6.6). Here, we use a Hardy inequality to gain the control $\|(1/r)u_k(t,\,\cdot\,)\|_2 \lesssim \|u'_k(t,\,\cdot\,)\|_2$. Similar arguments yield the given bounds for the second and third terms in (6.5).

By the inductive hypothesis, we see that (6.6) is controlled by

$$C_1\varepsilon^{1/2}(\log(2+T_\varepsilon))^{1/2}M_k(T_\varepsilon)+C_1\varepsilon^{3/2}. \tag{6.7}$$

This provides the necessary bound for the second, third, and fourth terms in (6.5).

Similarly, the fifth, sixth, and seventh terms in (6.5) are bounded by

$$\begin{aligned}&\sum_{|\mu|\le 9}\|\langle x\rangle^{-1/2}Z^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}\sum_{|\mu|\le 14}\|\langle x\rangle^{-1/2}Z^\mu u'_k\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}\\ &\qquad\qquad\times\sup_{0\le t\le T_\varepsilon}\sum_{|\mu|\le 14}\|Z^\mu u'_k(t,\,\cdot\,)\|_2^{1/2}\\ &\quad+\left(\sum_{|\mu|\le 10}\|\langle x\rangle^{-1/2}Z^\mu u'_k\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}+\sum_{|\mu|\le 9}\|\langle x\rangle^{-1/2}Z^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}\right)\\ &\qquad\times\sum_{|\mu|\le 14}\|\langle x\rangle^{-1/2}Z^\mu u'_{k-1}\|^{1/2}_{L^2_tL^2_x(S_{T_\varepsilon})}\left(\sup_{0\le t\le T_\varepsilon}\sum_{|\mu|\le 14}\|Z^\mu u'_k(t,\,\cdot\,)\|_2^{1/2}\right),\end{aligned}$$

which as above is controlled by

$$C_2\varepsilon^{1/2}(\log(2+T_\varepsilon))^{1/2}M_k(T_\varepsilon)+C_2\varepsilon^{3/2}. \tag{6.8}$$

By similar arguments, it follows that the last two terms of (6.5) are controlled by

$$C_3\Big(\varepsilon M_k(T_\varepsilon)+\varepsilon^2+\varepsilon(\log(2+T_\varepsilon))^{1/2}M_k(T_\varepsilon)+\varepsilon^2(\log(2+T_\varepsilon))^{1/2}\Big). \tag{6.9}$$

If we have

$$(C_1+C_2+C_3)\varepsilon^{1/2}(\log(2+T_\varepsilon))^{1/2}\le\frac{1}{2},$$

which indeed is the case if κ is chosen to be smaller than $\frac{1}{2(C_1+C_2+C_3)^2}$, we can bootstrap the terms in (6.7), (6.8), and (6.9) involving $M_k(T_\varepsilon)$. Thus, we see that

$$M_k(T_\varepsilon)\le 2[4C_0\varepsilon+(C_1+C_2+C_3\kappa^{1/2}(\log 2)^{1/2})\varepsilon^{3/2}].$$

Thus, if ε is small enough, we obtain (6.4) as desired.

If we define

$$\begin{aligned}A_k(T)=\sup_{0\le t\le T}\Bigg[&\sum_{|\mu|\le 14}\Big(\|\partial^\mu(u'_k-u'_{k-1})(t,\,\cdot\,)\|_2\\ &\qquad+(\log(2+t))^{-1/2}\|\langle x\rangle^{-1/2}\partial^\mu(u'_k-u'_{k-1})\|_{L^2_tL^2_x(S_t)}\Big)\\ &\qquad\qquad+\sum_{|\mu|\le 13}\Big(\|Z^\mu(u'_k-u'_{k-1})(t,\,\cdot\,)\|_2\\ &\qquad\qquad+(\log(2+t))^{-1/2}\|\langle x\rangle^{-1/2}Z^\mu(u'_k-u'_{k-1})\|_{L^2_tL^2_x(S_t)}\Big)\Bigg],\end{aligned}$$

similar arguments can be used to show that

$$A_k(T_\varepsilon)\le\frac{1}{2}A_{k-1}(T_\varepsilon).$$

Thus, we have that u_k converges to a solution of (1.1) satisfying (6.1), which completes the proof.

7. Global existence in higher dimensions. In this last section, we provide a few remarks that explain how to modify the proof in the previous section in order to obtain a proof of Theorem 1.2. Indeed, it is possible to show via iteration that a solution exists and satisfies

$$\sum_{|\mu|\le n+10}\Big(\|\partial^\mu u'(t,\,\cdot\,)\|_2+\|\langle x\rangle^{-(n-1)/4}\partial^\mu u'\|_{L^2_tL^2_x(S_t)}\Big) \\ +\sum_{|\mu|\le n+9}\Big(\|Z^\mu u'(t,\,\cdot\,)\|_2+\|\langle x\rangle^{-(n-1)/4}Z^\mu u'\|_{L^2_tL^2_x(S_t)}\Big)\le C\varepsilon \tag{7.1}$$

for any $T>0$. Here, we argue as in the previous section. When we apply the weighted Sobolev estimate, we get weights $\langle x\rangle^{-(n-1)/4}$. When $n\ge 4$, we have $(n-1)/4>1/2$, and thus, we may apply the bound for the first term in the left side of (5.1) rather than that for the second term. Since the first term in the left side of (5.1) does not require the loss of a $\log(2+T)^{1/2}$, we see immediately that we have no restriction on T_ε, which proves the desired global existence result.

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