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Global Existence of Solutions to Multiple Speed Systems of Quasilinear Wave Equations in Exterior Domains

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Abstract. In this paper we prove global existence for certain multispeed Dirichlet-wave equations with quadratic nonlinearities outside of obstacles. We assume the natural null condition for systems of quasilinear wave equations with multiple speeds. The null condition only puts restrictions on the self-interactions of each wave family. We use the method of commuting vector fields and weighted space-time L^2 estimates.

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1 Introduction

The goal of this paper is to prove global existence of solutions to quadratic quasilinear Dirichlet-wave equations exterior to a class of compact obstacles. As in Metcalfe-Sogge [22], the main condition that we require for our class of obstacles is exponential local energy decay (with a possible loss of regularity). Our result improves upon the earlier one of Metcalfe-Sogge [22] by allowing a more general null condition which only puts restrictions on the self-interaction of each wave family. The non-relativistic system that we study serves as a simplified model for the equations of elasticity. In Minkowski space, such equations were studied and shown to have global solutions by Sideris-Tu [29], Agemi-Yokoyama [1], and Kubota-Yokoyama [18].

The null condition we use here is the natural one for systems of quasilinear wave equations with multiple speeds. Following an observation of John and Shatah, this null condition is equivalent to the requirement that no plane wave solution of the system is genuinely nonlinear (see John [11], p. 23 for the single-speed case and Agemi and Yokoyama [1] for the multi-speed case). In order to allow the more general null condition, instead of just exploring a coupling between a low order dispersive estimate and higher order energy estimates as in Metcalfe-Sogge [22], we must first develop a low order energy estimate and couple this with a low order pointwise

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estimate on the gradient and higher order energy estimates. Thus, our approach is a blend of the ones using pointwise estimates based on fundamental solutions (see e.g., [12], [14], [18], [22], [32], [33]) and ones using more refined L^2 energy estimates for lower order terms (see, e.g. [5], [27], [28], [29]). As in the approach first developed in [13] weighted space-time L^2 estimates for lower order terms will also play a key role in our arguments.

We will be using an exterior domain analog of Klainerman's commuting vector fields method [16]. Here, we have to restrict to the collection of vector fields that are "admissible" for boundary value problems, $\{Z, L\}$, where Z denotes the generators of the spatial rotations and space-time translations

$$Z = \{\partial_i, x_j\partial_k - x_k\partial_j, 0 \le i \le 3, 1 \le j < k \le 3\} \tag{1.1}$$

and L is the scaling vector field

$$L = t\partial_t + r\partial_r. \tag{1.2}$$

Here and in what follows, $r = |x|$, and we will write

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \le i < j \le 3. \tag{1.3}$$

The generators of the hyperbolic rotations, $x_i\partial_t + t\partial_i$, have an associated speed and the coefficients are unbounded on the boundary of the obstacle, and thus, they do not seem appropriate for the problem in question.

In Minkowski space, since $[(\partial_t^2 - \Delta), Z] = 0$ and $[(\partial_t^2 - \Delta), L] = 2(\partial_t^2 - \Delta)$, we see that Z and L preserve the equation $(\partial_t^2 - \Delta)u = 0$. This is no longer the case in the exterior domain since the Dirichlet boundary condition is not preserved. For the vector fields Z, since their coefficients remain small in a neighborhood of our compact obstacle, this is fairly easy to get around. On the other hand, since the coefficients of L are large near the obstacle as $t \to \infty$, we must stick to estimates that require relatively few of the scaling vector field.

As in [13], [14], we will use weighted $L_t^2 L_x^2$ estimates where the weight is just a negative power of $\langle x \rangle = \langle r \rangle = \sqrt{1 + r^2}$. These estimates are useful for handling the lower order terms that arise in the study of such boundary value problems. They permit us to use pointwise estimates for linear, inhomogeneous wave equations with $O(\langle x \rangle^{-1})$ decay rather than the more standard $O(t^{-1})$ decay which is more difficult to prove in the obstacle setting. Additionally, such estimates allow us, as in [22], to handle the boundary terms that arise in the energy estimates if the obstacle is no longer assumed to be star-shaped. Here we exploit the fact that we are studying equations with quadratic nonlinearities.

Additionally, we will be developing exterior domain analogs of a class of weighted Sobolev estimates. The weights here will involve powers of r and $\langle t - r \rangle$. Specifically, we will be looking at estimates of Klainerman-Sideris [17] and Hidano-Yokoyama

[6]. We would additionally like to mention the works of Hidano [5], Kubota-Yokoyama [18], Sideris [27, 28], and Sideris-Tu [29] where similar estimates were used for the boundaryless case.

At this point, we wish to describe our assumptions on our obstacles $\mathscr{K} \subset \mathbb{R}^3$. We shall assume that $\mathscr{K}$ is smooth and compact, but not necessarily connected. By scaling, without loss of generality, we may assume

$$\mathscr{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$$

throughout. The only additional assumption states that there is exponential local energy decay with a possible loss of regularity. That is, if u is a solution to $\Box u = 0$ with Cauchy data $u(0,x), \partial_t u(0,x)$ supported in $|x| \le 4$, then there must be constants $c, C > 0$ so that

$$\left(\int_{\{x \in \mathbb{R}^3 \setminus \mathscr{K} : |x| < 4\}} |u'(t,x)|^2 \, dx\right)^{1/2} \le Ce^{-ct} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u'(0,\cdot)\|_2. \tag{1.4}$$

Here, and throughout, we are taking $\partial = \nabla_{t,x}$ to be the space-time gradient.

We note that we do not require exponential decay; in fact, $O((1+t)^{-1-\delta})$ may be sufficient with a tighter argument. For simplicity, we will assume (1.4). Currently, the authors are not aware of any 3-dimensional example that involves polynomial decay, but does not have exponential decay.

Notice that if the obstacle is assumed to be nontrapping, then a stronger version of (1.4) holds where $\alpha = 0$ on the right side (see, e.g., Morawetz-Ralston-Strauss [24]). If there are trapped rays, it was shown by Ralston [25] that (1.4) could not hold without a loss of regularity $\ell > 0$ in the right side. We will assume throughout that $\ell = 1$. This can be done without loss of generality since if $\ell > 1$, interpolation with the standard energy inequality will yield (1.4) (with a different constant c). In fact, we could take $\ell = \delta$ for any $\delta > 0$.

Ikawa [9], [10] showed that there was such exponential decay of local energy for certain finite unions of smooth, convex obstacles with a loss $\ell = 7$. In particular, using Ikawa's result, we have (1.4) for two disjoint convex obstacles or any number of sufficiently separated balls.

For such smooth, compact obstacles $\mathscr{K} \subset \mathbb{R}^3$ satisfying (1.4), we shall consider quadratic, quasilinear systems of the form

$$\begin{cases} \Box u = Q(du, d^2u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathscr{K} \\ u(t,\cdot)|_{\partial\mathscr{K}} = 0 \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases} \tag{1.5}$$

Here

$$(1.6) \qquad \Box = (\Box_{c_1}, \Box_{c_2}, \dots, \Box_{c_D})$$

is a vector-valued multiple speed d'Alembertian with

$$\Box_{c_I} = \partial_t^2 - c_I^2 \Delta.$$

We will assume that the wave speeds c_I are positive and distinct. This situation is referred to as the nonrelativistic case. Straightforward modifications of the argument give the more general case where the various components are allowed to have the same speed. Also, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the standard Laplacian. Additionally, when convenient, we will allow $x_0 = t$ and $\partial_0 = \partial_t$.

We shall assume that $Q(du, d^2u)$ is of the form

$$(1.7) \qquad Q^I(du, d^2u) = B^I(du) + \sum_{\substack{0 \le j,k,l \le 3 \\ 1 \le J,K \le D}} B_{K,l}^{IJ,jk} \partial_l u^K \partial_j \partial_k u^J, \quad 1 \le I \le D$$

where $B^I(du)$ is a quadratic form in the gradient of u and $B_{K,l}^{IJ,jk}$ are real constants satisfying the symmetry conditions

$$(1.8) \qquad B_{K,l}^{IJ,jk} = B_{K,l}^{JI,jk} = B_{K,l}^{IJ,kj}.$$

To obtain global existence, we shall also require that the equations satisfy the following null condition which only involves the self-interactions of each wave family. That is, we require that

$$(1.9) \qquad \sum_{0 \le j,k,l \le 3} B_{J,l}^{JJ,jk} \xi_j \xi_k \xi_l = 0 \quad \text{whenever } \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1, \dots, D.$$

To describe the null condition for the lower order terms, we expand

$$B^I(du) = \sum_{\substack{1 \le J,K \le D \\ 0 \le j,k \le 3}} A_{JK}^{I,jk} \partial_j u^J \partial_k u^K.$$

We then require that each component satisfies the similar null condition

$$(1.10) \qquad \sum_{0 \le j,k \le 3} A_{JJ}^{J,jk} \xi_j \xi_k = 0 \quad \text{whenever } \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1, \dots, D.$$

In order to solve (1.5) we must also assume that the data satisfies the relevant compatibility conditions. Since these are well-known (see, e.g., [12]), we shall only describe them briefly. To do so we first let $J_k u = \{\partial_x^\alpha u : 0 \le |\alpha| \le k\}$ denote the collection of all spatial derivatives of u of order up to k. Then if m is fixed and if u is a formal H^m solution of (1.5), we can write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$, for certain compatibility functions ψ_k which depend on the nonlinear term Q as well as $J_k f$ and $J_{k-1}g$. Having done this, the compatibility condition for (1.5) with $(f, g) \in H^m \times H^{m-1}$ is just the requirement that the ψ_k vanish on $\partial\mathscr{K}$ when $0 \le k \le m-1$. Additionally, we shall say that $(f, g) \in C^\infty$ satisfy the compatibility conditions to infinite order if this condition holds for all m.

We can now state our main result:

Theorem 1.1. *Let $\mathscr{K}$ be a fixed compact obstacle with smooth boundary that satisfies* (1.4). *Assume that $Q(du, d^2u)$ and $\Box$ are as above and that $(f, g) \in C^\infty(\mathbb{R}^3\backslash\mathscr{K})$ satisfy the compatibility conditions to infinite order. Then there is a constant $\varepsilon_0 > 0$, and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$, if*

$$(1.11)\qquad \sum_{|\alpha| \le N} \|\langle x\rangle^{|\alpha|}\partial_x^\alpha f\|_2 + \sum_{|\alpha| \le N-1} \|\langle x\rangle^{1+|\alpha|}\partial_x^\alpha g\|_2 \le \varepsilon$$

then (1.5) *has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3\backslash\mathscr{K})$.*

This paper is organized as follows. In the next section, we will recall some energy estimates from [22]. In §3, we will gather the pointwise estimates that we will require. In §4, we will collect some Sobolev-type estimates. Included are some bounds on the null forms which are exterior domain analogs of those from [29] and [32]. Finally, in §5, we will use these estimates to prove the global existence theorem via a continuity argument.

2 L^2 Estimates

In this section, we will recall some estimates of [22] related to the energy inequality. Unless stated otherwise, the proofs of the following results can be found in [22]. Specifically, we will be concerned with solutions $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3\backslash\mathscr{K})$ of the Dirichlet-wave equation

$$(2.1)\qquad \begin{cases} \Box_\gamma u = F \\ u|_{\partial\mathscr{K}} = 0 \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g \end{cases}$$

where

$$(\Box_\gamma u)^I = (\partial_t^2 - c_I^2\Delta)u^I + \sum_{J=1}^{D}\sum_{j,k=0}^{3} \gamma^{IJ,jk}(t,x)\partial_j\partial_k u^J, \quad 1 \le I \le D.$$

We shall assume that the $\gamma^{IJ,jk}$ satisfy the symmetry conditions

$$(2.2)\qquad \gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj}$$

as well as the size condition

$$(2.3)\qquad \sum_{I,J=1}^{D}\sum_{j,k=0}^{3}\|\gamma^{IJ,jk}(t,x)\|_\infty \le \delta$$

for δ sufficiently small (depending on the wave speeds). The energy estimate will involve bounds for the gradient of the perturbation terms

$$\|\gamma'(t,\cdot)\|_\infty = \sum_{I,J=1}^{D}\sum_{j,k,l=0}^{3}\|\partial_l\gamma^{IJ,jk}(t,\cdot)\|_\infty,$$

and the energy form associated with $\Box_\gamma$, $e_0(u) = \sum_{I=1}^{D} e_0^I(u)$, where

$$(2.4)\qquad e_0^I(u) = (\partial_0 u^I)^2 + \sum_{k=1}^{3} c_I^2(\partial_k u^I)^2 + 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0 u^I\partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}\partial_j u^I\partial_k u^J.$$

The most basic estimate will lead to a bound for

$$E_M(t) = E_M(u)(t) = \int\sum_{j=0}^{M} e_0(\partial_t^j u)(t,x)\,dx.$$

Lemma 2.1. *Fix $M = 0,1,2,\dots,$ and assume that the perturbation terms $\gamma^{IJ,jk}$ are as above. Suppose also that $u \in C^\infty$ solves* (2.1) *and for every t, $u(t,x) = 0$ for large x. Then there is an absolute constant C so that*

$$(2.5)\qquad \partial_t E_M^{1/2}(t) \le C\sum_{j=0}^{M}\|\Box_\gamma\partial_t^j u(t,\cdot)\|_2 + C\|\gamma'(t,\cdot)\|_\infty E_M^{1/2}(t).$$

Before stating the next result, let us introduce some notation. If $P = P(t,x,D_t,D_x)$ is a differential operator, we shall let

$$[P,\gamma^{kl}\partial_k\partial_l]u = \sum_{1\le I,J\le D}\sum_{0\le k,l\le 3}|[P,\gamma^{IJ,kl}\partial_k\partial_l]u^J|.$$

In order to generalize the above energy estimate to include the more general vector fields L, Z, we will need to use a variant of the scaling vector field L. We fix a bump function $\eta \in C^\infty(\mathbb{R}^3)$ with $\eta(x) = 0$ for $x \in \mathscr{K}$ and $\eta(x) = 1$ for $|x| > 1$. Then, set $\tilde{L} = \eta(x)r\partial_r + t\partial_t$. Using this variant of the scaling vector field and an elliptic regularity argument, one can establish

Proposition 2.2. *Suppose that the constant in* (2.3) *is small. Suppose further that*

$$\|\gamma'(t,\cdot)\|_\infty \le \delta/(1+t), \tag{2.6}$$

and

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} (\|\tilde{L}^\mu\partial_t^j\Box_\gamma u(t,\cdot)\|_2 + \|[\tilde{L}^\mu\partial_t^j,\gamma^{kl}\partial_k\partial_l]u(t,\cdot)\|_2) \tag{2.7}$$

$$\le \frac{\delta}{1+t}\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\|_2 + H_{\nu_0,N_0}(t),$$

where N_0 and ν_0 are fixed. Then

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'(t,\cdot)\|_2 \tag{2.8}$$

$$\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha\Box u(t,\cdot)\|_2$$

$$+ C(1+t)^{A\delta}\sum_{\substack{\mu+j\le N_0+\nu_0\\ \mu\le\nu_0}} \Big(\int e_0(\tilde{L}^\mu\partial_t^j u)(0,x)\,dx\Big)^{1/2}$$

$$+ C(1+t)^{A\delta}\Bigg(\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2\,ds + \int_0^t H_{\nu_0,N_0}(s)\,ds\Bigg)$$

$$+ C(1+t)^{A\delta}\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha u'(s,\cdot)\|_{L^2(|x|<1)}\,ds,$$

where the constants C and A are absolute constants.

In practice $H_{\nu_0,N_0}(t)$ will involve weighted L^2_x norms of $|L^\mu\partial^\alpha u'|^2$ with $\mu+|\alpha|$ much smaller than $N_0+\nu_0$, and so the integral involving H_{ν_0,N_0} can be dealt with using an inductive argument and weighted $L^2_tL^2_x$ estimates that will be presented at the end of this section.

In proving our existence results for (1.5), the key step will be to obtain a priori L^2-estimates involving $L^\mu Z^\alpha u'$. Begin by setting

$$Y_{N_0,\nu_0}(t) = \int \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{2.9}$$

We, then, have the following proposition which shows how the $L^\mu Z^\alpha u'$ estimates can be obtained from the ones involving $L^\mu \partial^\alpha u'$.

Proposition 2.3. *Suppose that the constant* δ *in* (2.3) *is small and that* (2.6) *holds. Then,*

$$\partial_t Y_{N_0,\nu_0} \le C Y_{N_0,\nu_0}^{1/2} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\Box_\gamma L^\mu Z^\alpha u(t,\cdot)\|_2 + C\|\gamma'(t,\cdot)\|_\infty Y_{N_0,\nu_0} + C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu \partial^\alpha u'(s,\cdot)\|^2_{L^2(|x|<1)}. \tag{2.10}$$

As in [13] and [14] we shall also require some weighted $L^2_t L^2_x$ estimates. They will be used, for example, to control the local L^2 norms such as the last term in (2.10). For convenience, for the remainder of this section, allow $\Box = \partial_t^2 - \Delta$ to denote the unit speed d'Alembertian. The transition from the following estimates to those involving (1.6) is straightforward. Also, allow

$$S_T = \{[0,T]\times \mathbb{R}^3\backslash\mathcal{K}\}$$

to denote the time strip of height T in $\mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K}$.

We, then, have the following proposition.

Proposition 2.4. *Fix* N_0 *and* ν_0. *Suppose that* $\mathcal{K}$ *satisfies the local exponential energy decay* (1.4). *Suppose further that* $u \in C^\infty$ *solves* (2.1) *and* $u(t,x) = 0$ *for* $t < 0$. *Then there is a constant* $C = C_{N_0,\nu_0,\mathcal{K}}$ *so that if u vanishes for large x at every fixed t*

$$(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_T)} \le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|\Box L^\mu \partial^\alpha u(s,\cdot)\|_2\, ds + C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\Box L^\mu \partial^\alpha u\|_{L^2(S_T)} \tag{2.11}$$

and

$$(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_T)} \le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|\Box L^\mu Z^\alpha u(s,\cdot)\|_2\, ds + C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\Box L^\mu Z^\alpha u\|_{L^2(S_T)}. \tag{2.12}$$

To be able to handle the last term in (2.8), we shall need the following.

Lemma 2.5. *Suppose that* (1.4) *holds, and suppose that* $u \in C^\infty$ *solves* (2.1) *and satisfies* $u(t,x) = 0$ *for* $t < 0$. *Then, for fixed* N_0 *and* ν_0 *and* $t > 2$,

$$(2.13)\qquad \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \int_0^t \|L^\mu \partial^\alpha u'(s,\cdot)\|_{L^2(|x|<2)}\, ds$$

$$\le C \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0+1 \\ \mu \le \nu_0}} \int_0^t \left(\int_0^s \|L^\mu \partial^\alpha \Box u(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\, d\tau \right) ds.$$

3 Pointwise Estimates

Here we will estimate solutions of the scalar inhomogeneous wave equation

$$(3.1)\qquad \begin{cases} (\partial_t^2 - \Delta) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathscr{K} \\ w(t,\cdot)|_{\partial \mathscr{K}} = 0 \\ w(t,x) = 0, & t \le 0. \end{cases}$$

If we assume, as before, that $\mathscr{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$ and that $\mathscr{K}$ satisfies (1.4), then we have the following result whose proof can be found in Metcalfe-Sogge [22].

Theorem 3.1. *Let* w *be a solution to* (3.1), *and suppose that the local energy decay bounds* (1.4) *hold for* $\mathscr{K}$. *Then,*

$$(3.2)\qquad (1+t+|x|)|L^\nu Z^\alpha w(t,x)| \le C \int_0^t \int_{\mathbb{R}^3 \backslash \mathscr{K}} \sum_{\substack{|\beta|+\mu \le |\alpha|+\nu+7 \\ \mu \le \nu+1}} |L^\mu Z^\beta F(s,y)| \frac{dy\, ds}{|y|}$$

$$+ C \int_0^t \sum_{\substack{|\beta|+\mu \le |\alpha|+\nu+4 \\ \mu \le \nu+1}} \|L^\mu \partial^\beta F(s,\cdot)\|_{L^2(|y|<2)}\, ds.$$

Here and throughout $\{|y| < 2\}$ is understood to mean $\{y \in \mathbb{R}^3 \backslash \mathscr{K} : |y| < 2\}$.

Additionally, we can prove the following improved pointwise bound for the gradient of the solution w. In this modified result, we are able to bring the gradient inside the main term (the first term) on the right side.

Theorem 3.2. *Let* w *be a solution to* (3.1), *and suppose that the local energy decay bounds* (1.4) *hold for* $\mathscr{K}$. *Suppose further that* $F(t,x) = 0$ *when* $|x| > 10t$. *Then, if* $|x| < t/10$ *and* $t > 1$,

$$(3.3)\quad (1+t+|x|)|L^\nu Z^\alpha w'(t,x)|$$

$$\leq C \sum_{\substack{\mu+|\beta|\leq \nu+|\alpha|+3\\ \mu\leq \nu+1}} \int_{t/100}^t \int_{\mathbb{R}^3\backslash\mathscr{K}} |L^\mu Z^\beta F'(s,y)|\frac{dy\,ds}{|y|}$$

$$+ C \sup_{0\leq s\leq t} (1+s) \sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+4\\ \mu\leq\nu}} \|L^\mu Z^\beta F(s,\cdot)\|_\infty$$

$$+ C \sup_{0\leq s\leq t} (1+s) \sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+7\\ \mu\leq\nu}} \int_0^s \int_{\substack{||y|-(s-\tau)|\leq 10\\ |y|\leq(600+\tau)/2}} |L^\mu Z^\beta F(\tau,y)|\frac{dy\,d\tau}{|y|}$$

$$+ C \sup_{0\leq s\leq t} \sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+8\\ \mu\leq\nu+1}} \int_{s/100}^s \int_{|y|\geq(1+\tau)/10} |L^\mu Z^\beta F(\tau,y)|\frac{dy\,d\tau}{|y|}.$$

The remainder of this section will be dedicated to the proof of (3.3).

The Minkowski space estimate we shall use says that if w_0 is a solution to the inhomogeneous wave equation

$$(3.4)\quad \begin{cases}(\partial_t^2-\Delta)w_0(t,x)=G(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\\ w_0(0,x)=\partial_t w_0(0,x)=0,\end{cases}$$

and if $G(s,y)=0$ when $|y|>10s$, then

$$(3.5)\quad (1+t)|L^\nu Z^\alpha w_0(t,x)|$$

$$\leq C\int_{t/100}^t\int_{\mathbb{R}^3}\sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+3\\ \mu\leq\nu+1}}|L^\mu Z^\beta G(s,y)|\frac{dy\,ds}{|y|},\quad |x|<t/10.$$

By using sharp Huygens principle, this follows from inequality (2.3) in [14] and the fact that $[\partial_t^2-\Delta,Z]=0$ and $[\partial_t^2-\Delta,L]=2(\partial_t^2-\Delta)$.

Recall that we are assuming that $\mathscr{K}\subset\{x\in\mathbb{R}^3 : |x|<1\}$. With this in mind, the first step is to see that (3.5) yields for $|x|<t/10$

$$(3.6)\quad (1+t)|L^\nu Z^\alpha w'(t,x)|\leq C\int_{t/100}^t\int_{\mathbb{R}^3\backslash\mathscr{K}}\sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+3\\ \mu\leq\nu+1}}|L^\mu Z^\beta F'(s,y)|\frac{dy\,ds}{|y|}$$

$$+C\sup_{|y|\leq 2,\,0\leq s\leq t}(1+s)\sum_{\substack{|\beta|+\mu\leq|\alpha|+\nu+1\\ \mu\leq\nu}}|L^\mu\partial^\beta w'(s,y)|.$$

The proof is exactly like that of Lemma 4.2 in [14].

As a result of (3.6), we would be done if we could show that

$$\sup_{0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu}}\|L^\mu\partial^\beta w'(s,\cdot)\|_{L^\infty(|x|<2)} \tag{3.7}$$

is controlled by the right side of (3.3).

To prove this, we shall need the following

Lemma 3.3. *Suppose that* w *is as above. Suppose further that* $(\partial_t^2-\Delta)w(s,y)=F(s,y)=0$ *if* $|y|>10$. *Then,*

$$(1+t)\sup_{|x|<2}|L^\nu\partial^\alpha w'(t,x)|\le C\sup_{0\le s\le t}\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+3\\ \mu\le\nu}}(1+s)\|L^\mu\partial^\beta F(s,\cdot)\|_2. \tag{3.8}$$

Proof of Lemma 3.3. By Sobolev estimates, the left side of (3.8) is dominated by

$$(1+t)\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+2\\ \mu\le\nu}}\|L^\mu\partial^\beta w'(t,\cdot)\|_{L^2(|x|<3)}.$$

By exponential energy decay and elliptic regularity (see Lemma 2.8 in [22]), there must be a constant $c>0$ so that this is controlled by

$$\begin{aligned}(1+t)&\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+1\\ \mu\le\nu}}\|L^\mu\partial^\beta F(t,\cdot)\|_2\\ &+(1+t)\int_0^t e^{-c(t-s)}\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+3\\ \mu\le\nu}}\|L^\mu\partial^\beta F(s,\cdot)\|_2\,ds\\ &\le C\sup_{0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+3\\ \mu\le\nu}}\|L^\mu\partial^\beta F(s,\cdot)\|_2,\end{aligned}$$

as desired. □

We also need an estimate for solutions whose forcing terms vanish near the obstacle. Suppose that w is as above, but now assume that $(\partial_t^2-\Delta)w(s,y)=F(s,y)=0$ when $|y|<5$. Then, write

$$w=w_0+w_r$$

where w_0 solves the boundaryless wave equation $(\partial_t^2-\Delta)w_0=F$ with zero initial data. Fix $\eta \in C_0^\infty(\mathbb{R}^3)$ satisfying $\eta(x) = 1$ for $|x| < 2$ and $\eta(x) = 0$ for $|x| \geq 3$. If we set $\tilde{w} = \eta w_0 + w_r$, then since $\eta F = 0$, $\tilde{w}$ solves the Dirichlet-wave equation

$$(\partial_t^2 - \Delta)\tilde{w} = G = -2\nabla_x \eta \cdot \nabla_x w_0 - (\Delta\eta) w_0$$

with zero initial data. This forcing term vanishes unless $2 \leq |x| \leq 3$. In this case, by Lemma 3.3,

$$(1+t) \sum_{\substack{|\beta|+\mu \leq |\alpha|+\nu+1 \\ \mu \leq \nu}} \sup_{|x|<2} |L^\mu \partial^\beta w'(t,x)| = (1+t) \sum_{\substack{|\beta|+\mu \leq |\alpha|+\nu+1 \\ \mu \leq \nu}} \sup_{|x|<2} |L^\mu \partial^\beta \tilde{w}'(t,x)|$$

$$\leq C \sup_{0 \leq s \leq t} \sum_{\substack{|\beta|+\mu \leq |\alpha|+\nu+4 \\ \mu \leq \nu}} (1+s) \|L^\mu \partial^\beta G(s,\cdot)\|_2$$

$$\leq C \sup_{0 \leq s \leq t} \sum_{\substack{|\beta|+\mu \leq |\alpha|+\nu+5 \\ \mu \leq \nu}} (1+s) \|L^\mu \partial^\beta w_0(s,\cdot)\|_{L^\infty(2<|x|<3)}.$$

Thus, if we could show the following lemma, we would have that the (3.7) is bounded by the right side of (3.3), which would complete the proof of Theorem 3.2.

Lemma 3.4. *Suppose that v is a solution to the free wave equation $(\partial_t^2 - \Delta)v = G$ and that v has vanishing Cauchy data. Suppose further that $G(t,x) = 0$ when $|x| > 10t$. Then,*

$$\sup_{2<|x|<3} |v(t,x)| \leq C \sum_{|\alpha| \leq 2} \int_0^t \int_{\substack{||y|-(t-s)|<10 \\ |y| \leq (600+s)/2}} |\Omega^\alpha G(s,y)| \frac{dy\,ds}{|y|} \tag{3.9}$$

$$+ \frac{C}{1+t} \sum_{\substack{|\alpha|+\mu \leq 3 \\ \mu \leq 1}} \int_{t/100}^t \int_{|y| \geq (1+s)/10} |L^\mu Z^\alpha G(s,y)| \frac{dy\,ds}{|y|}.$$

Proof of Lemma 3.4. This lemma is a consequence of the estimate

$$|x|\,|v(t,x)| \leq \frac{1}{2} \int_0^t \int_{|r-(t-s)|}^{r+t-s} \sup_{|\theta|=1} |G(s,\rho\theta)| \rho\, d\rho\, ds, \tag{3.10}$$

where $r = |x|$. See, e.g., (2.4) in [14].

We begin by choosing a cutoff function $\rho(x)$ satisfying $\rho(x) = 1$ for $|x| < 1/10$ and $\rho(x) = 0$ for $|x| > 1/2$. Set $G_1(t,x) = \rho(x/(1+t))G(t,x)$ and $G_2(t,x) = (1 - \rho(x/(1+t)))G(t,x)$, and for $j = 1,2$, let v_j solve the inhomogeneous wave equation $(\partial_t^2 - \Delta)v_j = G_j$ with zero initial data. Then, $v = v_1 + v_2$. Using (3.10) and the Sobolev estimate on the sphere, we have for $2 < |x| < 3$

$$|v_1(t,x)| \le C \sum_{|\alpha|\le 2} \int_0^t \int_{||y|-(t-s)|<10} |\Omega^\alpha G_1(s,y)| \frac{dy\,ds}{|y|}$$

$$\le C \sum_{|\alpha|\le 2} \int_0^t \int_{\substack{||y|-(t-s)|<10 \\ |y|\le(1+s)/2}} |\Omega^\alpha G(s,y)| \frac{dy\,ds}{|y|}.$$

By (3.5), if $2 < |x| < 3$ and $t > 10|x|$,

$$|v_2(t,x)| \le \frac{C}{1+t} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} \int_{t/100}^t \int |L^\mu Z^\alpha G_2(s,y)| \frac{dy\,ds}{|y|}$$

$$\le \frac{C}{1+t} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} \int_{t/100}^t \int_{|y|\ge(1+s)/10} |L^\mu Z^\alpha G(s,y)| \frac{dy\,ds}{|y|}.$$

Since $v = v_1 + v_2$, these two estimates yield (3.9) when $t > 10|x|$. The proof of the estimate for v_1 shows that for $0 < t < 10|x|$ the left side of (3.9) is dominated by the first term in the right. □

4 Estimates Related to the Null Condition and Sobolev-type Estimates

The first result of this section concerns bounds for the null forms. They must involve the weight $\langle c_J t - r\rangle$ since we are not using the generators of Lorentz rotations. The estimates will involve the admissible homogeneous vector fields that we are using $\{\Gamma\} = \{Z, L\}$. The proof of these estimates can be found in Sideris-Tu [29] and Sogge [32].

Lemma 4.1. *Suppose that the quasilinear null condition* (1.9) *holds. Then,*

$$(4.1)\quad \left|\sum_{0\le j,k,l\le 3} B_{J,l}^{JJ,jk} \partial_l u \partial_j \partial_k v\right|$$

$$\le C\langle r\rangle^{-1}(|\Gamma u|\,|\partial^2 v| + |\partial u|\,|\partial \Gamma v|) + C\frac{\langle c_J t - r\rangle}{\langle t+r\rangle}|\partial u|\,|\partial^2 v|.$$

Also, if the semilinear null condition (1.10) *holds*

$$(4.2)\quad \left|\sum_{0\le j,k\le 3} A_{JJ}^{J,jk} \partial_j u \partial_k v\right| \le C\langle r\rangle^{-1}(|\Gamma u|\,|\partial v| + |\partial u|\,|\Gamma v|) + C\frac{\langle c_J t - r\rangle}{\langle t+r\rangle}|\partial u|\,|\partial v|.$$

We shall also need the following Sobolev-type estimate. The first is an exterior domain analog of results of Klainerman-Sideris [17].

Lemma 4.2. *Suppose that* $u(t,x) \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3\backslash\mathscr{K})$ *vanishes for* $x \in \partial\mathscr{K}$. *Then if* $|\alpha| = M$ *and* ν *are fixed*

$$(4.3)\qquad \|\langle t-r\rangle L^\nu Z^\alpha \partial^2 u(t,\cdot)\|_2 \le C \sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le\nu+1}} \|L^\mu Z^\beta u'(t,\cdot)\|_2 + C \sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\cdot)\|_2 + C(1+t)\sum_{\mu\le\nu}\|L^\mu u'(t,\cdot)\|_{L^2(|x|<2)}.$$

Proof of Lemma 4.2. The first step is to show that

$$(4.4)\qquad \|\langle t-r\rangle L^\nu Z^\alpha \partial^2 u(t,\cdot)\|_2 \le C \sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le\nu+1}} \|L^\mu Z^\beta u'(t,\cdot)\|_2 + C \sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\cdot)\|_2 + C(1+t)\sum_{\substack{|\beta|+\mu\le M+\nu+2\\ \mu\le\nu}}\|L^\mu \partial^\beta u(t,\cdot)\|_{L^2(|x|<3/2)}$$

If one replaces the left side by the analogous expression with the norm taken over $|x| < 3/2$ then this term is dominated by the last term in (4.4) due to the fact that the coefficients of Z are bounded when $|x| < 3/2$.

To handle the part where $|x| > 3/2$ we shall use the following Minkowski space estimate

$$(4.5)\qquad \|\langle t-r\rangle L^\nu Z^\alpha \partial^2 h(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le C \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu+1}} \|L^\mu Z^\beta h'(t,\cdot)\|_{L^2(\mathbb{R}^3)} + C \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu\\ \mu\le\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)h(t,\cdot)\|_{L^2(\mathbb{R}^3)},$$

which is valid for $h \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$. This estimate follows from (2.10) and Lemma

3.1 of Klainerman and Sideris [17] if one uses the fact that $[(\partial_t^2 - \Delta), Z] = 0$ and $[(\partial_t^2 - \Delta), L] = 2(\partial_t - \Delta)$.

To use this, choose $\eta \in C_0^\infty(\mathbb{R}^3)$ so that $\eta(x) = 0$ for $|x| < 1$ and $\eta(x) = 1$ for $|x| > 3/2$. Then if we let $h(t,x) = \eta(x)u(t,x)$, we have

$$(\partial_t^2 - \Delta)h = \eta(x)(\partial_t^2 - \Delta)u - 2\nabla_x\eta \cdot \nabla_x u - (\Delta\eta)u.$$

Therefore since the last two terms are supported in $|x| < 3/2$, (4.5) yields

$$\begin{aligned}
\|\langle t-r\rangle L^\nu Z^\alpha \partial^2 u(t,\cdot)\|_{L^2(|x|>3/2)} &\le \|\langle t-r\rangle L^\nu Z^\alpha \partial^2 h(t,\cdot)\|_{L^2(\mathbb{R}^3)} \\
\le C \sum_{\substack{|\beta|+\mu \le M+\nu+1 \\ \mu \le \nu+1}} &\|L^\mu Z^\beta u'(t,\cdot)\|_2 \\
+ C \sum_{\substack{|\beta|+\mu \le M+\nu \\ \mu \le \nu}} &\|\langle t+r\rangle L^\mu Z^\beta (\partial_t^2 - \Delta)u(t,\cdot)\|_2 \\
+ C(1+t) \sum_{\substack{|\beta|+\mu \le M+\nu+1 \\ \mu \le \nu}} &\|L^\mu \partial^\beta u(t,\cdot)\|_{L^2(|x|<3/2)},
\end{aligned}$$

which completes the proof of (4.4).

In view of this inequality, to finish the proof of the lemma we need to show that if $1 < R < 2$ then

$$(1+t) \sum_{\substack{|\beta|+\mu \le M+\nu+2 \\ \mu \le \nu}} \|L^\mu \partial^\beta u(t,\cdot)\|_{L^2(|x|<R)} \tag{4.6}$$

is controlled by the right hand side of (4.3). However, if $1 < R < R_0 < 2$, by elliptic regularity this term is dominated by

$$\begin{aligned}
(1+t)\Bigg(\sum_{\substack{|\beta|+\mu \le M+\nu \\ \mu \le \nu}} \|L^\mu \partial^\beta \Delta u(t,\cdot)\|_{L^2(|x|<R_0)} + \sum_{\substack{|\beta|+\mu \le M+\nu \\ \mu \le \nu}} \|L^\mu \partial^\beta \partial_t u'(t,\cdot)\|_{L^2(|x|<R)} \\
+ \sum_{\substack{|\beta|+\mu \le (M-1)+\nu+2 \\ \mu \le \nu}} \|L^\mu \partial^\beta u(t,\cdot)\|_{L^2(|x|<R_0)} \Bigg).
\end{aligned}$$

Since Lemma 2.3 from [17] yields

$$\langle t-r\rangle\Big(\sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}}|L^\mu\partial^\beta\Delta u|+\sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}}|L^\mu\partial^\beta\partial_t u'|\Big)$$

$$\le C\sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le\nu+1}}|L^\mu Z^\beta u'|+C\langle t+r\rangle\sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}}|L^\mu Z^\beta(\partial_t^2-\Delta)u|,$$

the preceding inequality and an induction argument imply that (4.6) is dominated by the right hand side of (4.3) plus

$$(1+t)\sum_{|\beta|\le 1,\,\mu\le\nu}\|L^\mu\partial^\beta u(t,\cdot)\|_{L^2(|x|<\tilde R)},$$

where $\tilde R$ can be taken to satisfy $3/2<\tilde R<2$. Since $\partial_t^j u$ vanishes on $\partial\mathscr{K}$ one can use a similar induction argument to see that this is also dominated by the right hand side of (4.3) plus

$$(1+t)\sum_{\mu\le\nu}\|L^\mu\nabla_x u(t,\cdot)\|_{L^2(|x|<R_1)},$$

where $\tilde R<R_1<2$, which finishes the proof. □

The next lemma is an exterior domain analog of an estimate of Hidano-Yokoyama [6].

Lemma 4.3. *Suppose that $u(t,x)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3\backslash\mathscr{K})$ vanishes for $x\in\partial\mathscr{K}$. Then*

$$r^{1/2}\langle t-r\rangle|\partial L^\nu Z^\alpha u(t,x)| \tag{4.7}$$

$$\le C\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+2\\ \mu\le\nu+1}}\|L^\mu Z^\beta u'(t,\cdot)\|_2$$

$$+C\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu}}\|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\cdot)\|_2$$

$$+C(1+t)\sum_{\mu\le\nu}\|L^\mu u'(t,\cdot)\|_{L^\infty(|x|<2)}.$$

Proof of Lemma 4.3. Inequality (4.2) of Hidano-Yokoyama [6] implies that in Minkowski space

$$r^{1/2}\langle t-r\rangle|\partial L^\nu Z^\alpha h(t,x)| \tag{4.8}$$

$$\le C\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu}}(\|L^\mu Z^\beta h'(t,\cdot)\|_2+\|\langle t-r\rangle L^\mu Z^\beta\partial^2 h(t,\cdot)\|_2).$$

If we choose $\eta \in C_0^\infty(\mathbb{R}^3)$ so that $\eta(x) = 0$ for $|x| < 1$ and $\eta(x) = 1$ for $|x| > 5/4$, and let $h(t,x) = \eta(x)u(t,x)$, then we conclude that when $|x| > 5/4$,

$$\begin{aligned} &r^{1/2}\langle t-r\rangle|\partial L^\nu Z^\alpha u(t,x)| \\ &\quad \le C \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu}} (\|L^\mu Z^\beta u'(t,\cdot)\|_2 + \|\langle t-r\rangle L^\mu Z^\beta \partial^2 u(t,\cdot)\|_2) \\ &\qquad + C(1+t) \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+3\\ \mu\le\nu}} \|L^\mu \partial^\beta u(t,\cdot)\|_{L^2(|x|<3/2)}. \end{aligned}$$

By the Sobolev inequality, over $|x| < 5/4$ the left side of (4.7) is bounded by a similar inequality involving only the last term on the right.

If we use (4.3), we see that the second term in the right is dominated by the right side of (4.7). If we repeat the last part of the proof of Lemma 4.2, we conclude that the same is true for the last term in the preceding inequality. □

Finally, we will need the following now standard consequence of the Sobolev lemma (see [16]).

Lemma 4.4. *Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then, for $R > 2$,*

$$\|h\|_{L^\infty(R<|x|<R+1)} \le CR^{-1} \sum_{|\alpha|+|\beta|\le 2} \|\Omega^\alpha \partial_x^\beta h\|_{L^2(R-1<|x|<R+2)}.$$

5 Global Existence and the Continuity Argument

In this section, we will prove the main result, Theorem 1.1. We shall take $N = 101$ in its smallness hypothesis (1.11), but this is certainly not optimal.

To prove our global existence theorem, we shall need a standard local existence theorem.

Theorem 5.1. *Suppose that f and g are as in Theorem 1.1 with $N \ge 6$ in* (1.11). *Then there is a $T > 0$ so that the initial value problem* (1.5) *with this initial data has a C^2 solution satisfying*

$$u \in L^\infty([0,T]; H^N(\mathbb{R}^3\backslash\mathscr{K})) \cap C^{0,1}([0,T]; H^{N-1}(\mathbb{R}^3\backslash\mathscr{K})).$$

The supremum of such T is equal to the supremum of all T such that the initial value problem has a C^2 solution with $\partial^\alpha u$ bounded for $|\alpha| \le 2$. Also, one can take $T \ge 2$ if $\|f\|_{H^N} + \|g\|_{H^{N-1}}$ is sufficiently small.

This essentially follows from the local existence results Theorem 9.4 and Lemma 9.6 in [12]. The latter were only stated for diagonal single-speed systems; however, since the proof relied only on energy estimates, it extends to the multi-speed non-diagonal case if the symmetry assumptions (1.8) are satisfied.

Next, as in [14], in order to avoid dealing with compatibility conditions for the Cauchy data, it is convenient to reduce the Cauchy problem (1.5) to an equivalent equation with a nonlinear driving force but vanishing Cauchy data. We will then set up a continuity argument that utilizes the results of the previous three sections to show global existence and prove Theorem 1.1.

Recall that our smallness condition on the data is

$$\sum_{|\alpha|\le 101} \|\langle x\rangle^{|\alpha|}\partial_x^\alpha f\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|\le 100} \|\langle x\rangle^{1+|\alpha|}\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon. \tag{5.1}$$

To make the reduction to an equation with vanishing initial data, we will begin by noting that if the data satisfies (5.1) for ε sufficiently small, then we can find a solution u to (1.5) on a set of the form $0 < ct < |x|$ where $c = 5\max_I c_I$, and this solution satisfies

$$\sup_{0<t<\infty} \sum_{|\alpha|\le 101} \|\langle x\rangle^{|\alpha|}\partial^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|>ct)} \le C_0\varepsilon, \tag{5.2}$$

for an absolute constant C_0.

To prove this, we shall repeat the argument of Keel-Smith-Sogge [14]. By scaling in t, we may assume without loss that $\max_I c_I = 1/2$. Theorem 5.1 yields a solution u to (1.5) on the set $0 < t < 2$ which satisfies (5.2). We wish to show that this solution extends to the set $0 < ct < |x|$. To do so, let $R \ge 4$ and consider data (f_R, g_R) supported in the set $R/4 < |x| < 4R$ which agrees with the data (f, g) on the set $R/2 < |x| < 2R$. Let $u_R(t,x)$ satisfy the free wave equation

$$\Box u_R = Q(du_R, R^{-1}d^2u_R)$$

with Cauchy data $(f_R(R\cdot), Rg_R(R\cdot))$. The solution u_R then exists for $0 < t < 1$ by standard local existence theory (see, e.g., [7] and [31]) and satisfies

$$\begin{aligned}\sup_{0<t<1} &\|u_R(t,\cdot)\|_{H^{101}(\mathbb{R}^3)}\\ &\le C(\|f_R(R\cdot)\|_{H^{101}(\mathbb{R}^3)} + R\|g_R(R\cdot)\|_{H^{100}(\mathbb{R}^3)})\\ &\le CR^{-3/2}\Big(\sum_{|\alpha|\le 101}\|(R\partial_x)^\alpha f_R\|_{L^2(\mathbb{R}^3)} + R\sum_{|\alpha|\le 100}\|(R\partial_x)^\alpha g_R\|_{L^2(\mathbb{R}^3)}\Big).\end{aligned}$$

The smallness condition on $|u_R'|$ implies that the wave speeds for the quasilinear equation are bounded above by 1. A domain of dependence argument shows that the solutions $u_R(R^{-1}t, R^{-1}x)$ restricted to $\big||x| - R\big| < \frac{R}{2} - t$ agree on their overlaps, and also with the local solution, yielding a solution to (1.5) on the set $\{\mathbb{R}^3 \backslash \mathcal{K} : 2t < |x|\}$. An argument using a partition of unity now yields (5.2).

We are now ready to set up the continuity argument. We will use the local solution u to allow us to restrict to the case where the Cauchy data vanish. Fix a cutoff function $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi(s) = 1$ if $s \leq \frac{1}{2c}$ and $\chi(s) = 0$ if $s > \frac{1}{c}$. Set

$$u_0(t,x) = \eta(t,x)u(t,x), \quad \eta(t,x) = \chi(|x|^{-1}t).$$

Assuming as we may that $0 \in \mathcal{K}$, we have that $|x|$ is bounded below on the complement of $\mathcal{K}$ and the function $\eta(t,x)$ is smooth and homogeneous of degree 0 in (t,x). Additionally,

$$\Box u_0 = \eta Q(du, d^2u) + [\Box, \eta]u.$$

Thus, u solves $\Box u = Q(du, d^2u)$ for $0 < t < T$ if and only if $w = u - u_0$ solves

$$\begin{cases} \Box w = (1-\eta)Q(du, d^2u) - [\Box, \eta]u \\ w|_{\partial\mathcal{K}} = 0 \\ w(t,x) = 0, \quad t \leq 0 \end{cases} \tag{5.3}$$

for $0 < t < T$.

A key step in proving that (5.3) admits a global solution is to prove uniform energy and dispersive estimates for w on the interval of existence. First note that since $u_0 = \eta u$, by (5.2) and Lemma 4.4, there is an absolute constant C_1 so that

$$(1+t+|x|) \sum_{\mu+|\alpha|\leq 99} |L^\mu Z^\alpha u_0(t,x)| + \sum_{\mu+|\alpha|+|\beta|\leq 101} \|\langle t+r\rangle^{|\beta|} L^\mu Z^\alpha \partial^\beta u_0(t,\cdot)\|_2 \leq C_1\varepsilon. \tag{5.4}$$

Furthermore, if we let v be the solution of the linear equation

$$\begin{cases} \Box v = -[\Box, \eta]u \\ v|_{\partial\mathcal{K}} = 0 \\ v(t,x) = 0, \quad t \leq 0, \end{cases} \tag{5.5}$$

then we will show that (5.2) and Theorem 3.1 imply that there is an absolute constant C_2 so that

$$(1+t+|x|) \sum_{\mu+|\alpha|\leq 92} |L^\mu Z^\alpha v(t,x)| + \sum_{\mu+|\alpha|\leq 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2 \leq C_2\varepsilon. \tag{5.6}$$

Indeed, by (3.2), the first term on the left side of (5.6) is bounded by

$$\int_0^t \int_{|x|>cs} \sum_{\mu+|\alpha|\le 99} |L^\mu Z^\alpha([\Box,\eta]u)(s,x)| \frac{dx\,ds}{|x|}$$

$$+ \int_0^t \sum_{\mu+|\alpha|\le 96} \|L^\mu \partial^\beta([\Box,\eta]u)(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<2)}\,ds$$

which by the Schwarz inequality is bounded by

$$\sum_{\mu+|\alpha|\le 99} \sum_{j=0}^{\infty} \sup_{0<cs<2^{j+1}} \|\langle x\rangle^{3/2} L^\mu Z^\alpha[\Box,\eta]u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:2^j<|x|<2^{j+1})}.$$

Since this is dominated by

$$\sup_{0<t<\infty} \sum_{\mu+|\alpha|\le 99} \|\langle x\rangle^2 L^\mu Z^\alpha[\Box,\eta]u(t,\cdot)\|_2,$$

one gets that the first term on the left side of (5.6) is $O(\varepsilon)$ from (5.2) and the homogeneity of η.

For the second term on the left side of (5.6), if we argue as in the proof of (2.5) (except now for the linear wave equation), we see that

$$\partial_t \sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2^2$$

$$\le C\Bigl(\sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2\Bigr)\Bigl(\sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha \Box v(t,\cdot)\|_2\Bigr)$$

$$+ C \sum_{\mu+|\alpha|\le 90} \Bigl| \int_{\partial\mathcal{K}} \partial_0 L^\mu Z^\alpha v(t,\cdot) \nabla L^\mu Z^\alpha v(t,\cdot)\cdot n\,d\sigma \Bigr|,$$

where n is the outward normal at a given point on $\partial\mathcal{K}$. Since $\mathcal{K} \subset \{|x|<1\}$, it follows that

$$\partial_t \sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2^2$$

$$\le C\Bigl(\sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2\Bigr)\Bigl(\sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha \Box v(t,\cdot)\|_2\Bigr)$$

$$+ C \int_{\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<1\}} \sum_{\mu+|\alpha|\le 91} |L^\mu Z^\alpha v'(t,\cdot)|^2\,dx.$$

Thus, since $\Box v(s,y) = -[\Box,\eta]u(s,y)$, it follows that

$$(5.7)\qquad \sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha v'(t,\cdot)\|_2^2 \le C\Big(\int_0^t \sum_{\mu+|\alpha|\le 90} \|L^\mu Z^\alpha(-[\Box,\eta]u)(s,y)\|_2\,ds\Big)^2 + C\int_0^t \sum_{\mu+|\alpha|\le 91} \|L^\mu Z^\alpha v'(s,\cdot)\|^2_{L^2(|x|<1)}\,ds.$$

The first term on the right is $O(\varepsilon)$ by (5.2). Using the bound for the first term in the left of (5.6), it follows that the second term on the right of (5.7) is also $O(\varepsilon)$ as desired.

Using this, we are now ready to set up the continuity argument. If $\varepsilon > 0$ is as above, we shall assume that we have a solution of our equation (1.5) for $0 \le t \le T$ such that we have the following estimates

$$(5.8)\qquad \sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}} \|L^\nu Z^\alpha w'(t,\cdot)\|_2 \le A_0\varepsilon$$

$$(5.9)\qquad (1+t+r)\sum_{|\alpha|\le 40} |Z^\alpha w'(t,x)| \le A_1\varepsilon$$

$$(5.10)\qquad (1+t+r)\sum_{\substack{|\alpha|+\nu\le 55\\ \nu\le 3}} |L^\nu Z^\alpha(w-v)(t,x)| \le B_1\varepsilon^2(1+t)^{1/10}\log(2+t)$$

$$(5.11)\qquad \sum_{|\alpha|\le 100} \|\partial^\alpha u'(t,\cdot)\|_2 \le B_2\varepsilon(1+t)^{1/40}$$

$$(5.12)\qquad \sum_{\substack{|\alpha|+\nu\le 65\\ \nu\le 4}} \|L^\nu Z^\alpha u'(t,\cdot)\|_2 \le B_3\varepsilon(1+t)^{1/20}$$

$$(5.13)\qquad \sum_{\substack{|\alpha|+\nu\le 62\\ \nu\le 4}} \|\langle x\rangle^{-1/2} L^\nu Z^\alpha u'\|_{L^2(S_t)} \le B_4\varepsilon(1+t)^{1/20}(\log(2+t))^{1/2}.$$

Here, as before, the L^2 norms are taken over $\mathbb{R}^3\backslash\mathscr{K}$ and the weighted $L^2_tL^2_x$ norms are taken over $S_t = [0,t]\times\mathbb{R}^3\backslash\mathscr{K}$. In the main estimates (5.8) and (5.9), we can take $A_0 = A_1 = 4C_2$, where C_2 is the constant occurring in the bounds (5.6) for v.

Clearly if ε is small then all of these estimates are valid, if $T = 2$, by Theorem 5.1. Keeping this in mind, we shall then prove that, for $\varepsilon > 0$ smaller than some number depending on $B_1, \ldots, B_4$,

i.) (5.8) is valid with A_0 replaced by $A_0/2$.

ii.) Under the assumption of **(i.)**, that (5.9) is valid with A_1 replaced by $A_1/2$.

iii.) (5.10)–(5.13) are consequences of (5.8) and (5.9) for suitable constants B_i.

By the local existence theorem, it will follow that a solutions exists for all $t > 0$ if ε is small enough.

Before we begin the proof of (i.), we will set up some preliminary results under the assumption of (5.8)–(5.13). That is, we wish to show that

$$r^{1/2}\langle t-r\rangle |L^\nu Z^\alpha u'(t,x)| \le C\varepsilon(1+t)^{3/20}\log(2+t) \tag{5.14}$$

and

$$\|\langle t+r\rangle L^\nu Z^\alpha \Box u(t,\cdot)\|_2 \le C\varepsilon(1+t)^{3/20}\log(2+t) \tag{5.15}$$

for $\nu \le 2$ and $|\alpha| + \nu \le 63$. Notice that the first follows from the second by (4.7), (5.4), (5.6), (5.10), and (5.12). For (5.15), we expand $\Box u$ according to (1.7) to see that the left side is dominated by

$$\left\| \left(\langle t+r\rangle \sum_{\substack{|\alpha|+\mu\le 32\\ \mu\le 2}} |L^\mu Z^\alpha u'(t,\cdot)| \right) \sum_{\substack{|\alpha|+\mu\le 64\\ \mu\le 2}} |L^\mu Z^\alpha u'(t,\cdot)| \right\|_2 .$$

By (5.10) and (5.12), this is easily seen to be bounded by the right side of (5.15) as desired.

Since $|\Box(w-v)| \le C|\Box u|$, it is clear that the same proof also yields

$$r^{1/2}\langle t-r\rangle |L^\nu Z^\alpha (w-v)'(t,x)| \le C\varepsilon(1+t)^{3/20}\log(2+t). \tag{5.16}$$

Let's begin with (i.). Since v satisfies the better bound (5.6), it suffices to show

$$\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}} \|L^\nu Z^\alpha (w-v)'(t,\cdot)\|_2^2 \le C\varepsilon^3. \tag{5.17}$$

By the standard energy integral method (see, e.g., Sogge [31], p. 12), we have that the left side of (5.17) is bounded by

$$C \sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}} \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} |\langle \partial_0 L^\nu Z^\alpha (w-v), \Box L^\nu Z^\alpha (w-v)\rangle|\, dy\, ds$$

$$+ C \sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}} \left| \int_0^t \sum_{a=1}^3 \int_{\partial\mathcal{K}} \partial_0 L^\nu Z^\alpha (w-v) \partial_a L^\nu Z^\alpha (w-v) n_a\, d\sigma\, ds \right|$$

where $n = (n_1, n_2, n_3)$ is the outward normal at a given point on $\partial\mathcal{K}$ and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^D$. Since $\mathcal{K} \subset \{|x| < 1\}$ and since $|L^\nu Z^\alpha (w-v)'(t,x)| \le C\sum_{|\beta|\le|\alpha|,\mu\le\nu} |L^\mu \partial^\beta (w-v)'(t,x)|$ for $x \in \partial\mathcal{K}$, we have that the last term is bounded by

$$C\int_0^t\int_{\{x\in\mathbb{R}^3\backslash\mathscr{K}:|x|<1\}}\sum_{\substack{|\alpha|+\nu\le 53\\ \nu\le 2}}|L^\nu\partial^\alpha(w-v)'(s,y)|^2\,dy\,ds.$$

Since we also have that $[\Box, L] = 2\Box$ and $[\Box, Z] = 0$ and that $\Box(w-v) = (1-\eta)\Box u = (1-\eta)Q(du, d^2u)$, we see that the left side of (5.17) is thus controlled by

$$C\int_0^t\int_{\mathbb{R}^3\backslash\mathscr{K}}\left|\left\langle\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_0 L^\nu Z^\alpha(w-v),\ \sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}L^\nu Z^\alpha Q(du,d^2u)\right\rangle\right|dy\,ds$$

$$+\,C\int_0^t\int_{\{x\in\mathbb{R}^3\backslash\mathscr{K}:|x|<1\}}\sum_{\substack{|\alpha|+\nu\le 53\\ \nu\le 2}}|L^\nu\partial^\alpha(w-v)'(s,y)|^2\,dy\,ds.$$

By (1.7), this is dominated by:

$$(5.18)\quad C\int_0^t\int_{\mathbb{R}^3\backslash\mathscr{K}}\Bigg|\sum_{K=1}^{D}\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_0 L^\nu Z^\alpha(w-v)^K\sum_{0\le j,k,l\le 3}B^{KK,jk}_{K,l}\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_l L^\nu Z^\alpha u^K$$
$$\times\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_j\partial_k L^\nu Z^\alpha u^K\Bigg|\,dy\,ds$$

$$+\,C\int_0^t\int_{\mathbb{R}^3\backslash\mathscr{K}}\Bigg|\sum_{K=1}^{D}\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_0 L^\nu Z^\alpha(w-v)^K\sum_{0\le j,k\le 3}A^{K,jk}_{KK}\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_j L^\nu Z^\alpha u^K$$
$$\times\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}\partial_k L^\nu Z^\alpha u^K\Bigg|\,dy\,ds$$

$$+\,C\int_0^t\int_{\mathbb{R}^3\backslash\mathscr{K}}\sum_{\substack{1\le I,J,K\le D\\ (I,K)\ne(K,J)}}\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}|L^\nu Z^\alpha\partial(w-v)^K|\sum_{\substack{|\alpha|+\nu\le 52\\ \nu\le 2}}|L^\nu Z^\alpha\partial u^I|$$
$$\times\sum_{\substack{|\alpha|+\nu\le 53\\ \nu\le 2}}|L^\nu Z^\alpha\partial u^J|\,dy\,ds$$

$$+\,C\int_0^t\int_{\{x\in\mathbb{R}^3\backslash\mathscr{K}:|x|<1\}}\sum_{\substack{|\alpha|+\nu\le 53\\ \nu\le 2}}|L^\mu\partial^\alpha(w-v)'(s,y)|^2\,dy\,ds.$$

The first two terms in (5.18) satisfy the bounds of Lemma 4.1. The third term involves interactions between waves of different speeds.

When dealing with the first three terms of (5.18), depending on the linear estimates we shall employ, at times we shall use certain L^2 and L^∞ bounds for u while at other times, we shall use them for $w-v$. Since $u=(w-v)+v+u_0$ and u_0, v satisfy the bounds (5.4), (5.6) respectively, it will always be the case that bounds for $w-v$ will imply those for u and vice versa.

Let us first handle the null terms. By (4.1) and (4.2), the first two terms in (5.18) are controlled by

$$(5.19)\quad C\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 3}}|L^\mu Z^\alpha u| \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 3}}|L^\mu Z^\alpha u'| \sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 2}}|L^\mu Z^\alpha (w-v)'|\frac{dy\,ds}{|y|}$$

$$+C\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}\sum_{J=1}^{D}\frac{\langle c_J s-r\rangle}{\langle s+r\rangle}\sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 2}}|L^\mu Z^\alpha\partial(w-v)|\Big(\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 2}}|L^\mu Z^\alpha\partial u|\Big)^2\,dy\,ds.$$

To handle the contribution of the first term of (5.19), notice that by (5.4), (5.6), and (5.10) we have

$$\sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 3}}|L^\mu Z^\alpha u(s,y)|\le C\varepsilon\langle s+|y|\rangle^{-9/10}\log(2+s),$$

which means that the first term of (5.19) has a contribution to (5.18) which is dominated by

$$C\varepsilon\int_0^t\frac{\log(2+s)}{\langle s\rangle^{9/10}}\sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 3}}\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'(s,y)\|_2$$

$$\times\sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 2}}\|\langle y\rangle^{-1/2}L^\mu Z^\alpha(w-v)'(s,y)\|_2\,ds$$

by the Schwarz inequality. Thus, if we again apply the Schwarz inequality and (5.13), we see that this contribution is $O(\varepsilon^3)$.

We now want to show that the second term of (5.19) satisfies a similar bound. If we apply (5.16), we see that the second term of (5.19) is controlled by

$$(5.20)\quad C\varepsilon\int_0^t(1+s)^{3/20}\log(2+s)\int_{\mathbb{R}^3\backslash\mathcal{K}}\frac{1}{r^{1/2}\langle s+r\rangle}\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 2}}|L^\mu Z^\alpha\partial u|^2\,dy\,ds$$

$$\le C\varepsilon\int_0^t\frac{\log(2+s)}{(1+s)^{27/20}}\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 2}}\|L^\mu Z^\alpha u'(s,\cdot)\|^2_{L^2(|y|>s/2)}\,ds$$

$$+ C\varepsilon \int_0^t (1+s)^{3/20} \log(2+s) \int_{\mathbb{R}^3 \setminus \mathscr{K}, |y| \le s/2} \frac{1}{r^{1/2} \langle s+r \rangle} \times \sum_{\substack{|\alpha|+\mu \le 53 \\ \mu \le 2}} |L^\mu Z^\alpha u'(s,y)|^2 \, dy \, ds.$$

The first term on the right of (5.20) is clearly $O(\varepsilon^3)$ by (5.12). For the second term on the right of (5.20), we apply (5.14) to control it as follows

$$C\varepsilon^2 \int_0^t \frac{1}{\langle s \rangle^{(6/5)-2\delta}} \int_{\mathbb{R}^3 \setminus \mathscr{K}, |y| \le s/2} \frac{1}{r^{(3/2)+\delta}} \sum_{\substack{|\alpha|+\mu \le 53 \\ \mu \le 2}} |L^\mu Z^\alpha u'(s,y)| \, dy \, ds.$$

Thus, if δ is sufficiently small, the Schwarz inequality and (5.12) show that this term is also $O(\varepsilon^3)$. This concludes the proof that the contribution of the null forms enjoys an $O(\varepsilon^3)$ bound.

We now wish to show that the multi-speed terms

$$(5.21) \quad \int_0^t \int_{\mathbb{R}^3 \setminus \mathscr{K}} \sum_{\substack{|\alpha|+\mu \le 52 \\ \mu \le 2}} |\partial L^\mu Z^\alpha (w-v)^K| \sum_{\substack{|\alpha|+\mu \le 52 \\ \mu \le 2}} |\partial L^\mu Z^\alpha u^I| \sum_{\substack{|\alpha|+\mu \le 53 \\ \mu \le 2}} |\partial L^\mu Z^\alpha u^J| \, dy \, ds$$

with $(I,K) \neq (K,J)$ have the same contribution to (5.18). For simplicity, let us assume that $I \neq K$, $I = J$. A symmetric argument will yield the same bound for the remaining cases. If we set $\delta < |c_I - c_K|/2$, it follows that $\{|y| \in [(1-\delta)c_I s, (1+\delta)c_I s]\} \cap \{|y| \in [(1-\delta)c_K s, (1+\delta)c_K s]\} = \emptyset$. Thus, it will suffice to show the bound when the spatial integral is taken over the complements each of these sets separately. We will show the bound over $\{|y| \notin [(1-\delta)c_K s, (1+\delta)c_K s]\}$. The same argument will symmetrically yield the bound over the other set.

If we apply (5.16), we see that over $\{|y| \notin [(1-\delta)c_K s, (1+\delta)c_K s]\}$ (5.21) is bounded by

$$(5.22) \quad \varepsilon \int_0^t \frac{\log(2+s)}{\langle s \rangle^{17/20}} \int_{\mathbb{R}^3 \setminus \mathscr{K}, |y| \notin [(1-\delta)c_K s, (1+\delta)c_K s]} \frac{1}{r^{1/2}} \sum_{\substack{|\alpha|+\mu \le 53 \\ \mu \le 2}} |\partial L^\mu Z^\alpha u^I|^2 \, dy \, ds.$$

Arguing as above, it is easy to see that these multiple speed terms are also $O(\varepsilon^3)$.

Finally, we need to show that the last term in (5.18) enjoys an $O(\varepsilon^4)$ contribution. This is clear, however, since this term is bounded by

$$\int_0^t \sum_{\substack{|\alpha|+\mu \le 53 \\ \mu \le 2}} \|L^\mu \partial^\alpha (w-v)'(s,\cdot)\|_\infty^2 \, ds.$$

An application of (5.10) yields the desired bounds and completes the proof of (i.).

We are now ready to prove (ii.). That is, we want to show that we can prove (5.9) with A_1 replaced by $A_1/2$. In view of the bounds (5.6), we see that it suffices to prove

$$(5.23)\quad (1+t+r)\sum_{|\alpha|\le 40}|Z^\alpha(w-v)'(t,x)|\le C\varepsilon^{3/2}.$$

The estimate is straightforward when $|x|>t/10$. For then if we use Lemma 4.4 and (5.17), we get

$$(5.24)\quad (1+t+|x|)\sum_{|\alpha|\le 40}|Z^\alpha(w-v)'(t,x)|\le C\sum_{|\alpha|\le 42}\|Z^\alpha(w-v)'(t,\cdot)\|_2$$
$$\le C\varepsilon^{3/2},\quad |x|>t/10.$$

On account of this we only need to estimate the left side of (5.23) when $|x|<t/10$. Notice that $\Box(w-v)=(1-\eta)Q(du,d^2u)$ vanishes when $|x|>10t$. Thus, we can apply (3.3) to conclude that when $|x|<t/10$, the left side of (5.23) is dominated by

$$\sum_{\substack{|\beta|+\mu\le 43\\ \mu\le 1}}\int_{t/100}^t\int|L^\mu Z^\beta\partial[(1-\eta)Q(du,d^2u)]|\frac{dy\,ds}{|y|}$$
$$+C\sup_{0\le s\le t}(1+s)\sum_{|\beta|\le 44}\|Z^\beta[(1-\eta)Q(du,d^2u)](s,\cdot)\|_\infty$$
$$+C\sup_{0\le s\le t}(1+s)\sum_{|\beta|\le 47}\int_0^s\int_{\substack{||y|-(s-\tau)|<10\\ |y|\le(600+\tau)/2}}|Z^\beta[(1-\eta)Q(du,d^2u)](\tau,y)|\frac{dy\,d\tau}{|y|}$$
$$+C\sup_{0\le s\le t}\sum_{\substack{|\beta|+\mu\le 48\\ \mu\le 1}}\frac{1}{1+s}\int_0^s\int_{|y|\ge(1+\tau)/10}|L^\mu Z^\beta[(1-\eta)Q(du,d^2u)](\tau,y)|\,dy\,d\tau$$
$$=I+II+III+IV.$$

Terms II and IV are the easiest to handle. Since $u=(w-v)+v+u_0$, by using (5.4), (5.6), and (5.10), one finds that II is $O(\varepsilon^2)$. Additionally, since (5.4), (5.6), and (5.8) yield

$$\sum_{\substack{|\beta|+\mu\le 49\\ \mu\le 1}}\|L^\mu Z^\beta u'(\tau,\cdot)\|_2\le C\varepsilon,$$

we can conclude that IV is also $O(\varepsilon^2)$.

Similar considerations imply that

$$(5.25)\quad I \le \int_{t/100}^{t}\int_{|y|<s/2}\sum_{\substack{|\beta|+\mu\le 43\\ \mu\le 1}}|L^\mu Z^\beta \partial Q(du,d^2u)(s,y)|\frac{dy\,ds}{|y|}+C\varepsilon^2,$$

since $|y|^{-1}=O(1/t)$ when $|y|>s/2$ and $t/100<s<t$ and since $\partial\eta=O(1/t)$ when $t/100<s<t$. The first term on the right side of (5.25) is dominated by

$$(5.26)\quad \int_{t/100}^{t}\int_{|y|<s/2}\sum_{\substack{|\beta|+\mu\le 44\\ \mu\le 1}}|r^{1/2}\langle s-r\rangle L^\mu Z^\beta u'|\sum_{\substack{|\beta|+\mu\le 44\\ \mu\le 1}}|r\langle s-r\rangle L^\mu Z^\beta u''|\frac{dy\,ds}{|y|^3 s^{3/2}}.$$

If we apply Lemma 4.4 and (4.3), we see that

$$(5.27)\quad \sum_{\substack{|\beta|\le 44\\ \mu\le 1}}|r\langle s-r\rangle L^\mu Z^\beta u''(s,y)|\le C\varepsilon(1+s)^{3/20}\log(2+s)$$

by (5.4), (5.6), (5.8), and (5.15). Thus, it follows that (5.26), and hence I, is also $O(\varepsilon^2)$ using (5.14) and (5.27).

It remains to estimate III. If we use (4.7), we conclude that on the region of integration

$$(5.28)\quad \sum_{|\beta|\le 48}|Z^\beta u'(\tau,y)|$$
$$\le \frac{C}{r^{1/2}\tau}\Bigg[\sum_{\substack{|\beta|+\mu\le 50\\ \mu\le 1}}\|L^\mu Z^\beta u'(\tau,\cdot)\|_2+\sum_{|\beta|\le 49}\|\langle\tau+r\rangle Z^\beta\Box u(\tau,\cdot)\|_2$$
$$+(1+\tau)\|u'(\tau,\cdot)\|_\infty\Bigg]$$
$$\le \frac{C}{r^{1/2}\tau}\Bigg[\varepsilon+\sum_{|\beta|\le 49}\|\langle\tau+r\rangle Z^\beta\Box u(\tau,\cdot)\|_2\Bigg],$$

using (5.4), (5.6), (5.8), and (5.9) in the last step. If $|\beta|\le 49$

$$\langle\tau+r\rangle|Z^\beta\Box u(\tau,y)|\le\sum_{|\gamma|\le 50}|Z^\gamma u'(\tau,y)|\times\Bigg(\langle\tau+r\rangle\sum_{|\gamma|\le 25}|Z^\gamma u'(\tau,y)|\Bigg),$$

which by the low energy estimate (5.8) and the low dispersive estimate (5.9) gives

$$\|\langle\tau+r\rangle Z^\beta\Box u(\tau,\cdot)\|_2\le C\varepsilon\sup_y\langle\tau+r\rangle\sum_{|\gamma|\le 25}|Z^\gamma u'(\tau,y)|\le C\varepsilon^2.$$

Combining this with (5.28) and recalling that $(1-\eta(\tau,y))=0$ for $|y|>10\tau$, we get

$$(5.29)\quad III \le C\varepsilon^2 \sup_{0\le s\le t}(1+s)\int_{s/100}^{s}\int_{|r-(s-\tau)|<10}\left(\frac{1}{r^{1/2}\tau}\right)^2 r\,dr\,d\tau \le C\varepsilon^2$$

which completes the proof of (ii.).

To complete the proof of Theorem 1.1, we need to show how (5.8), (5.9) imply (5.10)–(5.13).

Since (5.9) has been established, the remainder of the argument follows nearly verbatim from the arguments of [22]. For completeness, we will sketch the argument here. We begin by using the above facts to prove (5.11). With notation as in §1–2, $\Box_\gamma u = B(du)$ with

$$\gamma^{IJ,jk} = -\sum_{\substack{0\le l\le 3\\ 1\le K\le D}} B^{IJ,jk}_{K,l}\partial_l u^K.$$

By (5.9), we have

$$\|\gamma'(s,\cdot)\|_\infty \le \frac{C\varepsilon}{1+s}.$$

Let us first show the estimates for the energy of $\partial_t^j u$ for $j\le M\le 100$. We shall use induction on M.

We first notice that by (2.5) and (5.9) we have

$$(5.30)\quad \partial_t E_M^{1/2}(u)(t) \le C\sum_{j\le M}\|\Box_\gamma\partial_t^j u(t,\cdot)\|_2 + \frac{C\varepsilon}{1+t}E_M^{1/2}(u)(t).$$

Note that for $M=1,2,\ldots$

$$\begin{aligned}
&\sum_{j\le M}|\Box_\gamma\partial_t^j u|\\
&\quad\le C\Big(\sum_{j\le M}|\partial_t^j u'| + \sum_{j\le M-1}|\partial_t^j\partial^2 u|\Big)\sum_{|\alpha|\le 40}|\partial^\alpha u'|\\
&\qquad + C\sum_{|\alpha|\le M-40}|\partial^\alpha u'|\sum_{41\le|\alpha|\le M/2}|\partial^\alpha u'|\\
&\quad\le \frac{C\varepsilon}{1+t}\Big(\sum_{j\le M}|\partial_t^j u'| + \sum_{j\le M-1}|\partial_t^j\partial^2 u|\Big) + C\sum_{|\alpha|\le M-40}|\partial^\alpha u'|\sum_{41\le|\alpha|\le M/2}|\partial^\alpha u'|
\end{aligned}$$

by (5.9) and (5.4). Also, if we use elliptic regularity and repeat this argument, we get

$$\sum_{j\le M-1}\|\partial_t^j\partial^2 u(t,\cdot)\|_2 \le C\sum_{j\le M}\|\partial_t^j u'(t,\cdot)\|_2 + C\sum_{j\le M-1}\|\partial_t^j\Box u(t,\cdot)\|_2$$

$$\le C\sum_{j\le M}\|\partial_t^j u'(t,\cdot)\|_2 + \frac{C\varepsilon}{1+t}\sum_{j\le M-1}\|\partial_t^j\partial^2 u(t,\cdot)\|_2$$

$$+ C\sum_{|\alpha|\le M-41,\,|\beta|\le M/2}\|\partial^\alpha u'(t,\cdot)\partial^\beta u'(t,\cdot)\|_2.$$

If ε is small, we can absorb the second to last term into the left side of the preceding inequality. Therefore, if we combine the last two inequalities, we conclude that

$$\sum_{j\le M}\|\Box_\gamma\partial_t^j u(t,\cdot)\|_2 \le \frac{C\varepsilon}{1+t}\sum_{j\le M}\|\partial_t^j u'(t,\cdot)\|_2$$

$$+ C\sum_{|\alpha|\le M-40,\,|\beta|\le M/2}\|\partial^\alpha u'(t,\cdot)\partial^\beta u'(t,\cdot)\|_2.$$

If we combine this with (5.30) we get that for small $\varepsilon > 0$

$$\partial_t E_M^{1/2}(u)(t) \le \frac{C\varepsilon}{1+t}E_M^{1/2}(u)(t) + C\sum_{|\alpha|\le M-40,\,|\beta|\le M/2}\|\partial^\alpha u'(t,\cdot)\partial^\beta u'(t,\cdot)\|_2, \tag{5.31}$$

since when ε is small, $\frac{1}{2}E_M^{1/2}(u)(t) \le \sum_{j\le M}\|\partial_t^j u'(t,\cdot)\|_2 \le 2E_M^{1/2}(u)(t)$.

For $M \le 52$, the energy estimate (5.11) follows from (5.8). When $M > 52$ we have to deal with the last term in (5.31). To do this we first note that by Lemma 4.4 we have

$$\sum_{|\alpha|\le M-40,\,|\beta|\le M/2}\|\partial^\alpha u'(t,\cdot)\partial^\beta u'(t,\cdot)\|_2$$

$$\le C\sum_{|\gamma|\le \max(M-38,\,2+M/2)}\|\langle x\rangle^{-1/2}Z^\gamma u'(t,\cdot)\|_2^2,$$

which means that for $40 \le M \le 100$, (5.31), (5.1), and Gronwall's inequality yield

$$E_M^{1/2}(u)(t) \le C(1+t)^{C\varepsilon}\Bigg[\varepsilon + \sum_{|\alpha|\le \max(M-38,\,2+M/2)}\|\langle x\rangle^{-1/2}Z^\alpha u'\|_{L^2(S_t)}^2\Bigg], \tag{5.32}$$

if, as before, $S_t = [0,t]\times\mathbb{R}^3\backslash\mathscr{K}$.

If we use (5.8) and (5.32) along with a simple induction argument we conclude that we would have the desired bounds

$$(5.33)\quad E_{100}^{1/2}(u)(t) \le C\varepsilon(1+t)^{C\varepsilon+\sigma}$$

for arbitrarily small $\sigma > 0$ if we apply the following lemma.

Lemma 5.2. *Under the above assumptions, if $M \le 100 - 8\mu$, $\mu \le 4$, and*

$$(5.34)\quad \sum_{\substack{|\alpha|+\nu\le M\\ \nu\le\mu}} \|L^\nu \partial^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\nu\le M-3\\ \nu\le\mu}} \|\langle x\rangle^{-1/2} L^\nu \partial^\alpha u'\|_{L^2(S_t)}$$
$$+ \sum_{\substack{|\alpha|+\nu\le M-4\\ \nu\le\mu}} \|L^\nu Z^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\nu\le M-6\\ \nu\le\mu}} \|\langle x\rangle^{-1/2} L^\nu Z^\alpha u'\|_{L^2(S_t)}$$
$$\le C\varepsilon(1+t)^{C\varepsilon+\sigma},$$

with $\sigma > 0$, then there is a constant C' so that

$$(5.35)\quad \sum_{\substack{|\alpha|+\nu\le M-2\\ \nu\le\mu}} \|\langle x\rangle^{-1/2} L^\nu \partial^\alpha u'\|_{L^2(S_t)}$$
$$+ \sum_{\substack{|\alpha|+\nu\le M-3\\ \nu\le\mu}} \|L^\nu Z^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\nu\le M-5\\ \nu\le\mu}} \|\langle x\rangle^{-1/2} L^\nu Z^\alpha u'\|_{L^2(S_t)}$$
$$\le C'\varepsilon(1+t)^{C'\varepsilon+C'\sigma}.$$

The proof of this lemma can be found in [22].

By elliptic regularity and (5.33), we get (5.11). Also, from Lemma 5.2, we get

$$(5.36)\quad \sum_{|\alpha|\le 98} \|\langle x\rangle^{-1/2}\partial^\alpha w'\|_{L^2(S_t)} + \sum_{|\alpha|\le 97} \|Z^\alpha w'(t,\cdot)\|_2$$
$$+ \sum_{|\alpha|\le 95} \|\langle x\rangle^{-1/2} Z^\alpha w'\|_{L^2(S_t)} \le C'\varepsilon(1+t)^{C'\varepsilon+C'\sigma},$$

since the same sort of bounds hold when w is replaced by u.

Here and in what follows σ denotes a small constant that must be taken to be larger and larger at each occurrence. Note that in terms of the number of Z derivatives (5.35) is considerably stronger than the variants of (5.12) and (5.13) where one just takes the terms with $\nu = 0$. This is because just as there is a loss of derivatives in going from (5.11) to (5.36), there will also be a loss of derivatives in going from L^2 bounds for terms of the form $L^\nu Z^\alpha u'$ to those of the form $L^{\nu+1} Z^\alpha u'$.

The proof of the estimates involving powers of L is a bit more complicated, but still follows the strategy above. First we will estimate $L^\nu \partial^\alpha u'$ in L^2 when α is small using

(5.9). Then we shall estimate the remaining parts of (5.12) and (5.13) using Lemma 5.2.

The main part of the next step is to show that

$$(5.37)\qquad \sum_{\substack{|\alpha|+\mu\le 92\\ \mu\le 1}} \|L^{\mu}\partial^{\alpha}u'(t,\cdot)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}.$$

For this we shall want to use (2.8). We must first establish appropriate versions of (2.7) for $N_0+v_0\le 92$, $v_0=1$. For this we note that for $M\le 92$

$$\begin{aligned}
&\sum_{\substack{j+\mu\le M\\ \mu\le 1}} (|\tilde{L}^{\mu}\partial_t^j\Box_\gamma u| + |[\tilde{L}^{\mu}\partial_t^j, \Box-\Box_\gamma]u|)\\
&\quad\le C\Bigg(\sum_{j\le M-1}|\tilde{L}\partial_t^j\partial u| + \sum_{j\le M-2}|\tilde{L}\partial_t^j\partial^2 u|\Bigg)\sum_{|\alpha|\le 40}|\partial^{\alpha}u'|\\
&\qquad + C\sum_{|\alpha|\le M-41}|L\partial^{\alpha}u'|\sum_{|\alpha|\le M}|\partial^{\alpha}u'| + C\sum_{|\alpha|\le M}|\partial^{\alpha}u'|\sum_{|\alpha|\le \max(M/2,\,M-40)}|\partial^{\alpha}u'|.
\end{aligned}$$

By this, (5.9), and elliptic regularity, we get that for $M\le 92$

$$\begin{aligned}
&\sum_{\substack{j+\mu\le M\\ \mu\le 1}} (\|\tilde{L}^{\mu}\partial_t^j\Box_\gamma u(t,\cdot)\|_2 + \|[\tilde{L}^{\mu}\partial_t^j, \Box-\Box_\gamma]u(t,\cdot)\|_2)\\
&\quad\le \frac{C\varepsilon}{1+t}\sum_{\substack{j+\mu\le M\\ \mu\le 1}}\|\tilde{L}^{\mu}\partial_t^j u'(t,\cdot)\|_2\\
&\qquad + C\sum_{|\alpha|\le M-41}\|\langle x\rangle^{-1/2}L\partial^{\alpha}u'(t,\cdot)\|_2\sum_{|\alpha|\le 94}\|\langle x\rangle^{-1/2}Z^{\alpha}u'(t,\cdot)\|_2\\
&\qquad + C\sum_{|\alpha|\le \max(M,\,2+M/2)}\|\langle x\rangle^{-1/2}Z^{\alpha}u'(t,\cdot)\|_2^2.
\end{aligned}$$

Based on this if ε is small then (2.7) holds with $\delta=C\varepsilon$ and

$$H_{1,M-1}(t) = C\sum_{|\alpha|\le M-41}\|\langle x\rangle^{-1/2}L\partial^{\alpha}u'(t,\cdot)\|_2^2 + C\sum_{|\alpha|\le 94}\|\langle x\rangle^{-1/2}Z^{\alpha}u'(t,\cdot)\|_2^2.$$

Therefore since the conditions on the data give $\int e_0(\tilde{L}^{\nu}\partial_t^j u)(0,x)\,dx\le C\varepsilon^2$ if $\nu+j\le 100$ it follows from (2.8) and (5.36) that for $M\le 92$

(5.38)
$$\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le 1}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma} + C(1+t)^{C\varepsilon} \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-1/2} L\partial^\alpha u'\|^2_{L^2(S_t)} + C(1+t)^{C\varepsilon}\int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha u'(s,\cdot)\|_{L^2(|x|<1)}\, ds.$$

If we apply (2.13) and (5.4) we get that the last integral is dominated by $\varepsilon\log(2+t)$ plus

$$\int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha w'(s,\cdot)\|_{L^2(|x|<1)}\, ds \le C\sum_{|\alpha|\le M+2}\int_0^t \left(\int_0^s \|\partial^\alpha \Box w(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\, d\tau\right) ds.$$

By (5.4) if we replace w by u_0 then the analog of the last term is $O(\varepsilon\log(2+t))$. We therefore conclude that

$$\sum_{|\alpha|\le M+1}\int_0^t \|\partial^\alpha u'(s,\cdot)\|_{L^2(|x|<1)}\, ds \le C\varepsilon\log(2+t) + C\sum_{|\alpha|\le M+2}\int_0^t \left(\int_0^s \|\partial^\alpha \Box u(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\, d\tau\right) ds.$$

Since

$$\sum_{|\alpha|\le M+2} |\partial^\alpha \Box u| \le C\sum_{|\alpha|\le M+3} |\partial^\alpha u'| \sum_{|\alpha|\le (M+3)/2} |\partial^\alpha u'|,$$

an application of Lemma 4.4 yields

$$\sum_{|\alpha|\le M+2} \|\partial^\alpha \Box u(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)} \le C\sum_{|\alpha|\le 95} \|\langle x\rangle^{-1/2} Z^\alpha u'\|^2_{L^2(||x|-(s-\tau)|<20)},$$

since $(3+M)/2\le 95$ if $M\le 92$. Since the sets $\{(\tau,x) : ||x|-(j-\tau)|<20\}$, $j=0,1,2,\dots$ have finite overlap, we conclude that for $M\le 92$

$$\sum_{|\alpha| \le M+1} \int_0^t \|\partial^\alpha u'(s,\cdot)\|_{L^2(|x|<1)}\, ds$$

$$\le C\varepsilon \log(2+t) + C \sum_{|\alpha| \le 95} \|\langle x\rangle^{-1/2} Z^\alpha u'\|^2_{L^2(S_t)}$$

$$\le C\varepsilon(1+t)^{C\varepsilon+\sigma}.$$

Therefore, by (5.38) we have that

$$\sum_{\substack{|\alpha|+\mu \le M \\ \mu \le 1}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2$$

$$\le C\varepsilon(1+t)^{C\varepsilon+\sigma} + C(1+t)^{C\varepsilon} \sum_{|\alpha| \le M-41} \|\langle x\rangle^{-1/2} L\partial^\alpha u'\|^2_{L^2(S_t)}.$$

This gives the desired bounds when $M \le 40$.

If we now use induction and Lemma 5.2, we get (5.37) as well as

$$\sum_{\substack{|\alpha|+\mu \le 90 \\ \mu \le 1}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)} + \sum_{\substack{|\alpha|+\mu \le 89 \\ \mu \le 1}} \|L^\mu Z^\alpha u'(t,\cdot)\|_2$$
$$+ \sum_{\substack{|\alpha|+\mu \le 87 \\ \mu \le 1}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}. \tag{5.39}$$

If we repeat this argument we can estimate $L^2 Z^\alpha u', L^3 Z^\alpha u'$, and $L^4 Z^\alpha u'$ for appropriate Z^α. Using (5.37), (5.39), and the last argument gives

$$\sum_{\substack{|\alpha|+\mu \le 84 \\ \mu \le 2}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\mu \le 82 \\ \mu \le 2}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)}$$
$$+ \sum_{\substack{|\alpha|+\mu \le 81 \\ \mu \le 2}} \|L^\mu Z^\alpha u'(t,\cdot)\|_2$$
$$+ \sum_{\substack{|\alpha|+\mu \le 79 \\ \mu \le 2}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}.$$

Then using the estimates for $L^\mu Z^\alpha u'$, $\mu \le 2$ we can argue as above to get

$$\sum_{\substack{|\alpha|+\mu\le 76\\ \mu\le 3}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\mu\le 74\\ \mu\le 2}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)}$$

$$+ \sum_{\substack{|\alpha|+\mu\le 73\\ \mu\le 3}} \|L^\mu Z^\alpha u'(t,\cdot)\|_2$$

$$+ \sum_{\substack{|\alpha|+\mu\le 71\\ \mu\le 3}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}.$$

Similarly, using the estimates for $L^\mu Z^\alpha u'$ for $\mu \le 3$ we finally get

$$\sum_{\substack{|\alpha|+\mu\le 68\\ \mu\le 4}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\mu\le 66\\ \mu\le 2}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)}$$

$$+ \sum_{\substack{|\alpha|+\mu\le 65\\ \mu\le 4}} \|L^\mu Z^\alpha u'(t,\cdot)\|_2$$

$$+ \sum_{\substack{|\alpha|+\mu\le 63\\ \mu\le 4}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}.$$

If we combine this with our earlier bounds, we conclude that (5.12) and (5.13) must be valid.

It remains to prove (5.10). This is straightforward. If we use Theorem 3.1 we find that its left side is dominated by the square of that of (5.13). Hence (5.13) implies (5.10) which completes the proof.

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