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GLOBAL EXISTENCE FOR SEMILINEAR WAVE EQUATIONS EXTERIOR TO NONTRAPPING OBSTACLES

JASON L. METCALFE

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Abstract. In this paper, we prove the existence of global small amplitude solutions to semilinear wave equations with quadratic nonlinearities exterior to a nontrapping obstacle. This generalizes the work of Hayashi in a domain exterior to a ball and of Shibata and Tsutsumi in spatial dimensions $n \geq 6$.

1. Introduction

The goal of this paper is to prove global existence of solutions to semilinear wave equations with quadratic nonlinearities exterior to a nontrapping obstacle. This extends results that were previously known in Minkowski space (see, e.g., [11]).

More precisely, let $\mathcal{K} \subset \mathbb{R}^n$ be a smooth, compact, nontrapping obstacle, and set $\Omega = \mathbb{R}^n \backslash \mathcal{K}$. We shall consider solutions to

$$\begin{cases} \Box u = \partial_t^2 u - \Delta_x u = Q(u'), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) = 0 \quad \text{for } x \in \partial\Omega \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases} \tag{1.1}$$

Here Q is a constant coefficient quadratic form in $u' = (\partial_t u, \nabla_x u)$.

In order to solve (1.1), we need to assume that the data satisfies certain compatibility conditions. To describe these briefly, let $J_k u = \{\partial_x^\alpha u : 0 \leq |\alpha| \leq k\}$. Then, for fixed m and u a formal H^m solution of (1.1), we can write $\partial_t^k u(0,\cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \leq k \leq m$, for compatibility functions ψ_k which depend on the

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nonlinear term Q, $J_k f$, and $J_{k-1}g$. The compatibility condition for (1.1) with $(f,g) \in H^m \times H^{m-1}$ requires that ψ_k vanish on $\partial\Omega$ when $0 \le k \le m-1$. Additionally, one says that $(f,g) \in C^\infty$ satisfy the compatibility condition to infinite order if this condition holds for all m. For more detail on compatibility conditions, see e.g., [3].

In proving the main theorem, we will need to use the invariance of the wave equation under translations and spatial rotations. Let $Z = \{\partial_t, \partial_k, x_i\partial_j - x_j\partial_i\}$, $1 \le k \le n$, $1 \le i < j \le n$. The vector fields of Z essentially preserve the Dirichlet boundary conditions. The Lorentz boosts $\Omega_{0k} = t\partial_k + x_k\partial_t$ do not have this property, and thus, do not seem appropriate for use in obstacle problems.

We are now ready to state the main result.

Theorem 1.1. *Suppose $n \ge 4$. Let Ω be a domain in $\mathbb{R}^n$ exterior to a smooth compact nontrapping obstacle and assume that $Q(u')$ is as above. Assume that $(f,g) \in C^\infty(\Omega)$ satisfies the compatibility conditions to infinite order. If $\varepsilon > 0$ is small and*

$$\sum_{|\alpha|\le n+2} \|Z^\alpha f\|_{L^2(\Omega)} + \sum_{|\alpha|\le n+1} \|Z^\alpha g\|_{L^2(\Omega)} \le \varepsilon, \tag{1.2}$$

then (1.1) has a unique global solution $u \in C^\infty(\mathbb{R}_+ \times \Omega)$.

Theorem 1.1 is an extension of previous results of Shibata and Tsutsumi [9] and Hayashi [1]. Shibata and Tsutsumi were able to prove Theorem 1.1 when $n \ge 6$. Their method, however, breaks down for the cases $n = 4, 5$ and requires that the nonlinearity be cubic. Hayashi was able to extend the result to $n \ge 4$ when the obstacle $\mathcal{K}$ is a ball. Here, we extend the result to all domains exterior to a nontrapping obstacle. The techniques that we use are similar to those used by Keel, Smith, and Sogge in [4, 5] to show that well-known almost global existence results in $n = 3$ for solutions of the Minkowski wave equation extend to exterior domains.

We will begin by showing that one can prove global existence in Minkowski space using these techniques. Specifically, we will consider

$$\begin{cases} \Box u = Q(u'), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases} \tag{1.3}$$

In this case, we shall prove

Theorem 1.2. *Let $n \geq 4$. Assume that $Q(u')$ is as above and that $(f, g) \in C^\infty(\mathbb{R}^n)$. If $\varepsilon > 0$ is small and*

$$\sum_{|\alpha| \leq n+2} \|Z^\alpha f\|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha| \leq n+1} \|Z^\alpha g\|_{L^2(\mathbb{R}^n)} \leq \varepsilon, \tag{1.4}$$

then (1.3) *has a unique global solution $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$.*

This paper is organized as follows. In the following section, we will prove the main estimates in Minkowski space that we will need. In Section 3, we will prove Theorem 1.2. In Section 4, we will extend the results of Section 2 and collect the main estimates that will be needed to show global existence in the exterior domain. Finally, in Section 5, we will prove Theorem 1.1.

In a future paper, we hope to extend these results via similar techniques to handle quasilinear equations.

2. Main estimates in free space.

We will begin by proving a few results related to the standard energy inequality. That is, if v is a solution to the wave equation $\Box v = G$ in $\mathbb{R}_+ \times \mathbb{R}^n$, we have

$$\|v'(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C\|v'(0, \cdot)\|_{L^2(\mathbb{R}^n)} + C \int_0^t \|G(s, \cdot)\|_{L^2(\mathbb{R}^n)}\, ds. \tag{2.1}$$

The first of these is a weighted energy estimate. This result follows from a simple modification of the arguments in Hörmander [2] (Lemma 6.3.5). A detailed proof can be found in [7].

Lemma 2.1. *Suppose that $n \geq 4$. Let $v(t, x)$ be a solution to the homogeneous Minkowski wave equation $\Box v = 0$ with initial data $f, g \in C_C^\infty(\mathbb{R}^n)$ supported in $\{|x| \leq R\}$. Then, the following estimate holds*

$$\int (t - |x| + 2)^2 \left(\left|\nabla_x v(t, x)\right|^2 + (\partial_t v(t, x))^2 \right)\, dx \leq C_R \left(\int \left|\nabla f\right|^2 + |g|^2\, dx \right).$$

We will also need $L^2_t L^2_x$ estimates. The first of these is an $L^2_t L^2_x \to L^2_t L^2_x$ estimate when the forcing term is supported in a ball of fixed radius for any time t.

Proposition 2.2. *Suppose $n \geq 4$. Let v be a solution of the wave equation $\Box v = G$ in $\mathbb{R}_+ \times \mathbb{R}^n$ with vanishing Cauchy data. Suppose, also, that $G(s, x) = 0$ when $|x| > 2$. Then,*

$$\sum_{|\alpha| \leq N} \|(1 + r)^{-(n-1)/4} Z^\alpha v'(s, x)\|_{L^2_{s,x}([0,t] \times \mathbb{R}^n)} \leq C \sum_{|\alpha| \leq N} \|Z^\alpha G\|_{L^2_{s,x}([0,t] \times \mathbb{R}^n)}$$

for a uniform constant C.

The proof of this proposition will be based on the following lemma.

Lemma 2.3. *Let v be a solution of the wave equation $\Box v = 0$ in $\mathbb{R}_+ \times \mathbb{R}^n$. Suppose further that $v(0,x) = 0$, $\partial_t v(0,x) = g(x)$, and supp $g(x) \subset \{|x| < 1\}$. Then,*

$$\|v'(t,\cdot)\|_{L^2(|x|<t/2)} \le C(1+t)^{-n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

Proof of Lemma 2.3. For $t < 3$, the lemma follows easily from (2.1). We will, thus, assume $t \ge 3$. We will show how to get the bound for $\partial_t v$. A similar argument can be used to bound $\partial_j v$ for $j = 1, 2, ..., n$. Since, by Hölder's inequality,

$$\|\partial_t v(t,\cdot)\|_{L^2(|x|<t/2)} \le Ct^{n/2}\|\partial_t v(t,\cdot)\|_{L^\infty(|x|<t/2)},$$

it will suffice to show

$$\|\partial_t v(t,\cdot)\|_{L^\infty(|x|<t/2)} \le Ct^{-n}\|g\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

Since $\partial_t v$ is a linear combination of $e^{\pm it\sqrt{-\Delta}}g$, it will suffice to show the bound for $e^{it\sqrt{-\Delta}}g$. The other piece will follow from the same argument.

We begin by fixing a smooth, radial cutoff χ such that $\chi(\xi) \equiv 1$ for $|\xi| \le 1$ and $\chi(\xi) \equiv 0$ for $|\xi| \ge 2$. Then, set

$$\beta(\xi) = \chi(\xi) - \chi(2\xi).$$

Thus, supp $\beta \subset \{1/2 \le |\xi| \le 2\}$, and we have a partition of unity

$$\chi(\xi) + \sum_{j=1}^{\infty} \beta(\xi/2^j) = 1$$

for all $\xi \ne 0$. We can then decompose the left side of (2.2) as

$$\begin{aligned} &\|e^{it\sqrt{-\Delta}}g\|_{L^\infty(|x|<t/2)} \\ &\quad\le \|e^{it\sqrt{-\Delta}}\chi(\sqrt{-\Delta})g\|_{L^\infty(|x|<t/2)} + \sum_{j=0}^{\infty}\|e^{it\sqrt{-\Delta}}\beta(\sqrt{-\Delta}/2^j)g\|_{L^\infty(|x|<t/2)} \end{aligned} \tag{2.3}$$

and examine the pieces on the right side separately.

Since g is supported in $\{|x| \le 1\}$, we see that

$$\begin{aligned} |e^{it\sqrt{-\Delta}}\beta(\sqrt{-\Delta}/2^j)g| &\le \int \left| \int e^{i(x-y)\cdot\xi} e^{it|\xi|}\beta(\xi/2^j)\, d\xi \right| |g(y)|\, dy \\ &\le \sup_{|y|\le 1} \left(\left| \int e^{i(x-y)\cdot\xi} e^{it|\xi|}\beta(\xi/2^j)\, d\xi \right| \right) \|g\|_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.4}$$

and similarly

$$|e^{it\sqrt{-\Delta}}\chi(\sqrt{-\Delta})g| \leq \sup_{|y|\leq 1}\left(\left|\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\chi(\xi)\,d\xi\right|\right)\|g\|_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

For (2.4), if we write the kernel in polar coordinates and do a change of variables, we have

$$\begin{aligned}\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\beta(\xi/2^j)\,d\xi &\leq \int_0^\infty\int_{S^{n-1}} e^{i\rho[(x-y)\cdot\omega+t]}\beta(\rho/2^j)\rho^{n-1}\,d\sigma(\omega)\,d\rho \\ &\leq 2^{jn}\int_0^\infty\int_{S^{n-1}} e^{i2^j\rho[(x-y)\cdot\omega+t]}a(\rho)\,d\sigma(\omega)\,d\rho\end{aligned}$$

where $a(\rho)$ is the smooth function, compactly supported away from 0 given by $\beta(\rho)\rho^{n-1}$. If we set

$$I_j = 2^{jn}\int_0^\infty e^{i\rho[2^j(x-y)\cdot\omega+2^jt]}a(\rho)\,d\rho$$

and integrate by parts N times, we see that

$$|I_j| \leq C\frac{2^{jn}}{2^{jN}\big|t-|x-y|\big|^N} \leq C2^{j(n-N)}t^{-N}$$

on $\{|x|<t/2\}\cap\{t\geq 3\}$. Thus, if we choose $N>n$, we see that

$$|e^{it\sqrt{-\Delta}}\beta(\sqrt{-\Delta}/2^j)g| \leq C2^{-jm}t^{-n}\|g\|_{L^2(\mathbb{R}^n)} \tag{2.6}$$

for some $m>0$.

For (2.5), if we write the kernel in polar coordinates, we have

$$\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\chi(\xi)\,d\xi \leq \int_0^\infty\int_{S^{n-1}} e^{i\rho[(x-y)\cdot\omega+t]}a_0(\rho)\,d\sigma(\omega)\,d\rho$$

where a_0 is the smooth function given by $\chi(\rho)\rho^{n-1}$. Here $\frac{\partial^N}{\partial^N\rho}a_0 = 0$ for $N<n-1$. Thus, if we set

$$I_0 = \int_0^\infty e^{i\rho[(x-y)\cdot\omega+t]}a_0(\rho)\,d\rho,$$

we can integrate by parts n times to get

$$|I_0| \leq \left|\frac{1}{(t+\omega\cdot(x-y))^n}\right| + \left|\int_0^\infty \frac{e^{i\rho[(x-y)\cdot\omega+t]}}{(t+\omega\cdot(x-y))^n}a_0^{(n)}(\rho)\,d\rho\right| \leq Ct^{-n}$$

on the set $\{|x|<t/2\}\cap\{t\geq 3\}$. Substituting this into (2.5), we have

$$|e^{it\sqrt{-\Delta}}\chi(\sqrt{-\Delta})g| \leq Ct^{-n}\|g\|_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

Plugging (2.6) and (2.7) into (2.3) yields (2.2) as desired. □

Proof of Proposition 2.2. Since the vector fields Z commute with $\Box$ and since $[Z,\partial_j]=\sum_{i=0}^n a_{ij}\partial_i$ for some constants a_{ij}, it will suffice to show the result for $N=0$.

Let $G_j(s,x)=\chi_{[j,j+1]}(s)G(s,x)$ where $\chi_{[j,j+1]}$ is the characteristic function of the interval $[j,j+1]$. Then, let v_j be the forward solution to $\Box v_j=G_j$ with vanishing Cauchy data. By the Cauchy-Schwartz inequality and finite propagation speed, we have

$$v=\sum_{j=0}^{\infty} v_j \le C\left(\sum_{j=0}^{\infty} |(s-j-|x|+2)v_j|^2\right)^{1/2}.$$

Thus, by Minkowski's integral inequality,

$$\begin{aligned}
&\|(1+r)^{-(n-1)/4}v'(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\\
&\quad\le C\sum_j \|(1+r)^{-(n-1)/4}(s-j-|x|+2)v_j'(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\\
&\quad\le C\sum_j (s-j+1)^{-(n-1)/2}\|(s-j-|x|+2)v_j'(s,\cdot)\|^2_{L^2(|x|>(s-j)/2)}\\
&\qquad\quad + C\sum_j (s-j+2)^2\|v_j'(s,\cdot)\|^2_{L^2(|x|<(s-j)/2)}.
\end{aligned}\tag{2.8}$$

By Lemma 2.1 and Duhamel's principle, the first term in the right side of (2.8) is bounded by

$$\begin{aligned}
(s-j+1)^{-(n-1)/2}&\left(\int_j^{j+1}\|G(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\,d\tau\right)^2\\
&\le C\int_j^{j+1}(s-\tau+1)^{-(n-1)/2}\|G(\tau,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned}$$

By Lemma 2.3 and Duhamel's principle, the second term in the right side of (2.8) is bounded by

$$\begin{aligned}
(s-j+1)^2&\left(\int_j^{j+1}(s-\tau+1)^{-n/2}\|G(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\,d\tau\right)^2\\
&\le C\int_j^{j+1}(s-\tau+1)^{-n+2}\|G(\tau,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned}$$

Thus, after summing, we have

(2.9)
$$\|(1+r)^{-(n-1)/4}v'(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le C\int (s-\tau+1)^{-(n-1)/2}\|G(\tau,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,d\tau + C\int (s-\tau+1)^{-n+2}\|G(\tau,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,d\tau.$$

If we integrate both sides of (2.9) from 0 to t and apply Young's inequality, we see that the result follows for $n \ge 4$. □

The next estimate is an $L^1_tL^2_x \to L^2_tL^2_x$ estimate when the forcing term is not assumed to be compactly supported in x. This follows from the arguments of Smith and Sogge [10] and the author [7]. From [7], we have

Lemma 2.4. *Let β be a smooth function supported in $\{|x| < 4\}$. Then,*

$$\int_{-\infty}^{\infty} \|\beta(\cdot)e^{it\sqrt{-\Delta}}f(\cdot)\|^2_{\dot H^1(\mathbb{R}^n)}\,dt \le C_\beta\|f\|^2_{\dot H^1(\mathbb{R}^n)}.$$

From this, we are able to deduce the following corollary using Duhamel's principle.

Corollary 2.5. *Let v be a solution to the wave equation $\Box v = G$ in $\mathbb{R}_+\times\mathbb{R}^n$. Then, for β a smooth function supported in $\{|x| \le 4\}$, we have*

$$\sup_{|\alpha|\le 1}\|\beta(x)\partial^\alpha_{t,x}v(s,x)\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)} \le C\|v'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^t \|G(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds.$$

And, from Corollary 2.5, we then easily get the following corollary.

Corollary 2.6. *Let v be a solution to the wave equation $\Box v = G$ in $\mathbb{R}_+\times\mathbb{R}^n$. Then, for β a smooth function supported in $\{R \le |x| \le 4R\}$ with $R > 1$, we have*

$$\|r^{-1/2}\beta(x)v'(s,x)\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)} \le C\|v'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^t \|G(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds.$$

Proof of Corollary 2.6. As above, setting $v_R(t,x) = v(Rt,Rx)$, $G_R(t,x) = R^2G(Rt,Rx)$, and $\beta_R(x) = \beta(Rx)$, we have $\Box v_R = G_R$ and $\operatorname{supp}\beta_R \subset \{1 \le |x| \le 4\}$. Thus, by Lemma 2.5,

$$\|r^{-1/2}\beta_R(x)v_R'(s,x)\|_{L^2_{s,x}([0,t/R]\times\mathbb{R}^n)} \le \|\beta_R(x)v_R'(s,x)\|_{L^2_{s,x}([0,t/R]\times\mathbb{R}^n)} \le C\|v_R'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^{t/R}\|G_R(s,\cdot)\|_{L^2_x(\mathbb{R}^n)}\,ds$$

or

$$R\|r^{-1/2}\beta(Rx)v'(Rs,Rx)\|_{L^2_{s,x}([0,t/R]\times\mathbb{R}^n)} \le CR\|v'(0,Rx)\|_{L^2_x(\mathbb{R}^n)} + CR^2\int_0^{t/R}\|G(Rs,Rx)\|_{L^2_x(\mathbb{R}^n)}\,ds.$$

After a change of variables, this becomes

$$\|r^{-1/2}\beta(x)v'(s,x)\|_{L_{s,x}([0,t]\times\mathbb{R}^n)} \le C\|v'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^t \|G(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds$$

as desired. □

We are now ready to prove our key radial decay estimate.

Proposition 2.7. *Suppose $n \ge 4$. Let v be a solution of the wave equation $\Box v = G$ in $\mathbb{R}_+ \times \mathbb{R}^n$. Then,*

$$\|(1+r)^{-(n-1)/4}v'(s,x)\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)} \le C\|v'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C\int_0^t \|G(s,\cdot)\|_{L^2_x(\mathbb{R}^n)}\,ds$$

for a uniform constant C.

Proof of Proposition 2.7. Fix χ to be a smooth, radial cutoff function where $\chi(x) \equiv 1$ when $|x| \le 1$ and $\chi(x) \equiv 0$ for $|x| > 2$. Additionally, set $\beta(x) = \chi(x) - \chi(2x)$. Thus, supp $\beta \subset \{1/2 \le |x| \le 2\}$ and

$$\begin{aligned}\|(1+r)^{-(n-1)/4}&v'(s,x)\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}\\ &\le \|(1+r)^{-(n-1)/4}\chi(x)v'(s,x)\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}\\ &\qquad + \sum_{j=0}^\infty \|(1+r)^{-(n-1)/4}\beta(x/2^j)v'(s,x)\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}\\ &\le \|\chi(x)v'(s,x)\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}\\ &\qquad + \sum_{j=0}^\infty 2^{-j(n-3)/4}\|(1+r)^{-1/2}\beta(x/2^j)v'(s,x)\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}.\end{aligned}$$

The result, then, follows from an application of Corollary 2.5 and Corollary 2.6. □

Since the vector fields Z commute with $\Box$ and since $[Z,\partial_j] = \sum_{i=0}^n a_{ij}\partial_i$ for some constants a_{ij}, Proposition 2.7 and (2.1) imply

Theorem 2.8. *Suppose that v is a solution of the wave equation $\Box v = G$ in $\mathbb{R}_+ \times \mathbb{R}^n$. Then, for any $N = 0, 1, 2, \ldots$*

$$\sum_{|\alpha|\le N} \Big(\|Z^\alpha v'(t,\cdot)\|_{L^2_x(\mathbb{R}^n)} + \|(1+r)^{-(n-1)/4} Z^\alpha v'(s,x)\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)} \Big)$$
$$\le C \sum_{|\alpha|\le N} \|Z^\alpha v'(0,\cdot)\|_{L^2(\mathbb{R}^n)} + C \sum_{|\alpha|\le N} \int_0^t \|Z^\alpha G(s,\cdot)\|_{L^2_x(\mathbb{R}^n)}\, ds$$

In addition to the L^2 estimates, we will need the following pointwise estimate. This is a weighted Sobolev estimate (see e.g., [4]).

Lemma 2.9. *Suppose that $h \in C^\infty(\mathbb{R}^n)$. Then, for $R > 1$,*

$$\|h\|_{L^\infty(R/2\le|x|\le R)} \le C R^{-(n-1)/2} \sum_{|\alpha|\le(n+2)/2} \|Z^\alpha h\|_{L^2(R/4\le|x|\le 2R)}.$$

Proof of Lemma 2.9. By Sobolev's lemma for $\mathbb{R}\times S^{n-1}$, we have

$$|h(x)| \le C \sum_{|\alpha|+j\le\frac{n+2}{2}} \left(\int_{|x|-1/4}^{|x|+1/4} \int_{S^{n-1}} |\partial_r^j \Omega^\alpha h(r\omega)|^2 \, d\sigma(\omega)\, dr \right)^{1/2}.$$

Thus,

$$\|h(x)\|_{L^\infty(R/2\le|x|\le R)}$$
$$\le C R^{-(n-1)/2} \sum_{|\alpha|+j\le\frac{n+2}{2}} \left(\int_{R/4}^{2R} \int_{S^{n-1}} |\partial_r^j \Omega^\alpha h(r\omega)|^2 r^{n-1}\, d\sigma(\omega)\, dr \right)^{1/2}$$
$$\le C R^{-(n-1)/2} \sum_{|\alpha|\le\frac{n+2}{2}} \|Z^\alpha h\|_{L^2(R/4\le|x|\le 2R)}.$$

□

3. Global existence in Minkowski space.

We now want to use Theorem 2.8 and Lemma 2.9 to prove Theorem 1.2. We will use an iteration to solve (1.3) and to show that

(3.1)
$$\sup_{0\le t\le T} \sum_{|\alpha|\le n+2} \Big(\|Z^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|(1+r)^{-(n-1)/4} Z^\alpha u'\|_{L^2_{s,x}([0,T]\times\mathbb{R}^n)} \Big) \le C\varepsilon$$

for any time T.

Proof of Theorem 1.2. Set $u_{-1} \equiv 0$. Then, define u_k recursively by setting it to be the solution of

$$\begin{cases} \Box u_k(t,x) = Q(u'_{k-1}(t,x)), & (t,x) \in [0,T_*] \times \mathbb{R}^n \\ u_k(0,\cdot) = f, \quad \partial_t u_k(0,\cdot) = g. \end{cases} \tag{3.2}$$

Let

$$M_k(T) = \sup_{0\le t\le T} \sum_{|\alpha|\le n+2} \|Z^\alpha u'_k(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha u'_k\|_{L^2_{s,x}([0,T]\times\mathbb{R}^n)}.$$

By (1.4) and Theorem 2.8, there is a constant C_0 such that

$$M_0(T) \le C_0\varepsilon$$

for all T. Our goal is to inductively prove that if $\varepsilon < \varepsilon_0$ is sufficiently small, then

$$M_k(T) \le 2C_0\varepsilon \tag{3.3}$$

for every $k = 1, 2, 3....$ To do so, we assume that this bound holds for $k-1$ and we will use the assumption to prove (3.3) for k. By Theorem 2.8, we have

$$M_k(T) \le C_0\varepsilon + C\sum_{|\alpha|\le n+2} \int_0^T \|Z^\alpha Q(u'_{k-1})(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds.$$

Since Q is quadratic, the bound

$$|Z^\alpha Q(u'_{k-1})(s,x)| \le C\left(\sum_{|\alpha|\le n+2} |Z^\alpha u'_{k-1}(s,x)|\right)\left(\sum_{|\alpha|\le (n+2)/2} |Z^\alpha u'_{k-1}(s,x)|\right)$$

holds for all $|\alpha| \le n+2$. Thus, applying Lemma 2.9, we have

$$\begin{aligned}
\sum_{|\alpha|\le n+2} &\|Z^\alpha Q({u'}_{k-1})(s,\cdot)\|_{L^2(\{|x|\in[2^j,2^{j+1}]\})} \\
&\le C \sum_{|\alpha|\le n+2} \|Z^\alpha u'_{k-1}(s,x)\|_{L^2(\{|x|\in[2^j,2^{j+1}]\})} \\
&\qquad \times \sum_{|\alpha|\le (n+2)/2} \|Z^\alpha u'_{k-1}(s,x)\|_{L^\infty(\{|x|\in[2^j,2^{j+1}]\})} \\
&\le C2^{-j(n-1)/2} \sum_{|\alpha|\le n+2} \|Z^\alpha u'_{k-1}(s,x)\|^2_{L^2(\{|x|\in[2^{j-1},2^{j+2}]\})} \\
&\le C \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha u'_{k-1}(s,x)\|^2_{L^2(\{|x|\in[2^{j-1},2^{j+2}]\})}
\end{aligned}$$

By the standard Sobolev lemma, we also have

$$\sum_{|\alpha|\le n+2} \|Z^\alpha Q(u'_{k-1})(s,\cdot)\|_{L^2(\{|x|<1\})} \le \sum_{|\alpha|\le n+2} \|Z^\alpha u'_{k-1}(s,\cdot)\|^2_{L^2(\{|x|<2\})}.$$

Hence,

$$\sum_{|\alpha|\le n+2} \|Z^\alpha Q(u'_{k-1})(s,\cdot)\|_{L^2(\mathbb{R}^n)} \le C \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha u'_{k-1}(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}$$

and using the inductive hypothesis,

$$\begin{aligned}
M_k(T) &\le C_0\varepsilon + C \sum_{|\alpha|\le n+2} \int_0^T \|Z^\alpha Q(u'_{k-1})(s,\cdot)\|_{L^2(\mathbb{R}^n)}\, ds \\
&\le C_0\varepsilon + C \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha u'_{k-1}(s,\cdot)\|^2_{L^2(\{(s,x):0\le s\le T\})} \\
&\le C_0\varepsilon + CM^2_{k-1}(T) \\
&\le C_0\varepsilon + 4C_0^2 C\varepsilon^2.
\end{aligned}$$

Thus, if $\varepsilon \le \varepsilon_0 = \frac{1}{8CC_0}$, then we see that (3.3) holds for any T.

We now need to show that the u_k converge to a solution. If we set

$$\begin{aligned}
\text{(3.4)} \quad A_k(T) = \sup_{0\le t\le T} &\sum_{|\alpha|\le n+2} \|Z^\alpha(u'_k - u'_{k-1})(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&+ \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha (u'_k - u'_{k-1})\|_{L^2(\{(s,x):0\le s\le T\})},
\end{aligned}$$

the proof will be completed if we can show

$$A_k(T) \le \frac{1}{2} A_{k-1}(T), \quad k = 1, 2, 3, ... \tag{3.5}$$

for any T. Since Q is quadratic, we have

$$|Q(u'_{k-1}) - Q(u'_{k-2})| \le C\left(|u'_{k-1}||u'_{k-1} - u'_{k-2}| + |u'_{k-2}||u'_{k-1} - u'_{k-2}|\right).$$

By repeating the previous arguments, we have

$$\begin{aligned} A_k(T) \le C \sum_{|\alpha|\le n+2} \int_0^T & \|(1+r)^{-(n-1)/4} Z^\alpha (u'_{k-1} - u'_{k-2})(s,\cdot)\|_{L^2(\mathbb{R}^n)} \\ & \times \Big(\|(1+r)^{-(n-1)/4} Z^\alpha u'_{k-1}(s,\cdot)\|_{L^2(\mathbb{R}^n)} \\ & \qquad + \|(1+r)^{-(n-1)/4} Z^\alpha u'_{k-2}(s,\cdot)\|_{L^2(\mathbb{R}^n)} \Big)\, ds. \end{aligned}$$

Thus, by the Schwarz inequality, we have

$$A_k(T) \le C[M_{k-1}(T) + M_{k-2}(T)]A_{k-1}(T) \le 4CC_0\varepsilon A_{k-1}(T)$$

for any T. For ε as above, we see that (3.5) holds which completes the proof. □

4. Main estimates in the exterior domain.

In the next section, we will prove Theorem 1.1. The first step in adapting the argument of the preceding section is to show that there are analogs of Theorem 2.8 and Lemma 2.9 that hold exterior to a nontrapping obstacle.

By scaling, we may assume that $\mathcal{K} \subset \{|x| < 1/2\}$.

The analog of Lemma 2.9 follows directly from the proof given above when $R > 2$. That is, if $h(x) = 0$ when $x \in \partial\mathcal{K}$, then

$$\|h\|_{L^\infty(R/2\le|x|\le R)} \le CR^{-(n-1)/2} \sum_{|\alpha|\le(n+2)/2} \|Z^\alpha h\|_{L^2(R/4\le|x|\le 2R)}. \tag{4.1}$$

When $R \le 2$, this follows from standard Sobolev estimates.

We will also need exterior domain analogs of the energy-type estimates. In order to avoid issues with compatibility conditions, we will restrict to the case where the initial data vanish. In proving Theorem 1.1, we will reduce to this situation. Thus, we will be looking at solutions of

$$\begin{cases} \Box w(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ w(0,x) = \partial_t w(0,x) = 0 \\ w(t,x) = 0 \quad \text{for } x \in \partial\Omega. \end{cases} \tag{4.2}$$

It is well known that analog of (2.1) holds in domains exterior to a nontrapping obstacle. Specifically, if w is given by (4.2), then

$$\|w'(t,\cdot)\|_{L^2(\Omega)} \le C\int_0^t \|F(s,\cdot)\|_{L^2(\Omega)}\,ds. \tag{4.3}$$

We now turn our attention to the exterior domain analog of our main estimate.

Theorem 4.1. *Suppose $n \ge 4$, and let w be the solution of (4.2). Then, for any $N = 0, 1, 2, ...$*

$$\sum_{|\alpha|\le N}\Big(\|Z^\alpha w'(t,\cdot)\|_{L^2(\Omega)} + \|(1+r)^{-(n-1)/4}Z^\alpha w'\|_{L^2_{s,x}([0,t]\times\Omega)}\Big)$$
$$\le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\,ds + C\sup_{0\le s\le t}\sum_{|\alpha|\le N-1}\|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}$$
$$+ C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}.$$

For any $R > 1/2$, set $B_R = \{|x| \le R\}\cap\Omega$ and $E_R = \{|x| \ge R\}$. We will prove Theorem 4.1 by proving the following four estimates:

$$\sum_{|\alpha|\le N}\|Z^\alpha w'(t,\cdot)\|_{L^2(B_2)} \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\,ds + C\sup_{0\le s\le t}\sum_{|\alpha|\le N-1}\|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}, \tag{4.4}$$

$$\sum_{|\alpha|\le N}\|(1+r)^{-(n-1)/4}Z^\alpha w'\|_{L^2_{t,x}([0,t]\times B_2)} \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\,ds + C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}, \tag{4.5}$$

$$\sum_{|\alpha|\le N}\|Z^\alpha w'(t,\cdot)\|_{L^2(E_2)} \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\,ds + C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}, \tag{4.6}$$

(4.7)

$$\sum_{|\alpha|\le N} \|(1+r)^{-(n-1)/4} Z^\alpha w'\|_{L^2_{t,x}([0,t]\times E_2)} \le C \sum_{|\alpha|\le N} \int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\,ds + C \sum_{|\alpha|\le N-1} \|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}.$$

Proof of Equation (4.4). Since

$$\sum_{|\alpha|\le N} \|Z^\alpha w'(t,\cdot)\|_{L^2(B_2)} \le C \sum_{|\alpha|\le N} \|\partial^\alpha_{t,x} w'(t,\cdot)\|_{L^2(\Omega)},$$

we need only show

$$\sum_{|\alpha|\le N} \|\partial^\alpha_{t,x} w'(t,\cdot)\|_{L^2(\Omega)} \le C \int_0^t \sum_{|\alpha|\le N} \|\partial^\alpha_{t,x} F(s,\cdot)\|_{L^2(\Omega)} + C \sum_{|\alpha|\le N-1} \|\partial^\alpha_{t,x} F(t,\cdot)\|_{L^2(\Omega)}. \tag{4.8}$$

We will prove (4.8) via induction. When $N = 0$, this follows from the standard energy inequality (4.3). Thus, we will assume that (4.8) holds for $N-1$ and prove that this implies the result for N.

Notice that since ∂_t preserves the boundary condition and that $[\Box, \partial_t] = 0$, the inductive hypothesis applied to $\partial_t w$ gives

$$\sum_{|\alpha|\le N-1} \|\partial^\alpha_{t,x}\partial_t w'(t,\cdot)\|_{L^2(\Omega)} \le C \sum_{|\alpha|\le N-1} \int_0^t \|\partial^\alpha_{t,x}\partial_t F(s,\cdot)\|_{L^2(\Omega)}\,ds + C \sum_{|\alpha|\le N-2} \|\partial^\alpha_{t,x}\partial_t F(t,\cdot)\|_{L^2(\Omega)}. \tag{4.9}$$

Thus, since $\|\partial^\alpha_{t,x}\partial_t^2 w(t,\cdot)\|_{L^2(\Omega)} \le \|\partial^\alpha_{t,x}\partial_t w'(t,\cdot)\|_{L^2(\Omega)}$ and $\partial_t^2 w = \Delta w + F$, we have

$$\sum_{|\alpha|\le N-1} \|\partial^\alpha_{t,x}\Delta w(t,\cdot)\|_{L^2(\Omega)} \le C \sum_{|\alpha|\le N} \int_0^t \|\partial^\alpha_{t,x} F(s,\cdot)\|_{L^2(\Omega)}\,ds + C \sum_{|\alpha|\le N-1} \|\partial^\alpha_{t,x} F(t,\cdot)\|_{L^2(\Omega)}. \tag{4.10}$$

Using elliptic regularity, we see that

$$\sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha w'(t,\cdot)\|_{L^2(\Omega)}$$
$$\le \|w'(t,\cdot)\|_{L^2(\Omega)} + \sum_{1\le|\alpha|\le N} \|\partial_x^\alpha w'(t,x)\|_{L^2(\Omega)} + \sum_{|\alpha|\le N-1} \|\partial_{t,x}^\alpha \partial_t w'(t,x)\|_{L^2(\Omega)}$$
$$\le C \sum_{|\alpha|\le N-1} \bigl(\|\partial_x^\alpha w'(t,\cdot)\|_{L^2(\Omega)} + \|\partial_x^\alpha \Delta w(t,x)\|_{L^2(\Omega)} + \|\partial_{t,x}^\alpha \partial_t w'(t,x)\|_{L^2(\Omega)}\bigr)$$

Thus, by the inductive hypothesis, (4.9) and (4.10), we see that the proof of (4.4) is complete. □

Since

$$\sum_{|\alpha|\le N} \|(1+r)^{-(n-1)/4} Z^\alpha w'\|_{L^2_{s,x}([0,t]\times B_2)} \le C \sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha w'\|_{L^2_{s,x}([0,t]\times B_2)},$$

we see that (4.5) follows easily from the following lemma.

Lemma 4.2. *Let w be as in* (4.2)*. Then, for any $N = 0, 1, 2, ...,$*

$$\sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha w'\|_{L^2_{s,x}([0,t]\times B_2)}$$
$$\le C \sum_{|\alpha|\le N} \int_0^t \|\partial_{t,x}^\alpha F(s,\cdot)\|_{L^2(\Omega)}\, ds + C \sum_{|\alpha|\le N-1} \|\partial_{t,x}^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}$$

Proof of Lemma 4.2. Following the same induction argument as above, it will suffice to show the $N = 0$ case,

$$\|w'\|_{L^2_{s,x}([0,t]\times B_2)} \le C \int_0^t \|F(s,\cdot)\|_{L^2(\Omega)}\, ds.$$

Suppose that $F(s,x) = 0$ when $|x| > 4$. In this case, since we are exterior to a nontrapping obstacle, the local energy decay of Melrose [6] and Duhamel's principle imply

$$\|w'(t,\cdot)\|_{L^2(B_2)} \le C \int_0^t [1 + (t-s)]^{-n/2} \|F(s,\cdot)\|_{L^2(\Omega)}\, ds$$

Thus, from Minkowski's integral inequality, we have

$$\|w'\|_{L^2_{s,x}([0,t]\times B_2)} \le C \int_0^t \|F(s,\cdot)\|_{L^2(\Omega)}\, ds$$

as desired.

Now suppose that $F(s,x)=0$ for $|x|\le 4$. Fix $\rho\in C_0^\infty$ such that $\rho\equiv 1$ when $|x|\le 2$ and $\rho\equiv 0$ when $|x|>4$. Let u_0 be the solution to the free wave equation $\Box u_0 = F$ with vanishing data. Here we have set F to zero on $\mathbb{R}^n\backslash\Omega$. Write

$$w = u_0 + u_r = (1-\rho)u_0 + [\rho u_0 + u_r]$$

and set $v = \rho u_0 + u_r$. By our assumption on F, we have

$$\Box v = \nabla_x\rho\cdot\nabla u_0 + (\Delta\rho)u_0$$

The argument in the preceding paragraph implies

$$\begin{aligned}\|w'\|_{L^2_{s,x}([0,t]\times B_2)} &= \|v'\|_{L^2_{s,x}([0,t]\times B_2)}\\ &\le \|\rho' u_0'\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)} + \|(\Delta\rho)u_0\|_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}.\end{aligned}$$

Thus, an application of Corollary 2.5 concludes the proof. $\Box$

Corollary 4.3. *Let w be as in* (4.2). *Then, for any $N=0,1,2,...,$*

$$\begin{aligned}&\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x}w\|_{L^2_{s,x}([0,t]\times B_2)}\\ &\qquad\le C\sum_{|\alpha|\le N}\int_0^t\|\partial^\alpha_{t,x}F(s,\cdot)\|_{L^2(\Omega)}\,ds + C\sum_{|\alpha|\le N-1}\|\partial^\alpha_{t,x}F\|_{L^2_{s,x}([0,t]\times\Omega)}\end{aligned}$$

Proof of Corollary 4.3. By the previous lemma, it will suffice to show

$$\|w\|_{L^2_{s,x}([0,t]\times B_2)}\le C\int_0^t\|F(s,\cdot)\|_{L^2(\Omega)}\,ds$$

When $F(s,x)=0$ for $|x|>4$, using a modification of local energy decay, see [7], we have

$$\begin{aligned}\|w(t,\cdot)\|_{L^2(B_2)} &\le C\int_0^t(1+t-s)^{-n/2}\|F(s,\cdot)\|_{\dot H^{-1}_D(\tilde\Omega)}\\ &\le C\int_0^t(1+t-s)^{-n/2}\|F(s,\cdot)\|_{L^2(\Omega)}\end{aligned}$$

where $\tilde\Omega$ is a compact manifold with boundary containing B_2. Thus, in this case, the result follows from Young's inequality.

When $F(s,x)=0$ for $|x|\le 4$, we can argue as in the previous lemma in order to complete the proof of the corollary. $\Box$

We can now conclude the proof of (4.1) by proving (4.6) and (4.7). Let's begin by fixing a $\beta\in C^\infty$ such that $\beta(x)\equiv 1$ for $|x|\ge 2$ and $\beta(x)\equiv 0$ for $|x|\le 1$.

Setting $v = \beta w$, we see that $v = w$ on $|x| > 2$ and that v solves the free wave equation

$$\Box v = \beta F + \nabla\beta \cdot \nabla_x w + (\Delta\beta) w$$

Decompose v into $v = v_1 + v_2$ where v_1 solves $\Box v_1 = \beta F$ and v_2 is the solution of $\Box v_2 = \nabla\beta \cdot \nabla_x w + (\Delta\beta) w$. Set $G = \nabla\beta \cdot \nabla_x w + (\Delta\beta) w$.

By Theorem 2.8, we have

$$\sum_{|\alpha|\le N} \Big(\|Z^\alpha v_1'(t,\cdot)\|_{L^2(E_2)} + \|(1+r)^{-(n-1)/4} Z^\alpha v_1'\|_{L^2_{s,x}([0,t]\times E_2)} \Big) \le \sum_{|\alpha|\le N} \int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\, ds$$

since for $|\alpha| \le N$,

$$\|Z^\alpha \beta(\cdot) F(s,\cdot)\|_{L^2(\Omega)} \le C \sum_{|\alpha|\le N} \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}.$$

Thus, it remains to show

$$\sum_{|\alpha|\le N} \|Z^\alpha v_2'(t,\cdot)\|_{L^2(E_2)} \le C \sum_{|\alpha|\le N} \int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\, ds + C \sum_{|\alpha|\le N-1} \|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}, \tag{4.11}$$

$$\sum_{|\alpha|\le N} \|(1+r)^{-(n-1)/4} Z^\alpha v_2'\|_{L^2_{t,x}([0,t]\times E_2)} \le C \sum_{|\alpha|\le N} \int_0^t \|Z^\alpha F(s,\cdot)\|_{L^2(\Omega)}\, ds + C \sum_{|\alpha|\le N-1} \|Z^\alpha F\|_{L^2_{s,x}([0,t]\times\Omega)}. \tag{4.12}$$

Proof of Equation (4.11). Let $G_j(s,x) = \chi_{[j,j+1]}(s) G(s,x)$ where $\chi_{[j,j+1]}$ is the characteristic function of the interval $[j, j+1]$. Then, let $v_{2,j}$ be the forward solution of $\Box v_{2,j} = G_j$ in free space with zero initial data. By finite propagation speed and the Cauchy-Schwartz inequality, we have

$$v_2 = \sum_{j=0}^\infty v_{2,j} \le C \left(\sum_{j=0}^\infty |(t-j-|x|+2) v_{2,j}|^2 \right)^{1/2}. \tag{4.13}$$

Thus, by the Minkowski integral inequality and Lemma 2.1,

$$\begin{aligned}\sum_{|\alpha|\le N}\|Z^\alpha v_2'(t,\cdot)\|^2_{L^2(E_2)} &\le C\sum_{|\alpha|\le N}\sum_j \|(t-j-|x|+2)Z^\alpha v_{2,j}'(t,\cdot)\|^2_{L^2(E_2)}\\ &\le C\sum_{|\alpha|\le N}\sum_j\Big(\int_j^{j+1}\|Z^\alpha G(s,\cdot)\|_{L^2(\mathbb{R}^n)}\,ds\Big)^2\\ &\le C\sum_{|\alpha|\le N}\|Z^\alpha G\|^2_{L^2_{s,x}([0,t]\times\mathbb{R}^n)}\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x}w'\|^2_{L^2_{s,x}([0,t]\times B_2)}+C\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x}w\|^2_{L^2_{s,x}([0,t]\times B_2)}\end{aligned}$$

Thus, (4.11) follows from Lemma 4.2 and Corollary 4.3. □

Proof of Equation (4.12). By Proposition 2.2, we have

$$\begin{aligned}\sum_{|\alpha|\le N}\|(1+r)^{-(n-1)/4}Z^\alpha v_2'\|_{L^2s,x([0,t]\times E_2)} &\le C\sum_{|\alpha|\le N}\|Z^\alpha G\|_{L^2_{s,x}(\mathbb{R}^n)}\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x}w'\|_{L^2_{s,x}([0,t]\times B_2)}+C\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x}w\|_{L^2_{s,x}([0,t]\times B_2)}\end{aligned}$$

Thus, (4.12) follows from Lemma 4.2 and Corollary 4.3. □

5. Global existence exterior to a nontrapping obstacle.

We now turn to the proof of Theorem 1.1. By scaling, we may assume that the obstacle $\mathcal{K}$ is contained in $\{|x| < 1/2\}$. It is convenient to show that one can instead show global existence for an equivalent nonlinear equation which has vanishing Cauchy data, as in Keel-Smith-Sogge [3]. This allows one to avoid the issues regarding the compatibility conditions. At this point, we can follow an iteration argument similar to that used to prove Theorem 1.2.

Proof of Theorem (1.1). We start by making the reduction mentioned above. Notice that if f, g satisfy (1.2), then we can find a local solution u to (1.1) in $0 < t < 1$. Moreover, if $\varepsilon > 0$ in (1.2) is sufficiently small, there is a constant C so that

(5.1)
$$\sup_{0\le t\le 1}\sum_{|\alpha|\le n+2}\|Z^\alpha u'(t,\cdot)\|_{L^2(\Omega)}+\sum_{|\alpha|\le n+2}\|(1+r)^{-(n-1)/4}Z^\alpha u'\|_{L^2_{s,x}([0,1]\times\Omega)}\le C\varepsilon$$

To see this, notice that local existence theory (see e.g., [3]) implies that (5.1) holds when ε is sufficiently small and the norms on the left side are taken over

$\{|x| < 10\}$. By finite propagation speed, on $\{0 < t < 1\} \times \{|x| \geq 10\}$, u agrees with a solution of the boundaryless wave equation $\Box u = Q(u')$ with data equal to a cutoff times the original data (f, g). Thus, in this case, (5.1) follows from (3.1).

We are now ready to set up the iteration. Fix $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) \equiv 1$ when $t \leq 1/2$ and $\eta(t) \equiv 0$ for $t > 1$. Let

$$u_0 = \eta u.$$

Thus,

$$\Box u_0 = \eta Q(u') + [\Box, \eta]u.$$

Hence, in order to show that there is a solution to $\Box u = Q(u')$ for all t, it will suffice to show that there is a solution $w = u - u_0$ of

$$\begin{cases} \Box w = (1-\eta)Q((u_0 + w)') - [\Box, \eta](u_0 + w) \\ w(t, x) = 0 \quad \text{for } x \in \partial\Omega \\ w(0, x) = \partial_t w(0, x) = 0. \end{cases} \tag{5.2}$$

In order to set up the iteration, as in the proof of Theorem 1.2, set $w_0 = 0$ and define w_k recursively by letting it be a solution of

$$\begin{cases} \Box w_k = (1-\eta)Q((u_0 + w_{k-1})') - [\Box, \eta](u_0 + w_k) \\ w_k(t, x) = 0 \quad \text{for } x \in \partial\Omega \\ w_k(0, x) = \partial_t w_k(0, x) = 0. \end{cases} \tag{5.3}$$

Also, as before, set

$$M_k(T) = \sup_{0 \leq t \leq T} \sum_{|\alpha| \leq n+2} \Big(\|Z^\alpha w_k'(t, \cdot)\|_{L^2(\Omega)} + \|(1+r)^{-(n-1)/4} Z^\alpha w_k'\|_{L^2_{s,x}([0,t]\times\Omega)} \Big).$$

Our first goal is to inductively prove that if $\varepsilon < \varepsilon_0$ is sufficiently small, then

$$M_k(T) \leq 4C_0\varepsilon \tag{5.4}$$

for every $k = 1, 2, 3, \ldots$. When $k = 1$, (5.4) follows from Gronwall's inequality. We, now, assume that the bound (5.4) holds for $k-1$. By Theorem 4.1 and (5.1), we then have

$$M_k(T) \le C \sum_{|\alpha|\le n+2} \int_0^T \|Z^\alpha (1-\eta)(s) Q((u_0+w_{k-1})')(s,\cdot)\|_{L^2(\Omega)}\,ds \tag{5.5}$$
$$+ C \sup_{0\le s\le T} \sum_{|\alpha|\le n+1} \|Z^\alpha (1-\eta)(s) Q((u_0+w_{k-1})')(s,\cdot)\|_{L^2(\Omega)}$$
$$+ C \sum_{|\alpha|\le n+1} \|Z^\alpha (1-\eta) Q((u_0+w_{k-1})')\|_{L^2_{s,x}([0,T]\times\Omega)}$$
$$+ 2C\varepsilon + C \sum_{|\alpha|\le n+2} \int_0^1 \|Z^\alpha w_k'(s,\cdot)\|_{L^2(\Omega)}\,ds.$$

Let's examine the pieces on the right separately.

Since Q is quadratic, for $|\alpha|\le n+2$, we have

$$|Z^\alpha Q((u_0+w_{k-1})')(s,x)| \tag{5.6}$$
$$\le C \left(\sum_{|\alpha|\le n+2} |Z^\alpha (u_0+w_{k-1})'(s,x)|\right)\left(\sum_{|\alpha|\le \frac{n+2}{2}} |Z^\alpha (u_0+w_{k-1})'(s,x)|\right).$$

Thus, by (4.1) and the standard Sobolev lemma, for $j=1,2,3,...$, we have

$$\|Z^\alpha Q((u_0+w_{k-1})')(s,x)\|_{L^2(\{2^j\le|x|\le 2^{j+1}\})}$$
$$\le C \sum_{\alpha\le n+2} \|r^{-(n-1)/4} Z^\alpha u_0'(s,\cdot)\|^2_{L^2(\{2^{j-1}\le|x|\le 2^{j+2}\})}$$
$$+ C\left(\sum_{|\alpha|\le n+2} \|Z^\alpha u_0'(s,\cdot)\|_{L^2(\Omega)}\right)\left(\sum_{|\alpha|\le n+2} \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\{2^j\le|x|\le 2^{j+1}\})}\right)$$
$$+ C\left(\sum_{|\alpha|\le n+2} \|Z^\alpha u_0'(s,\cdot)\|_{L^2(\{2^j\le|x|\le 2^{j+1}\})}\right)\left(\sum_{|\alpha|\le n+2} \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\Omega)}\right)$$
$$+ C \sum_{|\alpha|\le n+2} \|r^{-(n-1)/4} Z^\alpha w_{k-1}'(s,\cdot)\|^2_{L^2(\{2^{j-1}\le|x|\le 2^{j+2}\})}.$$

Since $u_0(s,x)$ vanishes for $s>1$, applying (5.1), gives

$$\int_0^T \|Z^\alpha (1-\eta)(s) Q((u_0+w_{k-1})')(s,\cdot)\|_{L^2(\Omega)}\,ds \le C(C_0\varepsilon + M_{k-1}(T))^2. \tag{5.7}$$

For the second term on the right of (5.5), by (5.6) and the standard Sobolev lemma, for $|\alpha| \le n+1$, we have

$$\|Z^\alpha (1-\eta)(s) Q((u_0 + w_{k-1})'(s,\cdot)\|_{L^2(\Omega)} \le C \sum_{n+1} \|Z^\alpha u_0'(s,\cdot)\|^2_{L^2(\Omega)} + 2C \left(\sum_{|\alpha|\le n+1} \|Z^\alpha u_0'(s,\cdot)\|_{L^2(\Omega)} \right) \left(\sum_{|\alpha|\le n+1} \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\Omega)} \right) + C \sum_{|\alpha|\le n+1} \|Z^\alpha w_{k-1}'(s,\cdot)\|^2_{L^2(\Omega)}.$$

Thus, by (5.1), we have

$$\|Z^\alpha (1-\eta)(s) Q((u_0 + w_{k-1})'(s,\cdot)\|_{L^2(\Omega)} \le C(C_0\varepsilon + M_{k-1}(T))^2. \tag{5.8}$$

Finally, for the third term on the right side of (5.5), again by (5.6),

$$\begin{aligned} &\|Z^\alpha Q((u_0 + w_{k-1})')(s,x)\|_{L^2(\{2^j \le |x| \le 2^{j+1}\})} \\ &\qquad \le C \sum_{\alpha \le n+2} \|r^{-(n-1)/4} Z^\alpha u_0'(s,\cdot)\|^2_{L^2(\{2^{j-1}\le|x|\le 2^{j+2}\})} \\ &+ C \left(\sum_{|\alpha|\le n+2} \|Z^\alpha u_0'(s,\cdot)\|_{L^2(\Omega)} \right) \left(\sum_{|\alpha|\le n+2} \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\{2^j\le|x|\le 2^{j+1}\})} \right) \\ &+ C \left(\sum_{|\alpha|\le n+2} \|Z^\alpha u_0'(s,\cdot)\|_{L^2(\{2^j\le|x|\le 2^{j+1}\})} \right) \left(\sum_{|\alpha|\le n+2} \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\Omega)} \right) \\ &\quad + C \sum_{|\alpha|\le n+2} 2^{-j(n-1)/4} \|r^{-(n-1)/4} Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\{2^{j-1}\le|x|\le 2^{j+2}\})} \\ &\qquad\qquad \times \|Z^\alpha w_{k-1}'(s,\cdot)\|_{L^2(\Omega)}. \end{aligned}$$

Thus,

$$\|Z^\alpha (1-\eta) Q((u_0 + w_{k-1})'\|_{L^2_{s,x}([0,T]\times\Omega)} \le C(C_0\varepsilon + M_{k-1}(T))^2. \tag{5.9}$$

By combining (5.5), (5.7), (5.8), and (5.9), we see that

$$M_k(T) \le 3C(C_0\varepsilon + M_{k-1}(T))^2 + 2C\varepsilon + C \sum_{|\alpha|\le n+2} \int_0^1 \|Z^\alpha w_k'(s,\cdot)\|_{L^2(\Omega)}\, ds.$$

Thus, if ε is small enough, (5.4) follows from Gronwall's Inequality.

Furthermore, if we set

$$A_k(T) = \sup_{0\le t\le T} \left(\sum_{|\alpha|\le n+2} \|Z^\alpha(w_k' - w_{k-1}')(t,\cdot)\|_{L^2(\Omega)} \right) + \sum_{|\alpha|\le n+2} \|(1+r)^{-(n-1)/4} Z^\alpha(w_k' - w_{k-1}')\|_{L^2_{s,x}([0,T]\times\Omega)}$$

and argue as in Section 3, we see that

$$A_k(T) \le \frac{1}{2} A_{k-1}(T)$$

if ε is small enough.

We have, thus, shown that w_k converge to a solution of (5.2) which satisfies

$$\sum_{|\alpha|\le n+2} \Big(\|Z^\alpha w'(t,\cdot)\|_{L^2(\Omega)} + \|(1+r)^{-(n-1)/4} Z^\alpha w'\|_{L^2_{s,x}([0,t]\times\Omega)} \Big) \le C\varepsilon$$

for any t. Thus, $u = u_0 + w$ is a solution to (1.1) satisfying an analogous bound. If the data satisfies the compatibility conditions to infinite order, the solution will be C^∞ on $\mathbb{R}_+ \times \Omega$ by standard local existence results (see e.g., [3]). This completes the proof of Theorem 1.1. □

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Georgia Institute of Technology, School of Mathematics, 686 Cherry Street, Atlanta, GA 30332-0160

E-mail address: metcalfe@math.gatech.edu