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General quasilinear wave equations with localized dissipation in exterior domains ☆

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Abstract

We show that the method of commuting vector fields can be applied to quasilinear wave equations with localized dissipations in general exterior domains. This allows us to show long time existence for general quasilinear wave equations with quadratic nonlinearities. Moreover, by assuming that the dissipation is effective in a certain neighborhood of the boundary, we need not place any assumption on the geometry of the domain.

1. Introduction

In this paper, we prove almost global existence for general quasilinear wave equations with localized dissipations in three-dimensional exterior domains. We make no geometric assumption on the obstacle, but require the dissipation to be effective in a certain neighborhood of the boundary.

Let us more precisely describe the problem at hand. To begin, we fix a smooth, compact obstacle $\mathcal{K}$.[2] Without loss of generality, we will assume that $0 \in \mathcal{K} \subset \{|x| < 1\}$. In the exterior of $\mathcal{K}$, we shall examine

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2 Notice that $\mathcal{K}$ is not necessarily assumed to be connected.

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$$\begin{cases} (\partial_t^2 - \Delta + a(x)\partial_t)u = Q(u', u''), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g. \end{cases} \tag{1.1}$$

As we are assuming that (1.1) is quasilinear, we expand Q as follows[3]

$$Q(u', u'') = A_\gamma^{\alpha\beta} \partial_\gamma u \partial_\alpha \partial_\beta u + R(u', u'), \tag{1.2}$$

where the $A_\gamma^{\alpha\beta}$ are real constants satisfying

$$A_\gamma^{\alpha\beta} = A_\gamma^{\beta\alpha}. \tag{1.3}$$

In order to describe $a(x)$, we first let

$$\Lambda = \{x \in \partial\mathcal{K}\colon x \cdot \nu(x) < 0\}, \tag{1.4}$$

where $\nu(x)$ is the outward unit normal to $\mathcal{K}$ at a point $x \in \partial\mathcal{K}$.[4] It is on this portion of the boundary that we will require that $a(x)$ is not vanishing. Indeed, we fix $a(x)$ to be a smooth, nonnegative function with $\operatorname{supp} a(\cdot) \subset \{|x| \leqslant L\}$ for some fixed $L > 0$. Moreover, we assume that $a(x) \geqslant \varepsilon_0 > 0$ for each $x \in \omega$ where ε_0 is fixed and $\omega \subset \mathbb{R}^3 \setminus \mathcal{K}$ with $\bar{\Lambda} \subset \omega$. For convenience, we will (without loss) take $L = 2$.

To solve (1.1), one must assume that the Cauchy data (f, g) satisfy the relevant compatibility conditions. We let $J_k u = \{\partial_x^\alpha u\colon 0 \leqslant |\alpha| \leqslant k\}$. We then note that if u is a formal H^m solution of (1.1), we can write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \leqslant k \leqslant m$, where ψ_k are called compatibility functions and depend on Q, $J_k f$, and $J_{k-1} g$. The compatibility condition for $(f, g) \in H^m \times H^{m-1}$ simply requires that ψ_k vanish on $\partial\mathcal{K}$ for $0 \leqslant k \leqslant m - 1$. Moreover, the compatibility conditions are said to hold to infinite order if this condition holds for all m.

Under the above assumptions, we prove the following long-time existence result.

Theorem 1.1. *Suppose $\mathcal{K}$ is a fixed compact obstacle with smooth boundary. Suppose that $Q(u', u'')$ and $a(x)$ are as above. Assume $(f, g) \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$ satisfy the compatibility condition to infinite order. Then, there are constants $\tilde{\varepsilon} > 0$, κ, and $N > 0$ so that if*[5]

$$\sum_{|\alpha| \leqslant N} \| \langle x \rangle^{|\alpha|} \partial_x^\alpha f \|_2 + \sum_{|\alpha| \leqslant N-1} \| \langle x \rangle^{1+|\alpha|} \partial_x^\alpha g \|_2 \leqslant \varepsilon, \tag{1.5}$$

for $\varepsilon \leqslant \tilde{\varepsilon}$, then (1.1) *has a unique solution $u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^3 \setminus \mathcal{K})$ where*

$$T_\varepsilon = \exp(\kappa/\varepsilon). \tag{1.6}$$

[3] Here and throughout we use the summation convention where repeated indices are summed. Greek indices are used to indicate that the summation runs from 0 to $n = 3$, while Latin indices indicate that the implicit summation instead runs from 1 to $n = 3$. We shall set $x_0 = t$, $\partial_0 = \partial_t$ when convenient. Additionally, we use $u' = (\partial_t u, \nabla_x u) = \partial u$ to denote the space–time gradient.

[4] If one does not assume that the origin is contained in $\mathcal{K}$ but instead one chooses some $x_0 \in \mathcal{K}$, then the definition of Λ must be altered so that it instead contains those points with $(x - x_0) \cdot \nu(x) < 0$.

[5] Here and throughout, we set $\langle x \rangle = \sqrt{1 + |x|^2}$.

We note that our techniques would allow us to handle multiple speed systems of wave equations with similar dissipations. By comparing our proof to that of [7], it will be clear to the interested reader how to make this adaptation. However, as this more general situation would further complicate the notation, we will concentrate on the scalar case as prescribed above. We note that as $a(x)$ is supported in $\{|x| \leqslant L\}$, blow-up in finite time is possible. This follows from finite propagation speed and the well-known counterexamples of John [4] and Sideris [20] for the free wave equation.

Here, we describe some related results. When $a(x)$ is effective on all of $\mathbb{R}^3 \setminus \mathcal{K}$, then global solutions to (1.1) are known to exist regardless of the geometry of $\mathcal{K}$. See Shibata [18].

When $\mathcal{K}$ is assumed to be star-shaped, then it is known that we may take $a(x) \equiv 0$ and still obtain almost global existence, as was first proved by Keel, Smith, and Sogge [7]. See also Metcalfe and Sogge [13]. This result, for $a(x) \equiv 0$, can be extended to any domain for which there is an exponential decay of local energy. Such domains include the exterior to nontrapping obstacles (see Morawetz, Ralston, and Strauss [15]) and the exterior to several convex bodies (see Ikawa [2,3]). While it was not explicitly shown, the proof of almost global existence in such domains follows from the techniques of Metcalfe and Sogge [11]. For related results on null form wave equations and higher-dimensional wave equations in such domains, see Keel, Smith, and Sogge [5]; Metcalfe and Sogge [11,12,14]; Metcalfe, Nakamura, and Sogge [9,10]; and Shibata and Tsutsumi [19].

In the current setting, long-time existence has been established for certain equations with special structure (e.g. Kirchhoff-type wave equations). See Nakao [17] and the references therein. In the current study, equipped with the knowledge of a local energy decay due to Nakao [16], we show that the method of commuting vector fields can be used to study general quasilinear wave equations. To do this, we will draw upon the original techniques of [7,10,11].

Our assumptions on the effectiveness of a were chosen merely to guarantee an exponential decay of local energy, as described in the next section. Our technique of proof is robust enough to allow any smooth, compact $\mathcal{K}$ and any smooth, nonnegative, compactly supported $a(x)$ for which there is a sufficiently rapid decay of local energy. Moreover, as was originally handled in [11] when $a(x) \equiv 0$, we may allow for a loss of regularity in the local energy decay. We refer the interested reader to, e.g., Christianson [1] where such decay is discussed for a different class of $(\mathcal{K}, a(x))$.

This article is organized as follows. In the remainder of this section, we will introduce some basic notations that will be used throughout. This includes the definition of the vector fields that shall be utilized. In the second section, we gather the L^2 estimates that we shall use. Included here are the exponential decay of local energy, some basic energy estimates, and a class of weighted mixed norm estimates, called Keel–Smith–Sogge (or KSS) estimates. The third section covers the decay estimates that we shall require. Finally, in the last section, we prove Theorem 1.1.

1.1. Vector fields

In the proof of Theorem 1.1, we will be using an adaptation of the method of commuting vector fields introduced in [6,7]. This argument relies on the invariance of the d'Alembertian under translations, spatial rotations, and scaling. The relevant vector fields associated to these operators are

$$\partial_t, \partial_j, \quad j = 1, 2, 3,$$

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \leqslant i < j \leqslant 3,$$
$$L = t\partial_t + r\partial_r,$$

respectively. As in previous works for the wave equation in exterior domains, we will require estimates involving relatively few of the scaling vector field. Thus, we denote the remaining admissible vector fields $Z = \{\partial, \Omega\}$. In particular, one should notice the absense of the Lorentz rotations, which due to their unbounded normal component on the boundary of a compact obstacle seem to be inadmissible for the study of such boundary-value problems. A key observation regarding the vector fields is their commutators[6]

$$[\Box, \partial] = 0, \qquad [\Box, \Omega] = 0, \qquad [\Box, L] = 2\Box.$$

While the commutators with the operator $\Box + a(x)\partial_t$ are not as nice, we will be able to gain sufficient control in order to utilize the vector fields.

2. L^2-estimates

In this section, we introduce the main L^2-estimates which will be required in the proof of Theorem 1.1. These include an exponential decay of local energy, standard energy inequalities, and Keel–Smith–Sogge estimates. Here, we also introduce the use of elliptic regularity in order to obtain energy estimates involving higher-order derivatives. Additionally, we must carefully examine the boundary terms (particularly when the scaling vector field is being utilized) when proving energy estimates involving all of the vector fields.

2.1. Local energy decay

As in previous works on wave equations in exterior domains (see, e.g., [6,7,9–12]), an important component of the following proof is the decay of local energy. In the current setting, this was established by Nakao [16] and states that for w satisfying

$$\begin{cases} \big(\Box + a(x)\partial_t\big)w = 0, \\ w|_{\partial\mathcal{K}} = 0, \\ w(0,\cdot) = f, \quad \partial_t w(0,\cdot) = g, \end{cases}$$

where f and g vanishes for $|x| \geqslant 5$,[7]

$$\big\|w'(t,\cdot)\big\|_{L^2(\{|x|\leqslant 10\})} \lesssim e^{-\lambda t}\big\|w'(0,\cdot)\big\|_2, \tag{2.1}$$

for some $\lambda > 0$.

In order to obtain higher-order estimates, we shall use elliptic regularity as is encapsulated below. The following lemma follows from straightforward modifications of arguments in [11], though similar estimates were used heavily in the previous works [19] and [7].

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6 Here and throughout, we use $\Box = \partial_t^2 - \Delta$ to denote the d'Alembertian.
7 We use $A \lesssim B$ to denote that there is an unspecified, positive constant C so that $A \leqslant CB$.

Lemma 2.1. *Let $\mathcal{K} \subset \{|x| < 1\} \subset \mathbb{R}^3$ be a compact obstacle with smooth boundary. Suppose that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K})$, $u|_{\partial\mathcal{K}} = 0$, and u vanishes for large $|x|$ for each t. Then,*

$$\sum_{|\alpha| \leqslant N_0} \left\| L^\nu \partial^\alpha u'(t,\cdot) \right\|_2 \lesssim \sum_{\substack{j+\mu \leqslant \nu+N_0 \\ \mu \leqslant \nu}} \left\| \left(L^\mu \partial_t^j u\right)'(t,\cdot) \right\|_2 + \sum_{\substack{|\alpha|+\mu \leqslant N_0+\nu-1 \\ \mu \leqslant \nu}} \left\| L^\mu \partial^\alpha \left(\Box + a(x)\partial_t\right) u(t,\cdot) \right\|_2 \tag{2.2}$$

for any fixed N_0 and ν.

When $\nu = 0$, this follows from standard elliptic regularity estimates. Indeed, we have that

$$\sum_{|\alpha| \leqslant N_0} \left\| \partial^\alpha u'(t,\cdot) \right\|_2 \leqslant \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \partial_x^2 u(t,\cdot) \right\|_2 + \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \partial_t u'(t,\cdot) \right\|_2 + \left\| u'(t,\cdot) \right\|_2.$$

Using elliptic regularity, it follows that

$$\begin{aligned}
\sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \partial_x^2 u(t,\cdot) \right\|_2 &\lesssim \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \Delta u(t,\cdot) \right\|_2 + \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha u'(t,\cdot) \right\|_2 \\
&\lesssim \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \partial_t^2 u(t,\cdot) \right\|_2 + \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha \left(\Box + a(x)\partial_t\right) u(t,\cdot) \right\|_2 \\
&\quad + \sum_{|\alpha| \leqslant N_0-1} \left\| \partial^\alpha u'(t,\cdot) \right\|_2.
\end{aligned}$$

Using this in the preceding equation and arguing recursively yields the $\nu = 0$ case of (2.2).

In order to complete the proof, we use induction on ν. Since $\mathcal{K} \subset \{|x| < 1\}$ and $\operatorname{supp} a(x) \subset \{|x| < 2\}$, it is easy to see that (2.2) reduces to showing the bound when the norm in the left is over $\{|x| < 4\}$.

To complete the proof, we notice that

$$\sum_{|\alpha| \leqslant N_0} \left\| L^\nu \partial^\alpha u'(t,\cdot) \right\|_{L^2(|x|<4)} \lesssim \sum_{\substack{|\alpha|+\mu \leqslant N_0+\nu \\ \mu \leqslant \nu}} t^\mu \left\| \partial_t^\mu \partial^\alpha u'(t,\cdot) \right\|_{L^2(|x|<4)}.$$

Since elliptic regularity yields[8]

[8] The notation $|x| < 4+$ is used to denote that the estimate holds for any ball of radius $4 + \delta$, $\delta > 0$. The implicit constant in our estimate depends on this δ, but this is permissible.

$$\sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \left\|\partial^\alpha\partial_t^\mu u'(t,\cdot)\right\|_{L^2(|x|<4)}$$
$$\lesssim \sum_{\substack{j+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \left\|\partial_t^{j+\mu} u'(t,\cdot)\right\|_{L^2(|x|<4+)} + \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-1\\ \mu\leqslant\nu}} \left\|\partial^\alpha\partial_t^\mu\big(\Box+a(x)\partial_t\big)u(t,\cdot)\right\|_{L^2(|x|<4+)}, \tag{2.3}$$

we see that

$$\sum_{|\alpha|\leqslant N_0} \left\|L^\nu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)}$$
$$\lesssim \sum_{j\leqslant N_0} \left\|L^\nu\partial_t^j u'(t,\cdot)\right\|_2 + \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu-1}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_2$$
$$+ \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-1\\ \mu\leqslant\nu}} \left\|L^\mu\partial^\alpha\big(\Box+a(x)\partial_t\big)u(t,\cdot)\right\|_2.$$

Using the inductive hypothesis, (2.2) clearly follows.

A first application of the preceding lemma allows us to obtain a higher-order decay of local energy. This, again, results from rather easy modifications of arguments from [11].

Lemma 2.2. *Let $\mathcal{K}$ be a smooth, compact obstacle, and assume that $a(x)$ is as above. Suppose that $(\Box+a(x)\partial_t)u(t,x)=0$ for $|x|>5$ and $u|_{\partial\mathcal{K}}=0$. Suppose further that $u(t,x)=0$ for $t\leqslant 0$. Then,*[9]

$$\sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)}$$
$$\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-1\\ \mu\leqslant\nu}} \left\|L^\mu\partial^\alpha\big(\Box+a(x)\partial_t\big)u(t,\cdot)\right\|_2$$
$$+\int_0^t e^{-(\lambda/2)(t-s)} \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \left\|L^\mu\partial^\alpha\big(\Box+a(x)\partial_t\big)u(s,\cdot)\right\|_2\,ds, \tag{2.4}$$

for any N_0 and ν.

By (2.1),

$$\sum_{\substack{j+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \langle t\rangle^\mu \left\|\partial_t^\mu\partial_t^j u'(t,\cdot)\right\|_{L^2(|x|<10)}$$

[9] The λ appearing here is as in (2.1).

is controlled by the last term in (2.4). The general result then follows from (2.3).

2.2. Energy estimates

We begin with the standard energy inequality. As opposed to the case $a(x) \equiv 0$, we will gain a term that is a result of the dissipation. In order to establish higher-order energy estimates, we will use elliptic regularity, cutoff tricks that resemble those from [11], and the mixed norm KSS estimates from the following section.

In this section, we will be concerned with smooth solutions to[10]

$$\begin{cases} \Box_h u + a(x)\partial_t u = F, \\ u|_{\partial\mathcal{K}} = 0, \\ u(t,x) = 0, \quad t \leqslant 0, \end{cases} \tag{2.5}$$

where[11]

$$\Box_h u = \left(\partial_t^2 - \Delta\right)u + h^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta u,$$

and

$$h^{\alpha\beta} = h^{\beta\alpha}. \tag{2.6}$$

We shall assume that the perturbation terms $h^{\alpha\beta}$ satisfy

$$\left|h^{\alpha\beta}(t,x)\right| \leqslant \frac{\delta}{1+t}, \quad 0 < \delta \ll 1 \tag{2.7}$$

and[12]

$$\sum_{\alpha,\beta,\gamma=0}^{3} \left\|\partial_\gamma h^{\alpha\beta}(t,x)\right\|_{L_t^1 L_x^\infty(S_{T_*})} \leqslant C_0. \tag{2.8}$$

We will begin by examining the energy-momentum vector associated to $\Box_h$:

$$e_\alpha[u] = \partial_\alpha u\partial_t u - \frac{1}{2}m_{\alpha 0}\left(\partial^\gamma u\partial_\gamma u\right) - m_{\alpha\gamma}h^{\gamma\delta}\partial_\delta u\partial_t u + \frac{1}{2}m_{\alpha 0}h^{\gamma\delta}\partial_\gamma u\partial_\delta u. \tag{2.9}$$

Here, $m_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric, and this is the metric which is used to raise indices. We have the well-known divergence property

$$\partial^\alpha e_\alpha[u] = -\partial_t u\Box_h u - \left(\partial_\gamma h^{\gamma\delta}\right)\partial_\delta u\partial_t u + \frac{1}{2}\left(\partial_t h^{\gamma\delta}\right)\partial_\gamma u\partial_\delta u. \tag{2.10}$$

If we integrate (2.10) over a time strip S_t, it follows that

[10] In the sequel, we shall make a reduction to the case of vanishing initial data.

[11] In the sequel, we shall set $h^{\alpha\beta} = -A_\gamma^{\alpha\beta}\partial_\gamma u$.

[12] We use the notation $S_T = [0,T] \times \mathbb{R}^3 \setminus \mathcal{K}$.

$$\int_{\mathbb{R}^3\backslash\mathcal{K}} e_0[u](t)\,dx + \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} a(x)(\partial_t u)^2\,dx\,dt$$

$$= \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \partial_t u\big(\Box_h + a(x)\partial_t\big)u\,dx\,dt + \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \big(\partial_\gamma h^{\gamma\delta}\big)\partial_\delta u \partial_t u\,dx\,dt$$

$$- \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \big(\partial_t h^{\gamma\delta}\big)\partial_\gamma u \partial_\delta u\,dx\,dt. \tag{2.11}$$

Here, we have used the fact that ∂_t preserves the Dirichlet boundary condition. If we apply (2.7), we see that

$$\|u'(t,\cdot)\|_2^2 + \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} a(x)(\partial_t u)^2\,dx\,dt$$

$$\lesssim \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \big|\partial_t u\big(\Box_h + a(x)\partial_t\big)u\big|\,dx\,dt + \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} |\partial h||u'|^2\,dx\,dt, \tag{2.12}$$

where

$$|\partial h| = \sum_{\alpha,\beta,\gamma=0}^{3} \big|\partial_\gamma h^{\alpha\beta}\big|.$$

By the Schwarz inequality, Gronwall's inequality, and (2.8), we obtain the following well-known result.

Lemma 2.3. *Assume that* (2.6)–(2.8) *hold. Suppose that $\mathcal{K}$ is a compact obstacle with C^1 boundary and that $a(x)$ is as above. Suppose further that u is a smooth solution to* (2.5) *and that u vanishes for large $|x|$ for each t. Then,*

$$\|u'(t,\cdot)\|_2 + \big\|\sqrt{a(x)}\partial_t u\big\|_{L_t^2 L_x^2(S_t)} \lesssim \int_0^t \big\|\big(\Box_h + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds \tag{2.13}$$

for any $t \leqslant T_*$.

If we apply elliptic regularity in a manner similar to above and use the fact that ∂_t preserves the Dirichlet boundary condition, we see that

Corollary 2.4. *Assume that* (2.6)–(2.8) *hold. Suppose that $\mathcal{K}$ is a compact obstacle with smooth boundary. Suppose further that u is a smooth solution to* (2.5) *and that u vanishes for large $|x|$ for each t. Then,*

$$\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha u'(t,\cdot)\big\|_2 \lesssim \sum_{j\leqslant N}\int_0^T \big\|\big(\Box_h + a(x)\partial_t\big)\partial_t^j u(t,\cdot)\big\|_2\,dt + \sum_{|\alpha|\leqslant N-1}\big\|\partial^\alpha\big(\Box + a(x)\partial_t\big)u(t,\cdot)\big\|_2 \tag{2.14}$$

for any $t \leqslant T_*$.

We next wish to show that a version of (2.13) holds if u is replaced by $L^\nu\partial^\alpha u$. To do this, we shall argue as in [11] by introducing a modified scaling vector field $\tilde{L} = t\partial_t + \beta(x)r\partial_r$ where $\beta \in C^\infty(\mathbb{R}^3)$ with $\beta(x) \equiv 0$, $x \in \mathcal{K}$ and $\beta(x) \equiv 1$ for $|x| > 2$. With this, we can now prove the following variant of the energy estimate.

Lemma 2.5. *Suppose* (2.6)–(2.8). *Let $\mathcal{K}$ and $a(x)$ be as above, and let $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K})$ be a solution to* (2.5) *which vanishes for large $|x|$ for each t. Then,*

$$\begin{aligned}
&\big\|\big(\tilde{L}^\nu\partial_t^j u\big)'(t,\cdot)\big\|_2 + \big\|\sqrt{a}\,\partial_t\big(\tilde{L}^\nu\partial_t^j u\big)\big\|_{L_t^2L_x^2(S_t)}\\
&\lesssim \int_0^t \big\|\tilde{L}^\nu\partial_t^j\big(\Box_h + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds + \int_0^t \big\|\big[\tilde{L}^\nu\partial_t^j, h^{\alpha\beta}\partial_\alpha\partial_\beta\big]u(t,\cdot)\big\|_2\,ds\\
&\quad + \sum_{\mu\leqslant\nu-1}\int_0^t \big\|L^\mu\partial_t^j\big(\Box + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds + \sum_{\substack{|\alpha|+\mu\leqslant j+\nu\\ \mu\leqslant\nu-1}}\int_0^t \big\|L^\mu\partial^\alpha u'(s,\cdot)\big\|_{L^2(\{|x|<2\})}\,ds
\end{aligned} \tag{2.15}$$

for all $t \leqslant T_*$.

As ∂_t and $\tilde{L}$ both preserve the Dirichlet boundary condition, we simply apply (2.13) with u replaced by $\tilde{L}^\nu\partial_t^j u$ to see that

$$\big\|\big(\tilde{L}^\nu\partial_t^j u\big)'(t,\cdot)\big\|_2 + \big\|\sqrt{a}\,\partial_t\big(\tilde{L}^\nu\partial_t^j u\big)\big\|_{L_t^2L_x^2(S_t)} \lesssim \int_0^t \big\|\big(\Box_h + a(x)\partial_t\big)\tilde{L}^\nu\partial_t^j u(s,\cdot)\big\|_2\,ds.$$

Noting that

$$\begin{aligned}
\big|\big(\Box_h + a(x)\partial_t\big)\tilde{L}^\nu\partial_t^j u\big| &\leqslant \big|\tilde{L}^\nu\partial_t^j\big(\Box_h + a(x)\partial_t\big)u\big| + \big|\big[\tilde{L}^\nu\partial_t^j, h^{\alpha\beta}\partial_\alpha\partial_\beta\big]u\big|\\
&\quad + \big|\big[\tilde{L}^\nu, \Box\big]\partial_t^j u\big| + \big|\big[\tilde{L}^\nu, a(x)\partial_t\big]\partial_t^j u\big|\\
&\leqslant \big|\tilde{L}^\nu\partial_t^j\big(\Box_h + a(x)\partial_t\big)u\big| + \big|\big[\tilde{L}^\nu\partial_t^j, h^{\alpha\beta}\partial_\alpha\partial_\beta\big]u\big| + \big|\big[L^\nu, \Box\big]\partial_t^j u\big|\\
&\quad + \big|\big[\tilde{L}^\nu - L^\nu, \Box\big]\partial_t^j u\big| + \big|\big[\tilde{L}^\nu, a(x)\partial_t\big]\partial_t^j u\big|
\end{aligned}$$

$$\lesssim \left|\tilde{L}^\nu \partial_t^j \left(\Box_h + a(x)\partial_t\right)u\right| + \left|\left[\tilde{L}^\nu \partial_t^j, h^{\alpha\beta}\partial_\alpha\partial_\beta\right]u\right| + \sum_{\mu\leqslant\nu-1} \left|L^\mu \partial_t^j \left(\Box + a(x)\partial_t\right)u\right| + \mathbf{1}_{\{|x|<2\}}(x) \sum_{\substack{|\alpha|+\mu\leqslant j+\nu\\ \mu\leqslant\nu-1}} \left|L^\mu \partial^\alpha u'\right|,$$

(2.15) follows immediately.

From the above lemma, we may now obtain the necessary energy inequality involving the scaling vector field and general translations. Here, we simply argue using (2.2) and (2.15).

Proposition 2.6. *Suppose* (2.6)–(2.8). *Assume that* $\mathcal{K}$ *and* $a(x)$ *are as above. Let* u *be a smooth solution to* (2.5) *which vanishes for large* $|x|$ *for each* t. *Then,*

$$\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \left\|L^\mu \partial^\alpha u'(t,\cdot)\right\|_2 \lesssim \sum_{\substack{j+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \int_0^t \left\|\tilde{L}^\mu \partial_t^j \left(\Box_h + a(x)\partial_t\right)u(s,\cdot)\right\|_2 ds + \sum_{\substack{j+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \int_0^t \left\|\left[\tilde{L}^\mu \partial_t^j, h^{\alpha\beta}\partial_\alpha\partial_\beta\right]u(s,\cdot)\right\|_2 ds + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu-1\\ \mu\leqslant\nu-1}} \int_0^t \left\|L^\mu \partial^\alpha \left(\Box + a(x)\partial_t\right)u(s,\cdot)\right\|_2 ds + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu-1}} \int_0^t \left\|L^\mu \partial^\alpha u'(s,\cdot)\right\|_{L^2(\{|x|<2\})} ds + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu-1\\ \mu\leqslant\nu}} \left\|L^\mu \partial^\alpha \left(\Box + a(x)\partial_t\right)u(t,\cdot)\right\|_2 \tag{2.16}$$

for any $t \leqslant T_*$.

Indeed, (2.16) would follow from (2.15) once the following modification of (2.2) is proved.

$$\sum_{|\alpha|\leqslant N_0} \left\|L^\nu \partial^\alpha u'(t,\cdot)\right\|_2 \lesssim \sum_{\substack{j+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu}} \left\|\left(\tilde{L}^\mu \partial_t^j u\right)'(t,\cdot)\right\|_2 + \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-1\\ \mu\leqslant\nu}} \left\|L^\mu \partial^\alpha \left(\Box + a(x)\partial_t\right)u(t,\cdot)\right\|_2. \tag{2.17}$$

Here, we may argue inductively (in ν), and it suffices to bound

$$\sum_{|\alpha|\leqslant N_0} \left\|L^{\nu-1}\partial^\alpha (\tilde{L}u)'(t,\cdot)\right\|_2.$$

By applying the inductive hypothesis to $v = \tilde{L}u$, which satisfies

$$\sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-2\\ \mu\leqslant\nu-1}} \left|L^\mu\partial^\alpha\big(\Box + a(x)\partial_t\big)(\tilde{L}u)\right| \lesssim \sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu-1\\ \mu\leqslant\nu}} \left|L^\mu\partial^\alpha\big(\Box + a(x)\partial_t\big)u\right| + \mathbf{1}_{\{|x|\leqslant 2\}}(x)\sum_{\substack{|\alpha|+\mu\leqslant N_0+\nu\\ \mu\leqslant\nu-1}} \left|L^\mu\partial^\alpha u'\right|,$$

as well as the Dirichlet boundary condition, one immediately sees that the result follows from the inductive hypothesis.

Finally, we shall need a version of the energy inequality that permits the use of all of the admissible vector fields.

Proposition 2.7. *Assume* (2.6)–(2.8). *Suppose that* $\mathcal{K}$ *and* $a(x)$ *are as above. Suppose further that* u *solves* (2.5) *and that* u *vanishes for large* $|x|$ *for each* t. *Then,*

$$\begin{aligned}&\sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \left\|L^\mu Z^\alpha u'(t,\cdot)\right\|_2 + \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \left\|\sqrt{a(x)}\partial_t L^\mu Z^\alpha u\right\|_{L^2_t L^2_x(S_t)} \\ &\quad\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \int_0^t \left\|L^\mu Z^\alpha\big(\Box_h + a(x)\partial_t\big)u(s,\cdot)\right\|_2 ds \\ &\qquad + \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \int_0^t \left\|\big[h^{\gamma\delta}\partial_\gamma\partial_\delta, L^\mu Z^\alpha\big]u(s,\cdot)\right\|_2 ds + \sum_{\substack{|\alpha|+\mu\leqslant N+1\\ \mu\leqslant\nu}} \left\|L^\mu\partial^\alpha u'\right\|_{L^2_t L^2_x([0,t]\times\{|x|<2\})},\end{aligned} \tag{2.18}$$

for any $N, \nu \geqslant 0$ *and any* $t \leqslant T_*$.

Here, we replace $e_\alpha[u]$ in (2.9), (2.10) by $e_{\alpha,\nu,N}[u] = \sum_{|\beta|+\mu\leqslant N,\mu\leqslant\nu} e_\alpha[L^\mu Z^\beta u]$. Arguing as in (2.11), we see that

$$\begin{aligned}&\int_{\mathbb{R}^3\backslash\mathcal{K}} e_{0,\nu,N}[u](t)\,dx + \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} a(x)\big(\partial_t L^\mu Z^\alpha u\big)^2\,dx\,ds \\ &\quad\leqslant \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \left|\partial_t L^\mu Z^\alpha u\right|\left|\big(\Box_h + a(x)\partial_t\big)L^\mu Z^\alpha u\right|\,dx\,ds \\ &\qquad + \int_0^t \left\|\partial h(s,\cdot)\right\|_\infty \int_{\mathbb{R}^3\backslash\mathcal{K}} e_{0,\nu,N}[u](s)\,dx\,ds + \int_0^t \int_{\partial\mathcal{K}} \left|e_{a,\nu,N}[u]\nu_a\right|\,d\sigma\,ds.\end{aligned} \tag{2.19}$$

Since the coefficients of Z are $O(1)$ on $\partial\mathcal{K}$, we see that the last term above is

$$\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N+1\\ \mu\leqslant\nu}} \left\| L^\mu \partial^\alpha u'\right\|^2_{L^2_t L^2_x([0,t]\times\{|x|<1\})}.$$

For the first term in the right-hand side of (2.19), we notice that it is

$$\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \sum_{\substack{|\beta|+\sigma\leqslant N\\ \sigma\leqslant\nu}} \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \left|\partial_t L^\mu Z^\alpha u\right| \left|L^\sigma Z^\beta \big(\Box_h + a(x)\partial_t\big)u\right| dx\, dt$$

$$+ \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \sum_{\substack{|\beta|+\sigma\leqslant N\\ \sigma\leqslant\nu}} \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \left|\partial_t L^\mu Z^\alpha u\right| \left|\big[h^{\gamma\delta}\partial_\gamma\partial_\delta, L^\sigma Z^\beta\big]u\right| dx\, dt$$

$$+ \sum_{\substack{|\alpha|+\mu\leqslant N\\ \mu\leqslant\nu}} \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}\cap\{|x|\leqslant 2\}} \left|\partial_t L^\mu \partial^\alpha u\right|^2 dx\, ds.$$

By combining these observations and applying the Schwarz inequality and Gronwall's inequality, we see that (2.18) follows.

2.3. *Keel–Smith–Sogge estimates*

In addition to the above estimates, we will use a weighted mixed-norm estimate called a Keel–Smith–Sogge estimate (or KSS estimate). These were first developed and used to study nonlinear equations by Keel, Smith, and Sogge [6]. Such estimates have played a prevalent role in previous studies of nonlinear wave equations in exterior domains and are especially useful for handling certain boundary terms that arise (see, e.g., those in (2.18)) and for dealing with certain technical aspects of the proof of Theorem 1.1 which concern the distribution of the scaling vector fields.

We first note that we have the following estimate for the boundaryless, free wave equation[13]

$$\big(\log(2+T)\big)^{-1/2}\left\|\langle x\rangle^{-1/2} v'\right\|^2_{L^2_t L^2_x(S_T)} + \left\|\langle x\rangle^{-1/2-} v'\right\|^2_{L^2_t L^2_x(S_T)} + \left\|\langle x\rangle^{-3/2-} v\right\|^2_{L^2_t L^2_x(S_T)}$$

$$\lesssim \left\|v'(0,\cdot)\right\|^2_2 + \int_0^T\int |\partial_{t,x} v\Box v|\, dx\, ds + \int_0^T \int \left|\langle x\rangle^{-1} v\Box v\right| dx\, ds. \tag{2.20}$$

See Rodnianski [23] and Metcalfe and Sogge [13]. This follows using the multiplier method with multiplier $X = \frac{r}{r+R}\partial_r$ where R ranges over the dyadic numbers with $R \leqslant T$. By the triangle inequality and the Schwarz inequality, we have that

[13] We use the notation $\langle x\rangle^{-1/2-}$ to indicate that the estimate holds with this weight replaced by $\langle x\rangle^{-1/2-\delta}$ for any $\delta > 0$. The implicit constant depends on δ, but this will not matter for our applications.

$$\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}v'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-1/2-}v'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-3/2-}v\big\|^2_{L^2_tL^2_x(S_T)}$$
$$\lesssim\big\|v'(0,\cdot)\big\|_2^2+\int_0^T\int\big|\partial_{t,x}v\big(\Box+a(x)\partial_t\big)v\big|\,dx\,ds+\int_0^T\int\big|\langle x\rangle^{-1}v\big(\Box+a(x)\partial_t\big)v\big|\,dx\,ds$$
$$+\sum_{|\alpha|\leqslant 1}\big\|\partial^\alpha v\big\|_{L^2_tL^2_x([0,T]\times\{|x|\leqslant 2\})}\big\|\sqrt{a(x)}\partial_t v\big\|_{L^2_tL^2_x(S_T)}. \tag{2.21}$$

Absorbing the first factor in the last term into the left-hand side and using the (boundaryless analog of the) energy inequality to control the last factor in the last term, we have

$$\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}v'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-1/2-}v'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-3/2-}v\big\|^2_{L^2_tL^2_x(S_T)}$$
$$\lesssim\big\|v'(0,\cdot)\big\|_2^2+\int_0^T\int\big|\partial_{t,x}v\big(\Box+a(x)\partial_t\big)v\big|\,dx\,ds+\int_0^T\int\big|\langle x\rangle^{-1}v\big(\Box+a(x)\partial_t\big)v\big|\,dx\,ds. \tag{2.22}$$

We now suppose that v solves the boundaryless equation

$$\begin{cases}\big(\Box+a(x)\partial_t\big)v=F+G, & (t,x)\in\mathbb{R}\times\mathbb{R}^3,\\ v(t,x)=0, & t\leqslant 0,\end{cases} \tag{2.23}$$

where $G(t,x)=0$ when $|x|>4$ for each t. Then, by using the Schwarz inequality, it is clear that the following estimate results from (2.22),

$$\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}v'\big\|_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-1/2-}v'\big\|_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-3/2-}v\big\|_{L^2_tL^2_x(S_T)}$$
$$\lesssim\int_0^T\big\|F(t,\cdot)\big\|_2\,dt+\|G\|_{L^2_tL^2_x(S_T)}. \tag{2.24}$$

We now show a higher-order KSS estimate for the boundaryless equation. The main idea here is to use induction to control the additional commutator terms that arise due to the localized dissipation.

Lemma 2.8. *Suppose that v solves* (2.23) *where G vanishes for $|x|\geqslant 4$. Then for any multiindex α and any $\mu\geqslant 0$, we have*

$$\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha v'\big\|_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-1/2-}L^\mu\partial^\alpha v'\big\|_{L^2_tL^2_x(S_T)}$$
$$+\big\|\langle x\rangle^{-3/2-}L^\mu\partial^\alpha v\big\|_{L^2_tL^2_x(S_T)}$$
$$\lesssim\sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu\\ \nu\leqslant\mu}}\int_0^T\big\|L^\nu\partial^\beta F(s,\cdot)\big\|_2\,ds+\sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu\\ \nu\leqslant\mu}}\big\|L^\mu\partial^\beta G\big\|_{L^2_tL^2_x(S_T)} \tag{2.25}$$

and

$$\begin{aligned}&\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}L^\mu Z^\alpha v'\big\|_{L_t^2L_x^2(S_T)}+\big\|\langle x\rangle^{-1/2-}L^\mu Z^\alpha v'\big\|_{L_t^2L_x^2(S_T)}\\&\quad+\big\|\langle x\rangle^{-3/2-}L^\mu Z^\alpha v\big\|_{L_t^2L_x^2(S_T)}\\&\quad\lesssim\sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu\\ \nu\leqslant\mu}}\int_0^T\big\|L^\nu Z^\beta F(s,\cdot)\big\|_2\,ds+\sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu\\ \nu\leqslant\mu}}\big\|L^\mu\partial^\beta G\big\|_{L_t^2L_x^2(S_T)}\end{aligned}\tag{2.26}$$

for any $T>0$.

Here, we shall focus on proving (2.26). The estimate that precedes it follows from the same argument.

To see (2.26), we notice that

$$\begin{aligned}\big[\Box+a(x)\partial_t,L^\mu Z^\alpha\big]v&=\big[\Box,L^\mu\big]Z^\alpha v+\big[a(x)\partial_t,L^\mu Z^\alpha\big]v\\&=\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha\Box v+\big[a(x)\partial_t,L^\mu Z^\alpha\big]v\\&=\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha\big(\Box+a(x)\partial_t\big)v\\&\quad+\Big[\big[a(x)\partial_t,L^\mu Z^\alpha\big]v-\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha a(x)\partial_t v\Big].\end{aligned}\tag{2.27}$$

Moreover, we notice that the bracketed quantity on the right-hand side is supported in $\{|x|\leqslant 2\}$. Thus, we may apply (2.24) with v replaced by $L^\mu Z^\alpha v$, F by

$$L^\mu Z^\alpha F+\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha F,$$

and G by

$$\begin{aligned}&L^\mu Z^\alpha G+\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha G+\big[a(x)\partial_t,L^\mu Z^\alpha\big]v\\&\quad-\sum_{j+\nu=\mu,\,j\geqslant1}2^j\binom{\mu}{j}L^\nu Z^\alpha a(x)\partial_t v.\end{aligned}$$

As the last two terms above are bounded by

$$\sum_{\substack{\nu+|\beta|\leqslant\mu+|\alpha|-1\\ \nu\leqslant\mu}}\big|L^\nu\partial^\beta\partial_t v(t,x)\big|,$$

the result follows from an inductive argument. Here we have used that the coefficients of Z are $O(1)$ on the support of $a(x)$.

We now establish an exterior domain analog of (2.22). To do so, we will use a decomposition of Smith and Sogge [21].

Proposition 2.9. *Suppose that $\mathcal{K}$ is a smooth, compact obstacle and that $a(x)$ is as above. Suppose further that $u \in C^\infty$ satisfies $u|_{\partial\mathcal{K}} = 0$, $u(t,x) = 0$ for $t \leqslant 0$, and u vanishes for large $|x|$ for each fixed t. Then, for fixed N, ν and any $T > 0$, we have that*

$$\begin{aligned}\big(\log(2+T)\big)^{-1/2} &\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\big\|_{L^2_t L^2_x(S_T)} \\ &\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \int_0^T \big\|L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds \\ &\quad + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu-1\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big)u\big\|_{L^2_t L^2_x(S_T)} \end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\big(\log(2+T)\big)^{-1/2} &\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\big\|_{L^2_t L^2_x(S_T)} \\ &\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \int_0^T \big\|L^\mu Z^\alpha \big(\Box + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds \\ &\quad + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu-1\\ \mu\leqslant\nu}} \big\|L^\mu Z^\alpha \big(\Box + a(x)\partial_t\big)u\big\|_{L^2_t L^2_x(S_T)}. \end{aligned} \tag{2.29}$$

We shall only prove (2.28) as the latter estimate follows from similar arguments. Here, we shall use the usual strategy of referring to local energy decay when x is near the obstacle and referring to the boundaryless estimate away from the obstacle.

We begin by showing that the following better bound is available near the obstacle

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha u'\big\|_{L^2_t L^2_x([0,T]\times\{|x|<2\})} &\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big)u(s,\cdot)\big\|_2\,ds \\ &\quad + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu-1\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big)u\big\|_{L^2_t L^2_x(S_T)}. \end{aligned} \tag{2.30}$$

When $(\Box + a(x)\partial_t)u$ vanishes for $|x| > 4$, this follows easily from (2.4). Thus, supposing $(\Box + a(x)\partial_t)u = 0$ for $|x| < 3$, we fix a smooth cutoff function ρ with $\rho \equiv 1$ for $|x| < 2$ and $\rho \equiv 0$ for $|x| > 3$. We then write $u = v + u_r$ where v is a solution to the boundaryless equation $(\Box + a(x)\partial_t)v = (\Box + a(x)\partial_t)u$ with vanishing Cauchy data. We then set $\tilde{u} = \rho v + u_r$ and notice that $u = \tilde{u}$ for $|x| < 2$ and

$$\big(\Box + a(x)\partial_t\big)\tilde{u} = -2\nabla\rho \cdot \nabla_x v - (\Delta\rho)v.$$

Applying (2.4), we see that

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha u'\big\|^2_{L^2_t L^2_x([0,T]\times\{|x|<2\})}\\
&\quad= \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha \tilde{u}'\big\|^2_{L^2_t L^2_x([0,T]\times\{|x|<2\})}\\
&\quad\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big)\tilde{u}\big\|^2_{L^2_t L^2_x(S_T)}\\
&\quad\lesssim \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha v'\big\|^2_{L^2_t L^2_x([0,T]\times\{|x|\leqslant 3\})} + \sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha v\big\|^2_{L^2_t L^2_x([0,T]\times\{|x|\leqslant 3\})}
\end{aligned}$$

in this case. Using (2.25), we complete the proof of (2.30).

It remains to show (2.28) when the norm in the left is taken over $[0, T] \times \{|x| > 2\}$. To do this, we fix $\beta \in C^\infty(\mathbb{R}^3)$ with $\beta(x) \equiv 1$ for $|x| > 2$ and $\beta(x) \equiv 0$ for $|x| \leqslant 3/2$. It, thus, suffices to show the estimate (2.28) with u replaced by $v = \beta u$, which solves the boundaryless[14] equation

$$\big(\Box + a(x)\partial_t\big)v = \beta\big(\Box + a(x)\partial_t\big)u - 2\nabla\beta \cdot \nabla_x u - (\Delta\beta)u.$$

If we apply (2.25) with $F = \beta(\Box + a(x)\partial_t)$ and $G = -2\nabla\beta \cdot \nabla_x u - (\Delta\beta)u$, we see that it suffices to establish a bound for

$$\sum_{\substack{|\alpha|+\mu\leqslant N+\nu\\ \mu\leqslant\nu}} \big\|L^\mu \partial^\alpha u'\big\|_{L^2_t L^2_x([0,T]\times\{|x|<2\})},$$

which follows from (2.30).

We also record the following KSS estimate which holds regardless of boundary condition.

Lemma 2.10. *Suppose that $\mathcal{K} \subset \{|x| < 1\}$ is a C^1, compact obstacle containing the origin and that $a(x)$ is as above. Let $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K})$ solve $(\Box + a(x)\partial_t)\phi = F + G$ where $G(s, y)$ vanishes for any $|y| \geqslant 2$. Suppose further ϕ vanishes for large $|x|$ for each fixed t. Then, for any $T > 0$,*

[14] Recall that we are assuming that $\mathcal{K} \subset \{|x| < 1\}$.

$$\begin{aligned}&\big(\log(2+T)\big)^{-1/2}\big\|\langle x\rangle^{-1/2}\phi'\big\|_{L^2_tL^2_x(S_T)}\\&\quad\lesssim \sup_{0\leqslant t\leqslant T}\big\|\phi'(t,\cdot)\big\|_2+\big\|\sqrt{a}\partial_t\phi\big\|_{L^2_tL^2_x(S_T)}+\int_0^T\big\|F(t,\cdot)\big\|_2\,dt+\|G\|_{L^2_tL^2_x(S_T)}\\&\qquad+\sup_{0\leqslant t\leqslant T}\biggl(\int_{\partial\mathcal{K}}\big|\phi(t,x)\big|^2\,d\sigma(x)\biggr)^{1/2}+\biggl(\int_0^T\int_{\partial\mathcal{K}}\bigl(\big|\phi'(t,x)\big|^2+\big|\phi(t,x)\big|^2\bigr)\,d\sigma(x)\,dt\biggr)^{1/2},\end{aligned}\tag{2.31}$$

where all of the L^2_x norms are over $\mathbb{R}^3\setminus\mathcal{K}$.

As above, this follows from the arguments in [13], which we shall only sketch. See also [23]. By defining the energy-momentum tensor $Q_{\alpha\beta}[\phi]$ associated to $\Box$,[15]

$$Q_{\alpha\beta}[\phi]=\partial_\alpha\phi\partial_\beta\phi-\frac{1}{2}m_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi,$$

and setting

$$\tilde{P}_\alpha[\phi,R]=\frac{r}{r+R}\frac{x^\beta}{r}Q_{\alpha\beta}[\phi]+\frac{1}{r+R}\phi\partial_\alpha\phi-\frac{1}{2}\partial_\alpha\biggl(\frac{1}{r+R}\biggr)\phi^2,$$

we may integrate the divergence of $\tilde{P}_\alpha[\phi,R]$ and sum over $R=2^j$, $j=0,1,2,\ldots,$ with $2^j\leqslant T$ to see

$$\begin{aligned}&\log(2+T)^{-1}\big\|\langle x\rangle^{-1/2}\phi'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-1/2-}\phi'\big\|^2_{L^2_tL^2_x(S_T)}+\big\|\langle x\rangle^{-3/2-}\phi\big\|^2_{L^2_tL^2_x(S_t)}\\&\quad\lesssim\big\|\phi'(0,\cdot)\big\|_2^2+\biggl\|\frac{1}{r}\phi(0,\cdot)\biggr\|_2^2+\big\|\phi'(T,\cdot)\big\|_2^2+\biggl\|\frac{1}{r}\phi(T,\cdot)\biggr\|_2^2\\&\qquad+\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}}\biggl(\big|\phi'(t,x)\big|+\frac{1}{\langle x\rangle}\big|\phi(t,x)\big|\biggr)\big|\Box\phi(t,x)\big|\,dx\,dt\\&\qquad+\int_0^T\int_{\partial\mathcal{K}}\bigl(\big|\phi'(t,x)\big|^2+\big|\phi(t,x)\big|^2\bigr)\,d\sigma(x)\,dt.\end{aligned}$$

Using the elementary identity

$$\partial_\beta\biggl(\frac{x^\beta}{r^2}\phi^2\biggr)=\frac{1}{r^2}\phi^2+\frac{2}{r}\phi\partial_r\phi,$$

[15] Here, as before, we use $m_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$ to denote the Minkowski metric, and we raise indices with this metric.

we easily see that

$$\left\| \frac{1}{r}\phi(t,\cdot) \right\|_2^2 \lesssim \int_{\partial\mathcal{K}} \left|\phi(t,x)\right|^2 d\sigma(x)\,dt + \left\|\partial_r \phi(t,\cdot)\right\|_2^2,$$

from which (2.31) follows.

2.4. Boundary term estimate

To handle the "boundary term"[16] appearing in (2.16), we develop an estimate for the dissipative wave equation which is analogous to that of [11] for the wave equation without dissipation. Let u solve

$$\begin{cases} \big(\Box + a(x)\partial_t\big)u = F, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(t,x) = 0, & t \leqslant 0. \end{cases} \tag{2.32}$$

Lemma 2.11. *Let $\mathcal{K}$ and $a(x)$ be as above. Suppose that $u \in C^\infty$ solves* (2.32). *Then,*

$$\sum_{\substack{|\alpha|+\mu \leqslant N+\nu \\ \mu \leqslant \nu}} \int_0^t \left\| L^\mu \partial^\alpha u'(s,\cdot) \right\|_{L^2(|x|<2)} ds$$
$$\lesssim \sum_{\substack{|\alpha|+\mu \leqslant N+\nu \\ \mu \leqslant \nu}} \int_0^t \left(\int_0^s \left\| L^\mu \partial^\alpha \big(\Box + a(x)\partial_t\big) u(\tau,\cdot) \right\|_{L^2(||x|-(s-\tau)|<10)} d\tau \right) ds \tag{2.33}$$

for any fixed $N, \nu \geqslant 0$ and $t > 2$.

When $(\Box + a(x)\partial_t)u = 0$ for $|x| > 4$, (2.33) follows from (2.4). Thus, for the remainder of the proof, we may assume that $(\Box + a(x)\partial_t)u$ vanishes for $|x| < 3$.

Here, we write $u = w + u_r$ where w solves the boundaryless equation $\Box w = F$ with vanishing initial data. Fixing a smooth cutoff ρ with $\rho(x) \equiv 1$ for $|x| < 2$ and $\rho(x) \equiv 0$ for $|x| \geqslant 3$, we set $\tilde{u} = \rho w + u_r$. It is easy to check that

$$\big(\Box + a(x)\partial_t\big)\tilde{u} = -2\nabla\rho \cdot \nabla_x w - (\Delta\rho) w,$$

since $\Box w$ is assumed to vanish on the support of ρ and since $\rho \equiv 1$ on the support of $a(x)$. As $u = \tilde{u}$ for $|x| \leqslant 2$, an application of (2.4) shows that it suffices to control

$$\sum_{\substack{|\alpha|+\mu \leqslant N+\nu \\ \mu \leqslant \nu}} \int_0^t \left\| L^\mu \partial^\alpha w'(s,\cdot) \right\|_{L^2(|x|<3)} ds + \sum_{\substack{|\alpha|+\mu \leqslant N+\nu \\ \mu \leqslant \nu}} \int_0^t \left\| L^\mu \partial^\alpha w(s,\cdot) \right\|_{L^2(|x|<3)} ds. \tag{2.34}$$

[16] We refer to the fourth term in the right of (2.16) as a boundary term, though it results from a commutator. However, this commutator is only necessary because of the boundary.

Since $\Box w = (\Box + a(x)\partial_t)u$, we may use the energy inequality associated to $\Box$ and sharp Huygens' principle to complete the proof.

3. Decay estimates

In this section, we gather the decay estimates that we shall require. The first is a rather standard weighted Sobolev estimate which provides decay in $|x|$. The latter is closely related to a L^1–L^∞ Hörmander-type estimate of Keel, Smith, and Sogge [7].

3.1. Decay in $|x|$

In the sequel, we shall use the following weighted Sobolev estimate from [8]. To prove the estimate, one simply applies Sobolev estimates on $\mathbb{R} \times S^2$. The decay results from comparing the volume elements of $\mathbb{R} \times S^2$ and $\mathbb{R}^3$.

Lemma 3.1. *Suppose that $h \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$. Then, for $R \geqslant 1$,*

$$\|h\|_{L^\infty(R/2<|x|<R)} \lesssim R^{-1} \sum_{|\alpha|\leqslant 2} \left\| Z^\alpha h \right\|_{L^2(R/4<|x|<2R)}. \tag{3.1}$$

3.2. Decay in $t + |x|$

In this section, we prove the main decay estimate that we shall require in order to show long-time existence. This estimate is an analog of the boundaryless estimate

$$\begin{aligned}\big(1+t+|x|\big)\big|v(t,x)\big| \lesssim{}& \sum_{|\alpha|\leqslant 4} \left\| \langle x\rangle^{|\alpha|}\partial^\alpha v(0,\cdot)\right\|_2 + \sum_{|\alpha|\leqslant 3} \left\| \langle x\rangle^{1+|\alpha|}\partial^\alpha \partial_t v(0,\cdot)\right\|_2 \\ &+ \sum_{\substack{|\alpha|+\mu\leqslant 3\\ \mu\leqslant 1}} \int_0^t \int_{\mathbb{R}^3} \left| L^\mu Z^\alpha \Box v(s,y)\right| \frac{dy\,ds}{\langle y\rangle}\end{aligned} \tag{3.2}$$

from [10]. This is a variant of a L^1–L^∞ Hörmander-type estimate. The original adaptations of these estimates to eliminate the dependence on the Lorentz boosts was due to Keel, Smith, and Sogge [7]. See also Sogge [22].

In this section, we shall estimate solutions to[17]

$$\begin{cases} \big(\Box + a(x)\partial_t\big) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\ w|_{\partial\mathcal{K}} = 0, \\ w(0,\cdot) = f, \quad \partial_t w(0,\cdot) = g, \\ \operatorname{supp} f, g \subset \big\{|x| \geqslant 5\big\}. \end{cases} \tag{3.3}$$

This yields the following proposition.

[17] We will reduce to this setting in the sequel.

Proposition 3.2. *Suppose that $\mathcal{K}$ and $a(x)$ are as above, and let w be a solution to* (3.3)*. Then,*

$$\begin{aligned}
&\big(1+t+|x|\big)\big|L^\mu Z^\alpha w(t,x)\big| \\
&\quad\lesssim \sum_{\substack{j+|\beta|+k\leqslant|\alpha|+\mu+7\\ j\leqslant 1}} \big\|\langle x\rangle^{j+|\beta|}\partial_x^\beta\partial_t^{k+j}w(0,\cdot)\big\|_2 + \int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu+6\\ \nu\leqslant\mu+1}} \big|L^\nu Z^\beta F(s,y)\big|\frac{dy\,ds}{|y|} \\
&\qquad + \int_0^t \sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu+3\\ \nu\leqslant\mu+1}} \big\|L^\nu\partial^\beta F(s,\cdot)\big\|_2\,ds
\end{aligned} \tag{3.4}$$

for any $|\alpha|=M$ and any μ.

Here, we argue as in the proof of Proposition 2.9. We first show that

$$\begin{aligned}
&\big(1+t+|x|\big)\big|L^\mu Z^\alpha w(t,x)\big| \\
&\quad\lesssim \sum_{\substack{j+|\beta|+k\leqslant|\alpha|+\mu+4\\ j\leqslant 1}} \big\|\langle x\rangle^{j+|\beta|}\partial_x^\beta\partial_t^{k+j}w(0,\cdot)\big\|_2 + \int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu+3\\ \nu\leqslant\mu+1}} \big|L^\nu Z^\beta F(s,y)\big|\frac{dy\,ds}{|y|} \\
&\qquad + \sup_{\substack{0\leqslant s\leqslant t\\ |y|\leqslant 4}}(1+s) \sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu+1\\ \nu\leqslant\mu}} \big|L^\nu\partial^\beta w(s,y)\big|.
\end{aligned} \tag{3.5}$$

Indeed, this estimate is trivial for $|x|\leqslant 4$ since the coefficients of Z are $O(1)$ on this set. To show the estimate when $|x|\geqslant 4$, we fix $\rho\in C^\infty(\mathbb{R})$ with $\rho(r)\equiv 1$ for $|x|\geqslant 4$ and $\rho(r)\equiv 0$ for $|x|<3$. It, thus, suffices to bound $w_0(t,x)=\rho(|x|)L^\mu Z^\alpha w(t,x)$ which solves the free (boundaryless) wave equation[18]

$$\Box w_0(t,x)=\rho\big(|x|\big)\Box L^\mu Z^\alpha w(t,x)-2\rho'\big(|x|\big)\frac{x}{|x|}\cdot\nabla_x L^\mu Z^\alpha w(t,x)-\big(\Delta\rho\big(|x|\big)\big)L^\mu Z^\alpha w(t,x).$$

As such, we may apply the argument of [7, Lemma 4.2].[19]

It remains to estimate the last term in (3.5). Using a smooth partition, it suffices to examine the cases:

Case 1: $F(s,y)=0$ if $|y|>6$ with vanishing data,
Case 2: $F(s,y)=0$ if $|y|<5$ with the data given in (3.3)

separately.

[18] Recall that, by assumption, $a(x)$ vanishes on the support of ρ.

[19] The argument in [7] is only given for vanishing data. Straightforward modifications yield the case of nonvanishing data. See, e.g., [10] for the required estimates.

In both cases, we shall use

$$t \sup_{|x|<4} \sum_{|\beta|\leqslant|\alpha|+1} \left|L^\mu \partial^\beta w(t,x)\right| \lesssim \int_0^t \sum_{\substack{|\beta|+\nu\leqslant|\alpha|+\mu+3\\ \nu\leqslant\mu+1}} \left\|L^\nu \partial^\beta w'(s,\cdot)\right\|_{L^2(\{|x|<4\})}\, ds,$$

which follows from the Fundamental Theorem of Calculus and the Sobolev lemma.[20] The bound for Case 1 follows from an application of (2.4).

To handle Case 2, we shall construct $\tilde{w}$ as in the proof of Lemma 2.11. That is, we write $w = v + w_r$ where v solves the free, boundaryless equation $\Box v = F$ with $(v(0,\cdot), \partial_t v(0,\cdot)) = (w(0,\cdot), \partial_t w(0,\cdot))$ which are supported in $\{|x| > 5\}$. For a smooth cutoff η with $\eta(x) \equiv 1$ for $|x| \leqslant 4$ and $\eta(x) \equiv 0$ for $|x| > 5$, we set $\tilde{w} = \eta v + w_r$ and notice that $w = \tilde{w}$ for $|x| \leqslant 4$, $\tilde{w}$ has vanishing Cauchy data, and[21]

$$\left(\Box + a(x)\partial_t\right)\tilde{w} = -2\nabla\eta \cdot \nabla_x v - (\Delta\eta)v,$$

which vanishes unless $4 \leqslant |x| \leqslant 5$. Thus, by Case 1, the last term in (3.5) is controlled in this case by

$$\int_0^t \sum_{\substack{|\beta|+\nu\leqslant M+\mu+4\\ \nu\leqslant\mu+1}} \left\|L^\nu \partial^\beta v(s,\cdot)\right\|_{L^\infty(4\leqslant|x|\leqslant5)}\, ds. \tag{3.6}$$

Using the identity

$$\begin{aligned}
|x|\left|v_0(t,x)\right| \lesssim{}& \sum_{|\gamma|\leqslant2} \int_{\{|y|\in[|t-|x||,t+|x|]\}} \left|\Omega^\gamma \nabla_x v_0(0,y)\right| \frac{dy}{|y|} \\
&+ \sum_{|\gamma|\leqslant2,\, j\leqslant1} \int_{\{|y|\in[|t-|x||,t+|x|]\}} \left|\left(|y|\partial_{|y|}\right)^j \Omega^\gamma v_0(0,y)\right| \frac{dy}{|y|^2} \\
&+ \sum_{|\gamma|\leqslant2} \int_{\{|y|\in[|t-|x||,t+|x|]\}} \left|\Omega^\gamma \partial_t v_0(0,y)\right| \frac{dy}{|y|} \\
&+ \int_0^t \int_{\{|y|\in[||x|-(t-s)|,|x|+(t-s)]\}} \sum_{|\gamma|\leqslant2} \left|\Omega^\gamma \Box v_0(s,y)\right| \frac{dy\,ds}{|y|}
\end{aligned} \tag{3.7}$$

for the free wave equation,[22] it follows that

[20] See [11].

[21] Recall that the support of $a(x)$ is contained in $\{|x| \leqslant 2\}$.

[22] See, e.g., [10].

$$\begin{aligned}\left\|L^\mu\partial^\beta v(s,\cdot)\right\|_{L^\infty(4\leqslant|x|\leqslant5)} \lesssim{}& \sum_{|\gamma|\leqslant2}\int\limits_{|s-|y||\leqslant5}\left|\left(\Omega^\alpha\nabla_x L^\mu\partial^\beta v\right)(0,y)\right|\frac{dy}{|y|}\\ &+\sum_{|\gamma|\leqslant2,\, j\leqslant1}\int\limits_{|s-|y||\leqslant5}\left|\left(\left(|y|\partial_{|y|}\right)^j\Omega^\alpha L^\mu\partial^\beta v\right)(0,y)\right|\frac{dy}{|y|^2}\\ &+\sum_{|\gamma|\leqslant2}\int\limits_{|s-|y||\leqslant5}\left|\left(\Omega^\alpha\partial_t L^\mu\partial^\beta v\right)(0,y)\right|\frac{dy}{|y|}\\ &+\sum_{|\gamma|\leqslant2}\int\limits_0^s\int\limits_{|s-\tau-|y||\leqslant5}\left|\Omega^\alpha\Box L^\mu\partial^\beta v(\tau,y)\right|\frac{dy\,d\tau}{|y|}.\end{aligned}$$

Since, e.g., the sets $\Lambda_s=\{(\tau,y)\colon\ 0\leqslant\tau\leqslant s,\ |s-\tau-|y||\leqslant5\}$ are disjoint if $|s-s'|\geqslant10$, the estimate for (3.6) follows after integration, application of the Schwarz inequality for the initial data terms, and recalling that $\Box v=(\Box+a(x)\partial_t)w=F$.

4. Almost global existence

In this section, we prove Theorem 1.1, though this is largely standard once the results of the preceding section have been established. In what follows, we shall take $N=60$, though this is far from optimal.

We shall first make a reduction to the case of vanishing Cauchy data, as in [10]. To do so, we note that, for ε in (1.5) small, we can find a solution to (1.1) on $0<t<2$. Moreover, we have

$$\sup_{0\leqslant t\leqslant2}\sum_{|\alpha|+\mu\leqslant59}\left\|L^\mu Z^\alpha u'(t,\cdot)\right\|_{L^2(\{|x|\leqslant10\})}\leqslant C_0\varepsilon. \tag{4.1}$$

Indeed, such an estimate follows from standard local existence theory arguments. See, e.g., [5]. Using finite propagation speed, it follows that over $\{t\in[0,2]\}\times\{|x|\geqslant5\}$, u corresponds to a solution of the free boundaryless wave equation $\Box u=Q(u',u'')$.[23] Thus, by free space estimates (see [7]), we have

$$\sup_{0\leqslant t\leqslant2}\sum_{|\alpha|+\mu\leqslant59}\left\|L^\mu Z^\alpha u'(t,\cdot)\right\|_{L^2(\{|x|\geqslant5\})}+\sup_{\substack{0\leqslant t\leqslant2\\|x|\geqslant5}}\left(1+|x|\right)\sum_{|\alpha|+\mu\leqslant51}\left|L^\mu Z^\alpha u(t,x)\right|\leqslant C_1\varepsilon. \tag{4.2}$$

We shall use this local solution to complete our reduction. Let $\eta\in C^\infty(\mathbb{R}\times\mathbb{R}^3)$ satisfy $\eta(t,x)\equiv1$ if $t\leqslant3/2$ and $|x|\leqslant5$, $\eta(t,\cdot)\equiv0$ for $t>2$, and $\eta(\cdot,x)\equiv0$ for $|x|>8$. We set $u_0=\eta u$ and notice that solving (1.1) on $[0,T_\varepsilon)$ is equivalent to solving

$$\begin{cases}\left(\Box+a(x)\partial_t\right)\tilde w=(1-\eta)Q\left((u_0+\tilde w)',(u_0+\tilde w)''\right)-\left[\Box+a(x)\partial_t,\eta\right](u_0+\tilde w),\\ \tilde w|_{\partial\mathcal{K}}=0,\\ \tilde w(0,\cdot)=(1-\eta)(0,\cdot)f,\\ \partial_t\tilde w(0,\cdot)=(1-\eta)(0,\cdot)g-\eta_t(0,\cdot)f\end{cases} \tag{4.3}$$

[23] Here we are using the assumptions that $\mathcal{K}\subset\{|x|<1\}$ and $\operatorname{supp}a\subset\{|x|<2\}$.

for $\tilde{w} = u - u_0$ on the same interval.

It will be convenient to make a further reduction. Here, we fix another smooth cutoff function β with $\beta(t) \equiv 1$ for $t \leqslant 1$ and $\beta(t) \equiv 0$ for $t \geqslant 3/2$. We then let v be the solution of

$$\begin{cases} (\Box + a(x)\partial_t)v = \beta(1-\eta)Q(u', u'') - [\Box + a(x)\partial_t, \eta]u, \\ v|_{\partial\mathcal{K}} = 0, \\ v(0,\cdot) = (1-\eta)(0,\cdot)f, \\ \partial_t v(0,\cdot) = (1-\eta)(0,\cdot)g - \eta_t(0,\cdot)f. \end{cases} \tag{4.4}$$

We then wish to show

$$\begin{aligned} &(1+t+|x|) \sum_{|\alpha|+\mu\leqslant 52} \left|L^\mu Z^\alpha v(t,x)\right| + \sum_{|\alpha|+\mu\leqslant 50} \left\|L^\mu Z^\alpha v'(t,\cdot)\right\|_2 \\ &\quad + \left(\log(2+t)\right)^{-1/2} \sum_{\mu+|\alpha|\leqslant 49} \left\|\langle x\rangle^{-1/2} L^\mu Z^\alpha v'\right\|_{L_t^2 L_x^2(S_t)} \leqslant C_2\varepsilon \end{aligned} \tag{4.5}$$

for any $0 \leqslant t \leqslant \infty$.

Proof of (4.5). We begin with the first term in the left-hand side. By (1.5) and (3.4), this term is

$$\begin{aligned} &\lesssim \varepsilon + \sum_{|\alpha|+\mu\leqslant 58} \int_0^t \int \left|L^\mu Z^\alpha \beta(s)(1-\eta)(s,y)Q(u',u'')(s,y)\right| \frac{dy\,ds}{|y|} \\ &\quad + \sum_{|\alpha|+\mu\leqslant 55} \int_0^t \left\|L^\mu Z^\alpha \beta(s)(1-\eta)(s,y)Q(u',u'')(s,\cdot)\right\|_2 ds \\ &\quad + \sum_{|\alpha|+\mu\leqslant 58} \int_0^t \int \left|L^\mu Z^\alpha [\Box + a(x)\partial_t, \eta]u(s,y)\right| \frac{dy\,ds}{|y|} \\ &\quad + \sum_{|\alpha|+\mu\leqslant 55} \int_0^t \left\|L^\mu Z^\alpha [\Box + a(x)\partial_t, \eta]u(s,\cdot)\right\|_2 ds. \end{aligned} \tag{4.6}$$

Using (4.1), that $\operatorname{supp}\beta \subset \{s \leqslant 3/2\}$, and the Schwarz inequality, the second term in (4.6) is $O(\varepsilon^2)$. Indeed, the second term in (4.6) is bounded by

$$\begin{aligned} &\sum_{|\alpha|+\mu\leqslant 30} \sum_{|\beta|+\nu\leqslant 59} \int_0^{3/2} \int_{\{|y|\geqslant 5\}} \left|L^\mu Z^\alpha u'(s,y)\right| \left|L^\nu Z^\beta u'(s,y)\right| \frac{dy\,ds}{|y|} \\ &\quad \lesssim \sup_{0\leqslant s\leqslant 3/2} \sum_{|\alpha|+\mu\leqslant 59} \left\|L^\mu Z^\alpha u'(s,\cdot)\right\|^2_{L^2(\{|x|\geqslant 5\})}, \end{aligned}$$

which is $O(\varepsilon^2)$ by (4.2). The third term in (4.6) can be shown to satisfy the same estimate using similar arguments. As last two terms in (4.6) are easily seen to be $\lesssim C_0\varepsilon$ using (4.1), we have completed the bound for the first term in (4.5)

To estimate the second term in the left-hand side of (4.5), we apply the standard energy integral method. This yields

$$\begin{aligned}
&\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha v'(t,\cdot)\big\|_2^2+\sum_{|\alpha|+\mu\leqslant 50}\big\|\sqrt{a(x)}\partial_t L^\mu Z^\alpha v\big\|^2_{L^2_tL^2_x(S_t)}\\
&\quad\lesssim \varepsilon^2+\int_0^t\int\sum_{|\alpha|+\mu\leqslant 50}\big|\big(\Box+a(x)\partial_t\big)L^\mu Z^\alpha v(s,y)\big|\big|\partial_t L^\mu Z^\alpha v(s,y)\big|\,dy\,ds\\
&\qquad+\sum_{|\alpha|+\mu\leqslant 50}\left|\int_0^t\int_{\partial\mathcal{K}}\partial_t L^\mu Z^\alpha v(s,\cdot)\nabla L^\mu Z^\alpha v(s,\cdot)\cdot\nu(x)\,d\sigma\,ds\right|.
\end{aligned}\tag{4.7}$$

Since $\mathcal{K}\subset\{|x|<1\}$ and since the coefficients of Z are $O(1)$ on $\partial\mathcal{K}$, a trace theorem implies that the last term is

$$\lesssim\int_0^t\sum_{|\alpha|+\mu\leqslant 51}\big\|L^\mu\partial^\alpha v'(s,\cdot)\big\|^2_{L^2(|x|<1)}\,ds,$$

which is $O(\varepsilon^2)$ using the preceding estimate for the first term in the left of (4.5). Using the Schwarz inequality and (2.27), this yields

$$\begin{aligned}
&\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha v'(t,\cdot)\big\|_2^2+\sum_{|\alpha|+\mu\leqslant 50}\big\|\sqrt{a(x)}\partial_t L^\mu Z^\alpha v\big\|^2_{L^2_tL^2_x(S_t)}\\
&\quad\lesssim \varepsilon^2+\left(\int_0^t\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha\big(\Box+a(x)\partial_t\big)v(s,\cdot)\big\|_2\,ds\right)^2\\
&\qquad+\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha\partial_t v\big\|^2_{L^2_tL^2_x([0,t]\times\{|x|<2\})}.
\end{aligned}\tag{4.8}$$

As above, the bound for the last term follows from that for the first term in the left of (4.5). The second term in the right of (4.8) is

$$\begin{aligned}
&\lesssim\left(\int_0^t\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha\beta(s)(1-\eta)(s,\cdot)Q(u',u'')(s,\cdot)\big\|_2\,ds\right)^2\\
&\quad+\left(\int_0^t\sum_{|\alpha|+\mu\leqslant 50}\big\|L^\mu Z^\alpha\big[\Box+a(x)\partial_t,\eta\big]u(s,\cdot)\big\|_2\,ds\right)^2.
\end{aligned}$$

The last term is clearly bounded by $C_0^2\varepsilon^2$ using (4.1). Arguing as above, it is clear that the first term is controlled by the square of the right-hand side of (4.5) using (4.2).

It only remains to bound the last term in the left of (4.5). To see this, we apply (2.31) with $\phi = \sum_{|\alpha|+\mu\leqslant 49} L^\mu Z^\alpha v$, which yields

$$\begin{aligned}
&\bigl(\log(2+t)\bigr)^{-1/2} \sum_{|\alpha|+\mu\leqslant 49} \bigl\|\langle x\rangle^{-1/2}\bigl(L^\mu Z^\alpha v\bigr)'\bigr\|^2_{L^2_t L^2_x(S_t)} \\
&\lesssim \varepsilon + \int_0^t \sum_{|\alpha|+\mu\leqslant 49} \bigl\|L^\mu Z^\alpha\bigl(\Box + a(x)\partial_t\bigr)v(s,\cdot)\bigr\|_2\,ds + \sum_{|\alpha|+\mu\leqslant 49} \bigl\|L^\mu Z^\alpha v\bigr\|_{L^2_t L^2_x(S_t)} \\
&\quad + \sup_{0\leqslant s\leqslant t}\Biggl(\int_{\partial\mathcal{K}} \sum_{|\alpha|+\mu\leqslant 49} \bigl|L^\mu Z^\alpha v(s,x)\bigr|^2\,d\sigma(x)\Biggr)^{1/2} \\
&\quad + \Biggl(\int_0^t\int_{\partial\mathcal{K}} \sum_{|\alpha|+\mu\leqslant 49} \bigl(\bigl|L^\mu Z^\alpha v'(s,x)\bigr|^2 + \bigl|L^\mu Z^\alpha v(s,x)\bigr|^2\bigr)\,d\sigma(x)\,ds\Biggr)^{1/2}.
\end{aligned} \tag{4.9}$$

Here, we have used the bound for the left-hand side of (4.8) to estimate the first two terms in the right of (2.31). We have also used (2.27) and chosen F and G in (2.31) in a fashion similar to that of the proof of Lemma 2.8. By a trace theorem and the estimate for the first term in (4.5), the bound for the last two terms follows. Since the second and third term in the right is each $O(\varepsilon)$ as above, the proof of (4.5) is complete. □

We now return to the task of solving (4.3). We will solve this equation using an iteration argument. Setting $w_0 = 0$ and recursively defining w_k to solve[24]

$$\begin{cases}
\bigl(\Box + a(x)\partial_t\bigr)w_k = (1-\beta)(1-\eta)Q\bigl((u_0+v+w_{k-1})', (u_0+v+w_k)''\bigr), \\
w_k|_{\partial\mathcal{K}} = 0, \\
w_k(t,x) = 0, \quad t\leqslant 0,
\end{cases} \tag{4.10}$$

we seek to show that

$$\begin{aligned}
M_k(T) &= \sum_{\mu=0}^1 \sup_{0\leqslant t\leqslant T}\Biggl(\sum_{|\alpha|\leqslant 50-3\mu} \bigl\|L^\mu\partial^\alpha w_k'(t,\cdot)\bigr\|_2 + \sum_{|\alpha|\leqslant 48-3\mu} \bigl\|L^\mu Z^\alpha w_k'(t,\cdot)\bigr\|_2\Biggr) \\
&\quad + \bigl(\log(2+T)\bigr)^{-1/2}\Biggl(\sum_{\mu=0}^1 \sum_{|\alpha|\leqslant 49-3\mu} \bigl\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha w_k'\bigr\|_{L^2_t L^2_x(S_T)} \\
&\quad + \sum_{\mu=0}^1 \sum_{|\alpha|\leqslant 47-3\mu} \bigl\|\langle x\rangle^{-1/2}L^\mu Z^\alpha w_k'\bigr\|_{L^2_t L^2_x(S_T)}\Biggr) \\
&\quad + \sup_{0\leqslant t\leqslant T}(1+t)\sum_{|\alpha|\leqslant 26} \bigl\|Z^\alpha w_k(t,\cdot)\bigr\|_\infty
\end{aligned} \tag{4.11}$$

[24] Roughly, we are setting $w = \tilde{w} - v$.

is uniformly bounded for $0 \leqslant T \leqslant T_\varepsilon$. We will denote these terms $I_{k,\mu}$, $II_{k,\mu}$, $III_{k,\mu}$, $IV_{k,\mu}$, and V_k respectively, where for example

$$I_{k,\mu}(T) = \sup_{0\leqslant t\leqslant T} \sum_{|\alpha|\leqslant 50-3\mu} \left\| L^\mu \partial^\alpha w_k'(t,\cdot)\right\|_2.$$

By fixing M_0 to be the quantity obtained by replacing w_k by $u_0 + v$ in (4.11), it follows that

$$M_0(T) \leqslant C_3\varepsilon, \quad 0 \leqslant T < \infty.$$

Here we have used (4.1) and (4.5). It remains to show inductively that

$$M_k(T_\varepsilon) \leqslant C_4 C_3 \varepsilon, \tag{4.12}$$

where C_4 is a fixed, uniform constant.

We begin with $I_{k,0}$ and $III_{k,0}$. By (2.14) and (2.28), it follows that

$$\begin{aligned}
&I_{k,0}(T_\varepsilon) + III_{k,0}(T_\varepsilon)\\
&\quad\lesssim \int_0^{T_\varepsilon} \sum_{|\alpha|\leqslant 50} \left\|\partial^\alpha\big(\Box_h + a(x)\partial_t\big)w_k(t,\cdot)\right\|_2 dt + \int_0^{T_\varepsilon} \sum_{|\alpha|\leqslant 50} \left\|\big[h^{\gamma\delta}\partial_\gamma\partial_\delta, \partial^\alpha\big]w_k(t,\cdot)\right\|_2 dt\\
&\qquad + \sum_{|\alpha|\leqslant 49} \int_0^{T_\varepsilon} \left\|\partial^\alpha\big(\Box + a(x)\partial_t\big)w_k(t,\cdot)\right\|_2 dt + \sum_{|\alpha|\leqslant 49} \left\|\partial^\alpha\big(\Box + a(x)\partial_t\big)w_k(T_\varepsilon,\cdot)\right\|_2\\
&\qquad + \sum_{|\alpha|\leqslant 48} \left\|\partial^\alpha\big(\Box + a(x)\partial_t\big)w_k\right\|_{L^2_t L^2_x(S_{T_\varepsilon})}.
\end{aligned} \tag{4.13}$$

The first three terms in the right can be controlled by

$$\begin{aligned}
&\lesssim \int_0^{T_\varepsilon} \Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\alpha|\leqslant 25} \left\|\partial^\alpha(v+w_{k-1})'(t,\cdot)\right\|_\infty\Big)\Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\beta|\leqslant 50}\left\|\partial^\beta(v+w_k)'(t,\cdot)\right\|_2\Big) dt\\
&\quad + \int_0^{T_\varepsilon} \Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\alpha|\leqslant 25} \left\|\partial^\alpha(v+w_k)'(t,\cdot)\right\|_\infty\Big)\Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\beta|\leqslant 50}\left\|\partial^\beta(v+w_{k-1})'(t,\cdot)\right\|_2\Big) dt\\
&\quad + \int_0^{T_\varepsilon} \Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\alpha|\leqslant 25} \left\|\partial^\alpha(v+w_{k-1})'(t,\cdot)\right\|_\infty\Big)\\
&\quad \times \Big(\mathbf{1}_{\{t\leqslant 2\}}\varepsilon + \sum_{|\beta|\leqslant 50}\left\|\partial^\beta(v+w_{k-1})'(t,\cdot)\right\|_2\Big) dt
\end{aligned}$$

using (4.1) and Sobolev embedding. By (4.5) and (4.11), this yields

$$
\begin{aligned}
&I_{k,0}(T_\varepsilon)+III_{k,0}(T_\varepsilon)\\
&\quad\lesssim \log(2+T_\varepsilon)\bigl[\varepsilon^2+\varepsilon M_{k-1}(T_\varepsilon)+\varepsilon M_k(T_\varepsilon)+M_{k-1}(T_\varepsilon)M_k(T_\varepsilon)+\bigl(M_{k-1}(T_\varepsilon)\bigr)^2\bigr]
\end{aligned}
\tag{4.14}
$$

since the same estimates for the last two terms in (4.13) follow from similar, but simpler, arguments.

To bound $II_{k,0}$ and $IV_{k,0}$, we argue similarly. We simply replace (2.14) and (2.28) by (2.18) and (2.29), respectively. Moreover, we notice that the last term in (2.18) is bounded by the right-hand side of (4.13) using (2.30). This immediately shows, using the argument above and (4.14), that

$$
\begin{aligned}
&II_{k,0}(T_\varepsilon)+IV_{k,0}(T_\varepsilon)\\
&\quad\lesssim \log(2+T_\varepsilon)\bigl[\varepsilon^2+\varepsilon M_{k-1}(T_\varepsilon)+\varepsilon M_k(T_\varepsilon)+M_{k-1}(T_\varepsilon)M_k(T_\varepsilon)+\bigl(M_{k-1}(T_\varepsilon)\bigr)^2\bigr].
\end{aligned}
\tag{4.15}
$$

Next, we shall estimate $I_{k,1}, III_{k,1}$. Using (2.16) and (2.28), we have that

$$
\begin{aligned}
&I_{k,1}(T_\varepsilon)+III_{k,1}(T_\varepsilon)\\
&\lesssim \sum_{\substack{|\alpha|+\mu\leqslant 48\\ \mu\leqslant 1}}\int_0^{T_\varepsilon}\bigl\|L^\mu\partial^\alpha\bigl(\Box_h+a(x)\partial_t\bigr)w_k(t,\cdot)\bigr\|_2\,dt\\
&\quad+\sum_{\substack{|\alpha|+\mu\leqslant 48\\ \mu\leqslant 1}}\int_0^{T_\varepsilon}\bigl\|\bigl[\tilde{L}^\mu\partial^\alpha, h^{\gamma\delta}\partial_\gamma\partial_\delta\bigr]w_k(t,\cdot)\bigr\|_2\,dt\\
&\quad+\sum_{\substack{|\alpha|+\mu\leqslant 47\\ \mu\leqslant 1}}\int_0^{T_\varepsilon}\bigl\|L^\mu\partial^\alpha\bigl(\Box+a(x)\partial_t\bigr)w_k(t,\cdot)\bigr\|_2\,dt\\
&\quad+\sum_{\substack{|\alpha|+\mu\leqslant 47\\ \mu\leqslant 1}}\bigl\|L^\mu\partial^\alpha\bigl(\Box+a(x)\partial_t\bigr)w_k(T_\varepsilon,\cdot)\bigr\|_2+\sum_{\substack{|\alpha|+\mu\leqslant 46\\ \mu\leqslant 1}}\bigl\|L^\mu\partial^\alpha\bigl(\Box+a(x)\partial_t\bigr)w_k\bigr\|_{L^2_tL^2_x(S_{T_\varepsilon})}\\
&\quad+\sum_{|\alpha|\leqslant 48}\int_0^{T_\varepsilon}\bigl\|\partial^\alpha w_k'(t,\cdot)\bigr\|_{L^2(\{|x|<2\})}\,dt.
\end{aligned}
\tag{4.16}
$$

The first three terms in the right-hand side are bounded by[25]

$$
\varepsilon^2+\varepsilon M_{k-1}(T_\varepsilon)+\varepsilon M_k(T_\varepsilon)
$$

plus[26]

[25] Using (4.1).

[26] Here, we use $|x|\approx 2^j$ to indicate $2^{j-1}\leqslant|x|\leqslant 2^j$.

$$\int_0^{T_\varepsilon} \sum_{|\alpha|\leqslant 25} \big\|\partial^\alpha (v+w_{k-1})'(t,\cdot)\big\|_\infty \sum_{\substack{|\beta|+\mu\leqslant 48\\ \mu\leqslant 1}} \big\|L^\mu \partial^\beta (v+w_k)'(t,\cdot)\big\|_2\,dt$$

$$+\int_0^{T_\varepsilon} \sum_j \sum_{\substack{|\alpha|+\mu\leqslant 25\\ \mu\leqslant 1}} \big\|L^\mu \partial^\alpha (v+w_{k-1})'(t,\cdot)\big\|_{L^\infty(|x|\approx 2^j)} \sum_{|\beta|\leqslant 48} \big\|\partial^\beta (v+w_k)'(t,\cdot)\big\|_{L^2(|x|\approx 2^j)}\,dt, \tag{4.17}$$

corresponding terms where the roles of $k-1$ and k are reversed, and terms where k is replaced by $k-1$. In order to estimate the last term in (4.17), we can apply (3.1) to see that this term is

$$\lesssim \sum_{\substack{|\alpha|+\mu\leqslant 27\\ \mu\leqslant 1}} \big\|\langle x\rangle^{-1/2} L^\mu Z^\alpha (v+w_{k-1})'\big\|_{L_t^2 L_x^2(S_{T_\varepsilon})} \sum_{|\beta|\leqslant 48} \big\|\langle x\rangle^{-1/2} \partial^\alpha (v+w_k)'\big\|_{L_t^2 L_x^2(S_{T_\varepsilon})}.$$

Thus, it follows from (4.5) and (4.11) that the first three terms in the right-hand side of (4.16) are

$$\lesssim \log(2+T_\varepsilon)\big[\varepsilon^2+\varepsilon M_{k-1}(T_\varepsilon)+\varepsilon M_k(T_\varepsilon)+M_{k-1}(T_\varepsilon)M_k(T_\varepsilon)+\big(M_{k-1}(T_\varepsilon)\big)^2\big]. \tag{4.18}$$

The fourth and fifth terms in the right-hand side of (4.16), as above, can easily be shown to satisfy the same bound.

It remains to estimate the last term in (4.16). To this end, we apply (2.33) to see that it is

$$\lesssim \sum_{|\alpha|\leqslant 48} \int_0^{T_\varepsilon}\int_0^s \big\|\partial^\alpha\big(\Box + a(x)\partial_t\big)w_k(\tau,\cdot)\big\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\,ds.$$

Since

$$\begin{aligned}
&\sum_{|\alpha|\leqslant 48} \big|\partial^\alpha Q\big((u_0+v+w_{k-1})',(u_0+v+w_k)''\big)\big| \\
&\quad\lesssim \sum_{|\beta|\leqslant 24} \big|\partial^\beta (u_0+v+w_{k-1})'\big| \sum_{|\alpha|\leqslant 49} \big|\partial^\alpha (u_0+v+w_k)'\big| \\
&\qquad+ \sum_{|\beta|\leqslant 25} \big|\partial^\beta (u_0+v+w_k)'\big| \sum_{|\alpha|\leqslant 48} \big|\partial^\alpha (u_0+v+w_{k-1})'\big| \\
&\qquad+ \sum_{|\beta|\leqslant 24} \big|\partial^\beta (u_0+v+w_{k-1})'\big| \sum_{|\alpha|\leqslant 48} \big|\partial^\alpha (u_0+v+w_{k-1})'\big|,
\end{aligned}$$

we can apply (3.1) to see that

$$
\begin{aligned}
&\sum_{|\alpha|\leqslant 48} \big\| \partial^\alpha Q\big((u_0+v+w_{k-1})', (u_0+v+w_k)''\big)\big\|_{L^2(||x|-(s-\tau)|<10)} \\
&\quad\lesssim \sum_{|\beta|\leqslant 26} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_{k-1})'\big\|_{L^2(||x|-(s-\tau)|<20)} \\
&\qquad \times \sum_{|\alpha|\leqslant 49} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_k)'\big\|_{L^2(||x|-(s-\tau)|<20)} \\
&\qquad + \sum_{|\beta|\leqslant 27} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_k')\big\|_{L^2(||x|-(s-\tau)|<20)} \\
&\qquad \times \sum_{|\alpha|\leqslant 48} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_{k-1})'\big\|_{L^2(||x|-(s-\tau)|<20)} \\
&\qquad + \sum_{|\beta|\leqslant 26} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_{k-1})'\big\|_{L^2(||x|-(s-\tau)|<20)} \\
&\qquad \times \sum_{|\alpha|\leqslant 48} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_{k-1})'\big\|_{L^2(||x|-(s-\tau)|<20)}.
\end{aligned}
$$

Since $\{(\tau, x)\colon ||x| - (j-\tau)| < 20\}$, $j = 0, 1, 2, \ldots,$ have finite overlap, we conclude that the last term in (4.16) is controlled by

$$
\begin{aligned}
&\sum_{|\beta|\leqslant 26} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_{k-1})'\big\|_{L^2_t L^2_x(S_{T_\varepsilon})} \sum_{|\alpha|\leqslant 49} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_k)'\big\|_{L^2_t L^2_x(S_{T_\varepsilon})} \\
&\quad + \sum_{|\beta|\leqslant 27} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_k')\big\|_{L^2_t L^2_x(S_{T_\varepsilon})} \\
&\quad \times \sum_{|\alpha|\leqslant 48} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_{k-1})'\big\|_{L^2_t L^2_x(S_{T_\varepsilon})} \\
&\quad + \sum_{|\beta|\leqslant 26} \big\| \langle x\rangle^{-1/2} Z^\beta (u_0+v+w_{k-1})'\big\|_{L^2_t L^2_x(S_{T_\varepsilon})} \\
&\quad \times \sum_{|\alpha|\leqslant 48} \big\| \langle x\rangle^{-1/2} \partial^\alpha (u_0+v+w_{k-1})'\big\|_{L^2_t L^2_x(S_{T_\varepsilon})}.
\end{aligned}
$$

Using (4.1), (4.5), and (4.11), we see that this term is also controlled by (4.18), which yields

$$
\begin{aligned}
&I_{k,1}(T_\varepsilon) + III_{k,1}(T_\varepsilon) \\
&\quad \lesssim \log(2+T_\varepsilon)\big[\varepsilon^2 + \varepsilon M_{k-1}(T_\varepsilon) + \varepsilon M_k(T_\varepsilon) + M_{k-1}(T_\varepsilon) M_k(T_\varepsilon) + \big(M_{k-1}(T_\varepsilon)\big)^2\big]. \qquad (4.19)
\end{aligned}
$$

Arguing as in (4.15), this also yields

$$
\begin{aligned}
&II_{k,1}(T_\varepsilon) + IV_{k,1}(T_\varepsilon) \\
&\quad \lesssim \log(2+T_\varepsilon)\big[\varepsilon^2 + \varepsilon M_{k-1}(T_\varepsilon) + \varepsilon M_k(T_\varepsilon) + M_{k-1}(T_\varepsilon) M_k(T_\varepsilon) + \big(M_{k-1}(T_\varepsilon)\big)^2\big]. \qquad (4.20)
\end{aligned}
$$

It remains to study V_k. We seek to show

$$V_k(T_\varepsilon) \lesssim \log(2+T_\varepsilon)\bigl[\varepsilon^2 + \varepsilon M_{k-1}(T_\varepsilon) + \varepsilon M_k(T_\varepsilon) + M_{k-1}(T_\varepsilon)M_k(T_\varepsilon) + \bigl(M_{k-1}(T_\varepsilon)\bigr)^2\bigr]. \tag{4.21}$$

By (3.4), we have that

$$\begin{aligned} V_k(T_\varepsilon) \lesssim &\int_0^{T_\varepsilon}\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\alpha|+\mu\leqslant 32\\ \mu\leqslant 1}} \bigl|L^\mu Z^\alpha Q\bigl((u_0+v+w_{k-1})', (u_0+v+w_k)''\bigr)\bigr|\frac{dy\,ds}{|y|} \\ &+ \int_0^{T_\varepsilon} \sum_{\substack{|\alpha|+\mu\leqslant 29\\ \mu\leqslant 1}} \bigl\|L^\mu \partial^\alpha Q\bigl((u_0+v+w_{k-1})', (u_0+v+w_k)''\bigr)(s,\cdot)\bigr\|_2\,ds. \end{aligned} \tag{4.22}$$

Since the first term is clearly

$$\begin{aligned} \lesssim &\sum_{\substack{|\alpha|+\mu\leqslant 32\\ \mu\leqslant 1}} \bigl\|\langle x\rangle^{-1/2} L^\mu Z^\alpha (u_0+v+w_{k-1})'\bigr\|^2_{L^2_t L^2_x(S_{T_\varepsilon})} \\ &+ \sum_{\substack{|\alpha|+\mu\leqslant 33\\ \mu\leqslant 1}} \bigl\|\langle x\rangle^{-1/2} L^\mu Z^\alpha (u_0+v+w_k)'\bigr\|^2_{L^2_t L^2_x(S_{T_\varepsilon})}, \end{aligned}$$

the bound (4.21) follows easily from (4.1), (4.5), and (4.11). Using arguments similar to those used for $I_{k,1} + III_{k,1}$, it follows that the last term in (4.22) satisfies the same bound.

By combining (4.14), (4.15), (4.19), (4.20), and (4.21), we see that

$$M_k(T_\varepsilon) \lesssim \log(2+T_\varepsilon)\bigl[\varepsilon^2 + \varepsilon M_{k-1}(T_\varepsilon) + \varepsilon M_k(T_\varepsilon) + M_{k-1}(T_\varepsilon)M_k(T_\varepsilon) + \bigl(M_{k-1}(T_\varepsilon)\bigr)^2\bigr].$$

Using the inductive hypothesis and (1.6), (4.12) follows for κ sufficiently small.[27]

As similar arguments may be used to show that

$$A_k(T_\varepsilon) \leqslant (1/2)A_{k-1}(T_\varepsilon),$$

where

$$\begin{aligned} A_k(T) = \sum_{\mu=0}^{1} \sup_{0\leqslant t\leqslant T}\biggl(&\sum_{|\alpha|\leqslant 49-3\mu} \bigl\|L^\mu \partial^\alpha (w_k - w_{k-1})'(t,\cdot)\bigr\|_2 \\ &+ \sum_{|\alpha|\leqslant 47-3\mu} \bigl\|L^\mu Z^\alpha (w_k - w_{k-1})'(t,\cdot)\bigr\|_2\biggr) \end{aligned}$$

[27] But independent of ε and k.

$$+ \big(\log(2+T)\big)^{-1/2}\Bigg(\sum_{\mu=0}^{1}\sum_{|\alpha|\leqslant 48-3\mu}\big\|\langle x\rangle^{-1/2}L^{\mu}\partial^{\alpha}(w_k-w_{k-1})'\big\|_{L_t^2L_x^2(S_T)}$$
$$+\sum_{\mu=0}^{1}\sum_{|\alpha|\leqslant 46-3\mu}\big\|\langle x\rangle^{-1/2}L^{\mu}Z^{\alpha}(w_k-w_{k-1})'\big\|_{L_t^2L_x^2(S_T)}\Bigg)$$
$$+\sup_{0\leqslant t\leqslant T}(1+t)\sum_{|\alpha|\leqslant 25}\big\|Z^{\alpha}(w_k-w_{k-1})(t,\cdot)\big\|_{\infty}, \tag{4.23}$$

we see that the sequence is Cauchy and thus converges, which completes the proof of Theorem 1.1.

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