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GLOBAL EXISTENCE FOR HIGH DIMENSIONAL QUASILINEAR WAVE EQUATIONS EXTERIOR TO STAR-SHAPED OBSTACLES

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Abstract. We prove global existence for quasilinear wave equations in high dimensional exterior domains with Dirichlet boundary conditions. In particular, we permit the nonlinear term to depend on the solution, not just its first and second derivatives. The key estimates are variants on localized energy estimates.

1. **Introduction.** The purpose of this article is to study long time existence for high dimensional quasilinear wave equations exterior to star-shaped obstacles. In particular, we seek to prove exterior domain analogs of the four dimensional results of [5] where the nonlinearity is permitted to depend on the solution not just its first and second derivatives. Previous proofs in exterior domains omitted this dependence as it did not mesh well with the energy methods in use. The main estimates used in the proof are the variable coefficient localized energy estimate of [12] as well as a constant coefficient variant of this estimate which was developed in [1], [3], and [4].

Let us more specifically describe the problem at hand. We fix a bounded set $\mathcal{K}$ which has smooth boundary and is star-shaped with respect to the origin. Without loss of generality, we shall assume that $\mathcal{K} \subset \{|x| < 1\}$. We then seek to solve the following boundary value problem

$$\begin{cases} \Box u = Q(u, u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ u(t,\,\cdot\,)|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g \end{cases} \tag{1}$$

where $\Box = \partial_t^2 - \Delta$ is the d'Alembertian. Here and throughout, we use $u' = (\partial_t u, \nabla_x u)$ to denote the space-time gradient. The nonlinear term Q is smooth in its arguments and has the form

$$Q(u, u', u'') = A(u, u') + B^{\alpha\beta}(u, u')\partial_\alpha\partial_\beta u \tag{2}$$

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with the convenient notation $\partial_0 = \partial_t$. Throughout this paper, we shall utilize the summation convention where repeated indices are summed. Greek indices α, β, γ are summed from 0 to the spatial dimension n, while Latin indices i, j, k are implicitly summed from 1 to n. We shall reserve μ, ν, and σ for multiindices. Here A is taken to vanish to second order at $(u, u') = (0, 0)$, and B vanishes to first order at the origin. We also assume the symmetry condition

$$B^{\alpha\beta}(u, u') = B^{\beta\alpha}(u, u'), \quad 0 \le \alpha, \beta \le n. \tag{3}$$

To solve (1), one must assume that the data satisfy some compatibility conditions. These are well known, and we shall only tersely describe them. A more detailed exposition is available in, e.g., [7]. We write $J_k u = \{\partial_x^\mu u : 0 \le |\mu| \le k\}$. For any formal H^m solution u, we can write $\partial_t^k u(0, \cdot\,) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$ for some compatibility functions ψ_k. The compatibility condition of order m for data $(f, g) \in H^m \times H^{m-1}$ simply requires that ψ_k vanishes on $\partial\mathcal{K}$ for $0 \le k \le m-1$. For $(f, g) \in C^\infty$, we say that the data satisfy the compatibility condition to infinite order if the above holds for all m.

Under these assumptions, we have small data global existence when $n \ge 5$.

Theorem 1.1. *Let $\mathcal{K} \subset \mathbb{R}^n$, $n \ge 5$ be a smooth, bounded, star-shaped obstacle, and let $Q(u, u', u'')$ be as above. Assume further that $(f, g) \in C^\infty(\mathbb{R}^n \backslash \mathcal{K})$ vanish for $|x| > R$ for fixed R and satisfy the compatibility conditions to infinite order. Then there is a constant ε_0 and a positive integer N so that if $0 < \varepsilon < \varepsilon_0$ and*

$$\sum_{|\mu| \le N+1} \|\partial_x^\mu f\|_{L^2(\mathbb{R}^n \backslash \mathcal{K})} + \sum_{|\mu| \le N} \|\partial_x^\mu g\|_{L^2(\mathbb{R}^n \backslash \mathcal{K})} \le \varepsilon, \tag{4}$$

then (1) has a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^n \backslash \mathcal{K})$.

When the $Q(u, u', u'') = Q(u', u'')$, such global existence results have previously been established in [12, 13] for $n \ge 4$. The geometrical restrictions on $\mathcal{K}$ in [13] are much less strict. When $n \ge 7$, the proof in [12] can easily be adapted to prove the theorem. We shall not discuss the $n = 5, 6$ result further as it follows from easy modifications of the arguments of [1] or of those in the sequel.

With a general nonlinearity such as above, in dimension 4, one expects almost global existence as in Hörmander [5] for the boundaryless case, and this was indeed proved in [1]. Similarly, in three dimensions, based on the boundaryless results of Lindblad [10], one expects a lifespan $T_\varepsilon \sim \varepsilon^{-2}$, and this was proved for star-shaped obstacles in [2].

The proofs of [5] and [10] for long time existence in the boundaryless case show an improved lifespan in $n = 3, 4$ if the additional restriction

$$(\partial_u^2 A)(0, 0, 0) = 0 \tag{5}$$

is imposed. With this additional restriction, the lifespan bounds which are proved are comparable to those which were previously available when the nonlinearity was not permitted to depend on u. That is, in three dimensions, solutions exist almost globally, and in four dimensions, there is global existence.

The exterior domain analog of this four dimensional global existence is the primary result of this article.

Theorem 1.2. *Let $\mathcal{K} \subset \mathbb{R}^4$ be a smooth, bounded, star-shaped obstacle, and let $Q(u, u', u'')$ satisfy (5) in addition to the assumptions of Theorem 1.1. Assume further that $(f, g) \in C^\infty(\mathbb{R}^4 \backslash \mathcal{K})$ vanish for $|x| > R$ for fixed R and satisfy the* *compatibility conditions to infinite order. Then there is a constant* ε_0 *and a positive integer* N *so that if* $0 < \varepsilon < \varepsilon_0$ *and*

$$\sum_{|\mu|\le N+1} \|\partial_x^\mu f\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} + \sum_{|\mu|\le N} \|\partial_x^\mu g\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} \le \varepsilon, \tag{6}$$

then (1) *has a unique global solution* $u \in C^\infty([0,\infty)\times\mathbb{R}^4\backslash\mathcal{K})$.

In the sequel, we shall take, without loss, $R = 1$ in Theorems 1.1, 1.2.

While we have only stated Theorem 1.1 and Theorem 1.2 for scalar equations, it is not difficult to extend these results to systems and even multiple speed systems. We do not expect the restriction to star-shaped obstacles to be optimal but impose this largely for simplicity of exposition. One might fully expect similar results to hold in domains such as those addressed in [11], that is, any domain for which there is a sufficiently rapid decay of local energy.

We also note here that we only examine the case of Dirichlet boundary conditions. While Neumann boundary conditions were permitted in [4], they are more difficult to handle for quasilinear equations. First of all, even proving energy estimates for small perturbations of the d'Alembertian in the exterior domain requires additional assumptions. In particular, one either needs to assume a nonlinear compatibility condition which is akin to what appears in [14] or more generally take the boundary condition to regard the conormal derivative as in [9]. Moreover, with Dirichlet boundary conditions and a star shaped obstacle, the boundary terms which appear in the localized energy estimates (see Proposition 5 and [12]) have a favorable sign. This is no longer the case with Neumann boundary conditions. By developing techniques that would allow more general obstacle geometries, it is possible that one may also permit Neumann boundary conditions, but we do not explore that here.

In hopes of making the arguments more transparent, we shall truncate the nonlinearity at the quadratic level. Since we are dealing with small amplitude solutions, the higher order terms are better behaved, and it is clear how to alter the proofs in the sequel to permit these terms. With such a truncation, we may now focus on

$$\Box u = a^\alpha u\partial_\alpha u + b^{\alpha\beta}\partial_\alpha u\partial_\beta u + A^{\alpha\beta}u\partial_\alpha\partial_\beta u + B^{\alpha\beta\gamma}\partial_\alpha u\partial_\beta\partial_\gamma u \tag{7}$$

with Dirichlet boundary conditions and small data.

Our proof is based on two key estimates. The first is a localized energy estimate which, beginning with [6], has played a key role in nearly every proof of long time existence for wave equations in exterior domains. In one of its simplest forms, it states

$$\|\langle x\rangle^{-1/2-}w'\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} \lesssim \|w'(0,\,\cdot)\|_2 + \int_0^T \|\Box w(s,\,\cdot)\|_2\,ds \tag{8}$$

in $\mathbb{R}_+\times\mathbb{R}^n$, $n\ge 3$. We shall also utilize a version of this for perturbations of the d'Alembertian which is from [12].

The second of the key estimates is a variant on this. It can be thought of as a generalization of the main new estimate used in [1]. It is also the $p = 2$ version of the weighted Strichartz estimates of [3], [4]. In $\mathbb{R}_+\times\mathbb{R}^4$, this states that

$$\||x|^{-1/2-\gamma}w\|_{L^2_{t,x}} \lesssim \|w'(0,\,\cdot)\|_{\dot H^{\gamma-1}} + \||x|^{-1-\gamma}\Box w\|_{L^1_tL^1_rL^2_\omega}, \quad 0<\gamma<1/2. \tag{9}$$

It is the $\gamma = 0$ version of this estimate which was used in [1] to prove almost global existence in $\mathbb{R}^4\backslash\mathcal{K}$ when (5) is not assumed. When $\gamma = 0$, there is a logarithmic blow up in t in the estimate, and this logarithm corresponds precisely to the exponential in the lifespan bound.

2. **Main estimates on $\mathbb{R}_+ \times \mathbb{R}^4$.** In this section, we gather the main boundaryless estimates. These estimates, for the most part, are not new. In the sequel, we shall apply a cutoff which vanishes near the boundary to the solution. What results solves a boundaryless wave equation to which these estimates may be applied.

The estimates which we explore here will be for linear wave equations in $\mathbb{R}_+ \times \mathbb{R}^4$. We let w solve

$$\begin{cases} \Box w = F, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^4, \\ w(0,\,\cdot\,) = w_0, \quad \partial_t w(0,\,\cdot\,) = w_1. \end{cases} \tag{10}$$

The first of these estimates is a localized energy estimate. This has become an increasingly standard tool in the study of nonlinear wave equations. In the next section, we shall also present a version of this estimate which holds for perturbations of the d'Alembertian

Proposition 1. *Let w be a smooth solution to (10) which vanishes for large $|x|$ for each t. Then, for any $T > 0$, we have*

$$\|\langle x\rangle^{-1/2-} w'\|_{L^2_{t,x}([0,T]\times\mathbb{R}^4)} + \|\langle x\rangle^{-3/2} w\|_{L^2_{t,x}([0,T]\times\mathbb{R}^4)} \lesssim \|w'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box w(t,\,\cdot\,)\|_2 \, dt \tag{11}$$

with constant independent of T.

This proposition, in fact, holds in any dimension $n \geq 3$, though in $n = 3$ the second term in the left side has a logarithmic divergence in T. A proof can be found in [12], though this estimate did not originate there. The interested reader can see the references therein for a more complete history. To prove the estimate, one uses a positive commutator argument with multiplier $f(r)\partial_r w + \frac{3}{2}\frac{f(r)}{r} w$ where $f(r) = \frac{r}{r+R}$ to get the estimate in a torus with radii $\approx R$. Summing over such dyadic radii yields the proposition.

The second estimate is from [3] and [4].

Proposition 2. *Let w be a smooth solution to (10). Then, for $0 < \gamma < \frac{1}{2}$, we have*

$$\||x|^{-\frac{1}{2}-\gamma} w\|_{L^2_{t,x}(\mathbb{R}_+\times\mathbb{R}^4)} \lesssim \|w_0\|_{\dot H^\gamma(\mathbb{R}^4)} + \|w_1\|_{\dot H^{\gamma-1}(\mathbb{R}^4)} + \||x|^{-1-\gamma} F\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^4)}. \tag{12}$$

This proposition follows in the homogeneous case by interpolating a trace lemma (on a sphere) and a variant of the localized energy estimate which follows by applying Plancherel's theorem in the t-variable. The inhomogeneous estimate follows, in turn, by using the dual estimate to the trace lemma which was applied. This yields a much wider class of estimates, which were dubbed weighted Strichartz estimates, for which we have only stated the $p = 2$ case.

The $\gamma = 0$ variant of this estimate, which involves a logarithmic blow-up in T when applied on $[0,T] \times \mathbb{R}^4$, laid at the heart of the proof of the four dimensional almost global analog of Theorem 1.1 in [1]. Here one alters (11) by applying it to the Reisz tranforms of the solution. A weighted variant of Sobolev's lemma is applied to the resulting Sobolev norm with negative index which contains the forcing term.

To obtain a boundaryless wave equation, one applies a cutoff to the solution in the exterior domain. In order to handle the resulting commutator term, we shall require a variant of these estimates that permits the forcing term to be taken in L^2_t provided that it is compactly supported in the spatial variable, uniformly in t.

Proposition 3. *Let w be a smooth solution to (10) with vanishing data ($w_0 = w_1 = 0$). Suppose that $F(t,x) = 0$ for $|x| > 2$. Then,*

$$\|\langle x\rangle^{-1/2-} w\|_{L^2_{t,x}([0,T]\times\mathbb{R}^4)} \lesssim \|F\|_{L^2_{t,x}([0,T]\times\mathbb{R}^4)}. \tag{13}$$

This proposition is from [1] and in fact holds for $n \geq 3$. The techniques described below, however, only work for $n \geq 4$. The $n = 3$ case was presented in [2] using ideas from [6]. When w is replaced by w' in the left side, this estimate follows from the proof of (11) described above. Rather than applying the Schwarz inequality in x and bounding the multiplier term using the energy inequality, one instead applies Schwarz in t and x while appropriately introducing a weight so that the multiplier term can be bootstrapped into the left side of the estimate. In order to obtain (13), we apply this to $\partial_j(\Delta^{-1}\partial_j w)$. Applying $\Delta^{-1}\partial_j$ to F does not maintain the compact support, but the kernel is $O(|x-y|^{-3})$ which remains sufficient to absorb the weight which was introduced.

We will utilize one additional variant of the localized energy estimates. This one is obtained using techniques akin to those which appeared in [5] and [10].

Proposition 4. *Let v be a smooth solution to*

$$\begin{cases} \Box v = \sum_0^4 a_j\partial_j G, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^4,\\ v(0,\,\cdot\,) = \partial_t v(0,\,\cdot\,) = 0. \end{cases} \tag{14}$$

Then,

$$\|\langle x\rangle^{-1/2-\delta} v\|_{L^2_{t,x}([0,T]\times\mathbb{R}^4)} \lesssim \|G(0,\,\cdot\,)\|_{\dot H^{\delta-1}} + \int_0^T \|G(t,\,\cdot\,)\|_2\,dt \tag{15}$$

for $0<\delta<1/2$.

Proof. We let v_1 solve $\Box v_1 = G$ with vanishing data, and let v_0 solve the homogeneous equation $\Box v_0 = 0$ with $v_0(0,\,\cdot\,) = 0$, $\partial_t v_0(0,\,\cdot\,) = G(0,\,\cdot\,)$. Then,

$$v = \sum_0^4 a_j\partial_j v_1 - a_0 v_0.$$

Thus,

$$\|\langle x\rangle^{-1/2-\delta} v\|_{L^2_{t,x}} \lesssim \|\langle x\rangle^{-1/2-\delta} v_1'\|_{L^2_{t,x}} + \|\langle x\rangle^{-1/2-\delta} v_0\|_{L^2_{t,x}}.$$

For the first term in the right side, we simply apply (11), and to the second term we apply (12). ∎

We end this section with our principal source of decay. Here, as was initiated in [6], we use the localized energy estimates so as to obtain long time existence from decay in the $|x|$ variable rather than decay in the t variable, which is more standard in the boundaryless case but much more difficult to prove when there is a boundary. The decay that we obtain is based on the vector fields that generate translations and rotations. To that end, we set

$$\{\Omega\} = \{\Omega_{jk} = x_j\partial_k - x_k\partial_j\},\quad 1\leq j<k\leq 4$$

and

$$\{Z\} = \{\partial_\alpha, \Omega_{jk}\}, \quad 0 \le \alpha \le 4,\, 1 \le j < k \le 4.$$

Lemma 2.1. *For $h \in C^\infty(\mathbb{R}^4)$ and $R > 1$,*

$$\|h\|_{L^\infty(\{|x|\in[R,2R]\})} \lesssim R^{-3/2} \sum_{|\mu|\le 2, j\le 1} \|\Omega^\mu \nabla_r^j h\|_{L^2(\{|x|\in[R/2,4R]\})}. \tag{16}$$

This lemma is proved by apply Sobolev embeddings on $\mathbb{R}\times S^3$ after localizing to the annulus. The decay results from the difference in the volume element for $\mathbb{R}\times S^3$ versus that of $\mathbb{R}^4$ in polar coordinates. See [8].

3. **Main estimates on $\mathbb{R}_+ \times \mathbb{R}^4\backslash\mathcal{K}$.** The main estimate here is a localized energy estimate which holds for perturbations of the d'Alembertian. The technique of proof described in the previous section continues to hold. The boundary term that arises due to the obstacle, thanks to the star-shapedness assumption, has a favorable sign and can simply be dropped.

In order to handle the highest order terms of the quasilinear equation, it is beneficial to have an analog of the localized energy estimate for perturbations of the d'Alembertian. We suppose that $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^4\backslash\mathcal{K})$ solves

$$\begin{cases} \Box_h \phi = F, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^4\backslash\mathcal{K}, \\ \phi|_{\partial\mathcal{K}} = 0, \\ \phi(0,\,\cdot\,) = f, \quad \partial_t\phi(0,\,\cdot\,) = g \end{cases} \tag{17}$$

where f, g are smooth, supported in $|x| < 1$[1], and satisfy the smallness condition (6). Here

$$\Box_h \phi = (\partial_t^2 - \Delta)\phi + h^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta\phi,$$

and

$$h^{\alpha\beta}(t,x) = h^{\beta\alpha}(t,x) \tag{18}$$

as well as

$$|h| = \sum_{\alpha,\beta=0}^{4} |h^{\alpha\beta}(t,x)| \le \delta \ll 1. \tag{19}$$

We shall also utilize the notation

$$|\partial h| = \sum_{\alpha,\beta,\gamma=0}^{4} |\partial_\gamma h^{\alpha\beta}(t,x)|$$

and

$$S_T^{\mathcal{K}} = [0,T] \times \mathbb{R}^4\backslash\mathcal{K}.$$

For such a ϕ, we have the following estimates from [12].

Proposition 5. *Suppose that $\mathcal{K} \subset \{|x| < 1\}$ is a smooth, bounded, star-shaped obstacle as above. Let $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^4\backslash\mathcal{K})$ solve (17) with data satisfying (6) and the compatibility conditions. Suppose further that*

$$\sum_{|\mu|\le N} \|\partial^\mu \Box\phi(0,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} \le C\varepsilon.$$

[1] Recall that we are taking $R = 1$.

Suppose that $h^{\alpha\beta}$ satisfies (18) and (19) for a sufficiently small choice of δ. Then for any nonnegative integer N and for any $T > 0$,

$$\sum_{|\mu|\le N} \|\langle x\rangle^{-1/2-}\partial^\mu \phi'\|_{L^2(S_T^{\mathcal{K}})} + \sum_{|\mu|\le N} \|\langle x\rangle^{-3/2}\partial^\mu \phi\|_{L^2(S_T^{\mathcal{K}})} \\ + \sum_{|\mu|\le N} \|\partial^\mu \phi'(T,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} \lesssim \varepsilon + \sum_{j\le N}\int_0^T \|\Box_h \partial_t^j \phi(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\ + \sum_{j\le N}\int_0^T \Big\|\Big(|\partial h| + \frac{|h|}{r}\Big)|\nabla \partial_t^j \phi|\Big\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\ + \sum_{|\mu|\le N-1} \|\Box \partial^\mu \phi\|_{L^2(S_T^{\mathcal{K}})} + \sum_{|\mu|\le N-1} \|\Box\partial^\mu \phi(T,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}, \tag{20}$$

and

$$\sum_{|\mu|\le N} \|\langle x\rangle^{-1/2-} Z^\mu \phi'\|_{L^2(S_T^{\mathcal{K}})} + \sum_{|\mu|\le N} \|\langle x\rangle^{-3/2} Z^\mu \phi\|_{L^2(S_T^{\mathcal{K}})} \\ + \sum_{|\mu|\le N} \|Z^\mu \phi'(T,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} \lesssim \varepsilon + \sum_{|\mu|\le N}\int_0^T \|\Box_h Z^\mu \phi(t,\,\cdot\,)\|_2\,dt \\ + \sum_{|\mu|\le N}\int_0^T \Big\|\Big(|\partial h| + \frac{|h|}{r}\Big)|\nabla Z^\mu \phi|\Big\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\ + \sum_{|\mu|\le N+1} \|\partial_x^\mu \phi'\|_{L^2([0,T]\times\{|x|<1\})}. \tag{21}$$

In $n = 3$, there is logarithmic blow up associated to the second term in the left side of (20) and (21). In higher dimensions, this can be avoided. Indeed, there is a second term with favorable sign in (4.23) of [12] which permits one to recover the estimates as given.

As stated above, the description of the proof in the previous sections remains valid when $N = 0$. For the higher order cases, one uses the fact that ∂_t commutes with $\Box$ and preserves the Dirichlet boundary conditions. Then, by using elliptic regularity and relating the Laplacian to time derivatives via the equation, one can obtain (20). To prove (21), one argues as in the $N = 0$ case and applies a trace theorem to the resulting boundary terms. Here we note that the coefficients of Z are $O(1)$ on $\partial\mathcal{K}$. Also note that the last term in (21) is controlled by the left side of (20).

4. **Proof of Theorem 1.2.** With the estimates of the previous sections in hand, we now proceed to the proof of the main long time existence result.

We solve the nonlinear equation via an iteration. We set $u_0 \equiv 0$ and recursively define u_l to solve

$$\begin{cases} \Box u_l = a^\alpha u_{l-1}\partial_\alpha u_{l-1} + b^{\alpha\beta}\partial_\alpha u_{l-1}\partial_\beta u_{l-1} \\ \qquad\qquad + A^{\alpha\beta} u_{l-1}\partial_\alpha\partial_\beta u_l + B^{\alpha\beta\gamma}\partial_\alpha u_{l-1}\partial_\beta\partial_\gamma u_l \\ u_l|_{\partial\mathcal{K}} = 0 \\ u_l(0,\,\cdot\,) = f, \quad \partial_t u_l(0,\,\cdot\,) = g. \end{cases} \tag{22}$$

For any fixed $0 < \delta \le 1/8$, set

$$\begin{aligned} M_l(T) = \sum_{|\mu|\le 50} \Big[\sup_{t\in[0,T]} \|(\partial^\mu u_l)'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\setminus\mathcal{K})} + \|\langle x\rangle^{-1/2-\delta}(\partial^\mu u_l)'\|_{L^2(S_T^{\mathcal{K}})}\Big] \\ + \sum_{|\mu|\le 49} \Big[\|\langle x\rangle^{-1/2-\delta}(Z^\mu u_l)'\|_{L^2(S_T^{\mathcal{K}})} + \|\langle x\rangle^{-1/2-2\delta} Z^\mu u_l\|_{L^2([0,T]\times\{|x|>2\})} \\ + \sup_{t\in[0,T]} \|(Z^\mu u_l)'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\setminus\mathcal{K})}\Big] + \|u_l\|_{L^2([0,T]\times\{|x|<3\})}. \end{aligned} \tag{23}$$

Our first goal is to show that $M_l(T)$ is bounded uniformly in l and T.

We claim first that there is a C_0 so that $M_1(T) \le C_0\varepsilon$ for any $T > 0$. Such boundedness follows easily from Proposition 5 (with $h^{\alpha\beta} = 0$) and (6) for every term in $M_1(T)$ except for the fourth term. For this fourth term, we fix a smooth test function ρ which is identically 1 on $\{|x| < 1\}$ and vanishes outside of $\{|x| < 2\}$. Since $\mathcal{K}$ and the supports of the data are contained in $\{|x| < 1\}$, we then have that $(1-\rho)Z^\mu u_1$ solves the boundaryless wave equation

$$\Box(1-\rho)Z^\mu u_1 = [\Delta, \rho]Z^\mu u_1$$

with vanishing data. Since the commutator has compact support, we may apply (13). What results in (13) from the commutator is easily bounded using (20) as above.

We now inductively show that

$$M_l(T) \le 10C_0\varepsilon, \quad l = 2, 3, \dots.$$

We begin with bounding everything but the fourth term of $M_l(T)$ using Proposition 5. Here, we set

$$h^{\alpha\beta} = -A^{\alpha\beta}u_{l-1} - B^{\gamma\alpha\beta}\partial_\gamma u_{l-1}.$$

It then follows that terms I, II, III, V, and VI of (23) are

$$\begin{aligned} \le C_0\varepsilon + C\sum_{|\mu|\le 50}\int_0^T \|\partial^\mu \Box_h u_l(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\setminus\mathcal{K})}\,dt \\ + C\sum_{|\mu|\le 50}\int_0^T \|[\Box_h, \partial^\mu]u_l(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\setminus\mathcal{K})}\,dt \\ + C\sum_{\substack{|\mu|\le 50\\ |\sigma|\le 2}}\int_0^T \|(\partial^\sigma u_{l-1})(\partial^\mu u_l')\|_{L^2(\mathbb{R}^4\setminus\mathcal{K})}\,dt \end{aligned} \tag{24}$$

$$
\begin{aligned}
&+ C \sum_{|\mu|\le 49} \int_0^T \|Z^\mu \Box_h u_l(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\
&+ C \sum_{|\mu|\le 49} \int_0^T \|[\Box_h, Z^\mu]u_l(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\
&+ C \sum_{\substack{|\mu|\le 49\\ |\sigma|\le 2}} \int_0^T \|(Z^\sigma u_{l-1})(Z^\mu u_l')\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt \\
&+ C \sum_{|\mu|\le 49} \Big[\sup_{t\in[0,T]} \|\partial^\mu \Box u_l(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} + \|\partial^\mu \Box u_l\|_{L^2(S_T^{\mathcal{K}})}\Big].
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{|\mu|\le 50} \Big(|\partial^\mu \Box_h u_l| + |[\partial^\mu, \Box_h]u_l|\Big) &\lesssim \sum_{|\mu|\le 26} |\partial^\mu u_{l-1}| \sum_{|\nu|\le 50} |\partial^\nu u_l'| \\
&+ \sum_{|\mu|\le 27} |\partial^\mu u_l| \sum_{|\nu|\le 50} |\partial^\nu u_{l-1}'| + \sum_{|\mu|\le 26} |\partial^\mu u_{l-1}| \sum_{|\nu|\le 50} |\partial^\nu u_{l-1}'|
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\sum_{|\mu|\le 49} \Big(|Z^\mu \Box_h u_l| + |[Z^\mu, \Box_h]u_l|\Big) &\lesssim \sum_{|\mu|\le 26} |Z^\mu u_{l-1}| \sum_{|\nu|\le 49} |Z^\nu u_l'| \\
&+ \sum_{|\mu|\le 27} |Z^\mu u_l| \sum_{\substack{|\nu|\le 49\\ |\sigma|\le 1}} |Z^\nu \partial^\sigma u_{l-1}| + \sum_{|\mu|\le 26} |Z^\mu u_{l-1}| \sum_{\substack{|\nu|\le 49\\ |\sigma|\le 1}} |Z^\nu \partial^\sigma u_{l-1}|,
\end{aligned}
$$

we may apply (16) and the Schwarz inequality to control the second through seventh terms in (24) by

$$
\begin{aligned}
&\sum_{|\mu|\le 29} \|\langle x\rangle^{-3/4} Z^\mu u_{l-1}\|_{L^2(S_T^{\mathcal{K}})} \\
&\qquad\times \Big(\sum_{|\nu|\le 50} \|\langle x\rangle^{-3/4} \partial^\nu u_l'\|_{L^2(S_T^{\mathcal{K}})} + \sum_{|\nu|\le 49} \|\langle x\rangle^{-3/4} Z^\nu u_l'\|_{L^2(S_T^{\mathcal{K}})}\Big) \\
&+ \sum_{|\mu|\le 30} \|\langle x\rangle^{-3/4} Z^\mu u_l\|_{L^2(S_T^{\mathcal{K}})} \\
&\qquad\times \Big(\sum_{|\nu|\le 50} \|\langle x\rangle^{-3/4} \partial^\nu u_{l-1}'\|_{L^2(S_T^{\mathcal{K}})} + \sum_{\substack{|\nu|\le 49\\ |\sigma|\le 1}} \|\langle x\rangle^{-3/4} Z^\nu \partial^\sigma u_{l-1}\|_{L^2(S_T^{\mathcal{K}})}\Big) \\
&+ \sum_{|\mu|\le 29} \|\langle x\rangle^{-3/4} Z^\mu u_{l-1}\|_{L^2(S_T^{\mathcal{K}})} \\
&\qquad\times \Big(\sum_{|\nu|\le 50} \|\langle x\rangle^{-3/4} \partial^\nu u_{l-1}'\|_{L^2(S_T^{\mathcal{K}})} + \sum_{\substack{|\nu|\le 49\\ |\sigma|\le 1}} \|\langle x\rangle^{-3/4} Z^\nu \partial^\sigma u_{l-1}\|_{L^2(S_T^{\mathcal{K}})}\Big).
\end{aligned}
$$

The last two terms of (24) are controlled similarly, though more simply. Here, for the pointwise in time terms, it is worth noting the following variant of (16).

Proposition 6. *If $h \in C_0^\infty(\mathbb{R}^4)$ and $R > 0$, then*

$$R\|h\|_{L^\infty(\{R/2<|x|<R\})} \lesssim \sum_{|\mu|\le 2} \|\nabla_x \Omega^\mu h\|_{L^2(\mathbb{R}^4)}.$$

See, e.g., [1, Lemma 2.1]. It is also necessary to invoke a Hardy inequality. These details are left to the reader.

With this, we have seen that terms I, II, III, V, and VI of (23) are

$$\le C_0\varepsilon + CM_{l-1}(T)M_l(T) + C(M_{l-1}(T))^2.$$

It remains only to bound term IV of (23). For this, for a solution u_l to (22), we examine $(1-\rho)Z^\mu u_l$ where $|\mu| \le 49$ and ρ is a smooth cutoff function as above. We have that $(1-\rho)Z^\mu u_l$ solves the boundaryless wave equation

$$\begin{aligned}\Box(1-\rho)Z^\mu u_l = [\Delta,\rho]Z^\mu u_l &+ (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} a^\alpha_{\mu\nu\sigma}\partial_\alpha(Z^\nu u_{l-1}Z^\sigma u_{l-1})\\ &+ (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} b^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_{l-1}\\ &+ (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} A^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha(Z^\sigma u_{l-1}\partial_\beta Z^\nu u_l)\\ &+ (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} \tilde{A}^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_l\\ &+ (1-\rho)\sum_{\substack{|\sigma|+|\nu|\le|\mu|\\|\sigma|>0}} B^{\alpha\beta\gamma}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta\partial_\gamma Z^\nu u_l\\ &+ (1-\rho)B^{\alpha\beta\gamma}_{\mu\mu0}\partial_\beta(\partial_\alpha u_{l-1}\partial_\gamma Z^\mu u_l) + (1-\rho)\tilde{B}^{\alpha\beta\gamma}_{\mu\mu0}\partial_\alpha\partial_\beta u_{l-1}\partial_\gamma Z^\mu u_l.\end{aligned}$$

for appropriate coefficients $a^\alpha_{\mu\nu\sigma}$, $b^{\alpha\beta}_{\mu\nu\sigma}$, $A^{\alpha\beta}_{\mu\nu\sigma}$, etc. based on those in (7). Here we have used the fact that $[\partial, Z]$ is in the span of $\{\partial\}$, and the new coefficients depend on these commutators as well as the appropriate binomial coefficients. We further decompose the right side into

$$\begin{aligned}&[\Delta,\rho]Z^\mu u_l + \sum_{|\sigma|+|\nu|\le|\mu|} a^\alpha_{\mu\nu\sigma}\partial_\alpha((1-\rho)Z^\mu u_{l-1}Z^\sigma u_{l-1})\\ &\quad+ \sum_{|\sigma|+|\nu|\le|\mu|} A^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha((1-\rho)Z^\sigma u_{l-1}\partial_\beta Z^\nu u_l) + B^{\alpha\beta\gamma}_{\mu\mu0}\partial_\beta((1-\rho)\partial_\alpha u_{l-1}\partial_\gamma Z^\mu u_l)\\ &+(1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} b^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_{l-1} + (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} \tilde{A}^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_l\\ &\quad+ (1-\rho)\sum_{\substack{|\sigma|+|\nu|\le|\mu|\\|\sigma|>0}} B^{\alpha\beta\gamma}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta\partial_\gamma Z^\nu u_l + (1-\rho)\tilde{B}^{\alpha\beta\gamma}_{\mu\mu0}\partial_\alpha\partial_\beta u_{l-1}\partial_\gamma Z^\mu u_l\\ &\quad+ \sum_{|\sigma|+|\nu|\le|\mu|} C^\alpha_{\mu\nu\sigma}(\partial_\alpha\rho)Z^\nu u_{l-1}Z^\sigma u_{l-1} + \sum_{\substack{|\sigma|+|\nu|\le|\mu|\\ \delta\le 1}} \tilde{C}^{\alpha\beta\gamma}_{\mu\nu\sigma\delta}(\partial_\alpha\rho)\partial^\delta_\beta Z^\sigma u_{l-1}\partial_\gamma Z^\nu u_l.\end{aligned}$$

We write $(1-\rho)Z^\mu u_l = v_1 + v_2 + v_3$ where v_1 solves $\Box v_1 = [\Delta, \rho]Z^\mu u_l$ and

$$\Box v_2 = \sum_{|\sigma|+|\nu|\le|\mu|} a^{\alpha}_{\mu\nu\sigma}\partial_\alpha((1-\rho)Z^\nu u_{l-1}Z^\sigma u_{l-1})$$
$$+ \sum_{|\sigma|+|\nu|\le|\mu|} A^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha((1-\rho)Z^\sigma u_{l-1}\partial_\beta Z^\mu u_l) + B^{\alpha\beta\gamma}_{\mu\mu 0}\partial_\beta((1-\rho)\partial_\alpha u_{l-1}\partial_\gamma Z^\nu u_l).$$

Both v_1 and v_2 are taken to have vanishing initial data. It then follows that v_3 solves

$$\Box v_3 = (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} b^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_{l-1}$$
$$+ (1-\rho)\sum_{|\sigma|+|\nu|\le|\mu|} \tilde{A}^{\alpha\beta}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta Z^\nu u_l$$
$$+ (1-\rho)\sum_{\substack{|\sigma|+|\nu|\le|\mu| \\ |\sigma|>0}} B^{\alpha\beta\gamma}_{\mu\nu\sigma}\partial_\alpha Z^\sigma u_{l-1}\partial_\beta\partial_\gamma Z^\nu u_l + (1-\rho)\tilde{B}^{\alpha\beta\gamma}_{\mu\mu 0}\partial_\alpha\partial_\beta u_{l-1}\partial_\gamma Z^\mu u_l$$
$$+ \sum_{|\sigma|+|\nu|\le|\mu|} C^{\alpha}_{\mu\nu\sigma}(\partial_\alpha\rho)Z^\nu u_{l-1}Z^\sigma u_{l-1} + \sum_{\substack{|\sigma|+|\nu|\le|\mu| \\ \delta\le 1}} \tilde{C}^{\alpha\beta\gamma}_{\mu\nu\sigma\delta}(\partial_\alpha\rho)\partial^\delta_\beta Z^\sigma u_{l-1}\partial_\gamma Z^\nu u_l$$

with vanishing Cauchy data.

To bound

$$\|\langle x\rangle^{-1/2-2\delta}v_1\|_{L^2([0,T]\times\mathbb{R}^4)},$$

we apply (13). It then remains to bound

$$\|Z^\mu u_l\|_{L^2([0,T]\times\{1<|x|<2\})} + \|(Z^\mu u_l)'\|_{L^2([0,T]\times\{1<|x|<2\})}$$

which we have done in the previous step.

To establish a bound for v_2 in the same space, we may apply (15). We have

$$\|\langle x\rangle^{-1/2-2\delta}v_2\|_{L^2([0,T]\times\mathbb{R}^4)} \le C\sum_{|\nu|\le 49,|\sigma|\le 25}\int_0^T \|Z^\sigma u_{l-1}Z^\nu u_{l-1}\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt$$
$$+ C\sum_{|\nu|\le 49,|\sigma|\le 25}\int_0^T \|Z^\sigma u_{l-1}\partial Z^\nu u_l\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt$$
$$+ C\sum_{|\nu|\le 49,|\sigma|\le 25}\int_0^T \|Z^\sigma\partial u_l Z^\nu u_{l-1}\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\,dt.$$

For each of the latter three terms, we apply (16) to the lower order term on each dyadic annulus. By applying the Schwarz inequality and summing over the dyadic intervals, we see that this is

$$\le C(M_{l-1}(T))^2 + CM_{l-1}(T)M_l(T)$$

provided $\delta \le 1/8$.

For v_3, we use (12) to obtain

$$
\begin{aligned}
&\|\langle x\rangle^{-1/2-2\delta} v_3\|_{L^2([0,T]\times\mathbb{R}^4)} \\
&\qquad \le C \sum_{|\nu|\le 49, |\sigma|\le 25} \|\langle x\rangle^{-1-2\delta}(\partial Z^\sigma u_{l-1})(\partial Z^\nu u_{l-1})\|_{L^1_t L^1_r L^2_\omega} \\
&\qquad + C \sum_{|\nu|\le 49, |\sigma|\le 25} \|\langle x\rangle^{-1-2\delta}(\partial Z^\sigma u_{l-1})(\partial Z^\nu u_l)\|_{L^1_t L^1_r L^2_\omega} \\
&\qquad + C \sum_{|\nu|\le 49, |\sigma|\le 25} \|\langle x\rangle^{-1-2\delta}(\partial Z^\sigma u_l)(\partial Z^\nu u_{l-1})\|_{L^1_t L^1_r L^2_\omega} \\
&\qquad + C \sum_{|\nu|\le 49, |\sigma|\le 25} \|Z^\sigma u_{l-1} Z^\nu u_{l-1}\|_{L^1_t L^1_r L^2_\omega([0,T]\times\{1<|x|<2\})} \\
&\qquad + C \sum_{|\nu|\le 49, |\sigma|\le 26} \|Z^\sigma u_{l-1} \partial Z^\nu u_l\|_{L^1_t L^1_r L^2_\omega([0,T]\times\{1<|x|<2\})} \\
&\qquad\qquad + C \sum_{\substack{|\nu|\le 49, |\sigma|\le 26 \\ \delta\le 1}} \|Z^\sigma u_l \partial^\delta Z^\nu u_{l-1}\|_{L^1_t L^1_r L^2_\omega([0,T]\times\{1<|x|<2\})}.
\end{aligned}
$$

By applying Sobolev embeddings on S^3 to the lower order term in each of the last six terms and utilizing the Schwarz inequality, we again have that this is controlled by

$$C(M_{l-1}(T))^2 + CM_{l-1}(T)M_l(T).$$

By combining the bounds for each of these pieces, we see that

$$M_l(T) \le C_0\varepsilon + C(M_{l-1}(T))^2 + CM_{l-1}(T)M_l(T).$$

If we apply the inductive hypothesis $M_{l-1}(T) \le 10C_0\varepsilon$, it indeed follows that

$$M_l(T) \le 10C_0\varepsilon$$

provided that ε is sufficiently small.

It remains to show that $\{u_l\}$ is a Cauchy sequence in similar spaces. To this end, we set

$$
\begin{aligned}
A_l(T) = &\sum_{|\mu|\le 49} \Big[\sup_{t\in[0,T]} \|(\partial^\mu(u_l - u_{l-1}))'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})} \\
&\qquad\qquad + \|\langle x\rangle^{-1/2-\delta}(\partial^\mu(u_l-u_{l-1}))'\|_{L^2(S_T^{\mathcal{K}})}\Big] \\
&+ \sum_{|\mu|\le 48} \Big[\|\langle x\rangle^{-1/2-\delta}(Z^\mu(u_l-u_{l-1}))'\|_{L^2(S_T^{\mathcal{K}})} \\
&\qquad\qquad + \|\langle x\rangle^{-1/2-2\delta} Z^\mu(u_l-u_{l-1})\|_{L^2([0,T]\times\{|x|>2\})} \\
&\qquad\qquad + \sup_{t\in[0,T]} \|(Z^\mu(u_l-u_{l-1}))'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^4\backslash\mathcal{K})}\Big] \\
&+ \|u_l - u_{l-1}\|_{L^2([0,T]\times\{|x|<3\})}.
\end{aligned} \tag{25}
$$

Using quite similar arguments and the $O(\varepsilon)$ bound on $M_l(T)$, one can prove

$$A_l(T) \le \frac{1}{2}A_{l-1}(T).$$

This suffices to show that the sequence converges. Its limit is indeed the solution u, which completes the proof.

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