
Decay Estimates for Variable Coefficient Wave Equations in Exterior Domains

Jason Metcalfe¹ and Daniel Tataru^{2*}

¹ Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
metcalfe@email.unc.edu

² Mathematics Department, University of California, Berkeley, CA 94720-3840, USA
tataru@math.berkeley.edu

Summary. In this article we consider variable coefficient, time-dependent wave equations in exterior domains $\mathbb{R} \times (\mathbb{R}^n \setminus \Omega)$, $n \geq 3$. We prove localized energy estimates if Ω is star-shaped, and global in time Strichartz estimates if Ω is strictly convex.

2000 AMS Subject Classification: Primary: 35L20, Secondary: 34L25

Key words: Wave equation, local energy decay, exterior domains

1 Introduction

Our goal, in this article, is to prove analogs of the well-known Strichartz estimates and localized energy estimates for variable coefficient wave equations in exterior domains. We consider long-range perturbations of the flat metric, and we take the obstacle to be star-shaped. The localized energy estimates are obtained under a smallness assumption for the long-range perturbation. Global-in-time Strichartz estimates are then proved assuming the local-in-time Strichartz estimates, which are known to hold for strictly convex obstacles.

For the constant coefficient wave equation $\square = \partial_t^2 - \Delta$ in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, we have that solutions to the Cauchy problem

$$\square u = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (1.1)$$

*The work of the first author was supported in part by NSF grant DMS0800678. The work of the second author was supported in part by NSF grants DMS0354539 and DMS0301122.

satisfy the Strichartz estimates¹

$$\| |D_x|^{-\rho_1} \nabla u \|_{L^{p_1} L^{q_1}} \lesssim \| \nabla u(0) \|_{L^2} + \| |D_x|^{\rho_2} \square u \|_{L^{p_2'} L^{q_2'}},$$

for Strichartz admissible exponents (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) . Here, exponents (ρ, p, q) are called Strichartz admissible if $2 \leq p, q \leq \infty$,

$$\rho = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \quad \frac{2}{p} \leq \frac{n-1}{2} \left(1 - \frac{2}{q}\right),$$

and $(\rho, p, q) \neq (1, 2, \infty)$ when $n = 3$.

The Strichartz estimates follow via a TT^* argument and the Hardy–Littlewood–Sobolev inequality from the dispersive estimates,

$$\| |D_x|^{-\frac{n+1}{2}(1-\frac{2}{q})} \nabla u(t) \|_{L^q} \lesssim t^{-\frac{n-1}{2}(1-\frac{2}{q})} \| u_1 \|_{L^{q'}}, \quad 2 \leq q < \infty$$

for solutions to (1.1) with $u_0 = 0$, $f = 0$. This in turn is obtained by interpolating between an $L^2 \rightarrow L^2$ energy estimate and an $L^1 \rightarrow L^\infty$ dispersive bound which provides $O(t^{-(n-1)/2})$ type decay. Estimates of this form originated in the work [25], and as stated are the culmination of several subsequent works. The endpoint estimate $(p, q) = \left(2, \frac{2(n-1)}{n-3}\right)$ was most recently obtained in [8], and we refer the interested reader to the references therein for a more complete history.

The second estimate which shall be explored is the localized energy estimate, a version of which states

$$\begin{aligned} \sup_j \| \langle x \rangle^{-1/2} \nabla u \|_{L^2(\mathbb{R} \times \{|x| \in [2^{j-1}, 2^j]\})} \\ \lesssim \| \nabla u(0) \|_{L^2} + \sum_k \| \langle x \rangle^{1/2} \square u \|_{L^2(\mathbb{R} \times \{|x| \in [2^{k-1}, 2^k]\})} \end{aligned} \quad (1.2)$$

in the constant coefficient case. These estimates can be proved using a positive commutator argument with a multiplier which is roughly of the form $f(r)\partial_r$ when $n \geq 3$ and are quite akin to the bounds found in, e.g., [16], [24], [9], [20], [7], and [23]. See also [1], [12], [13] for certain estimates for small perturbations of the d'Alembertian.

Variants of these estimates for constant coefficient wave equations are also known in exterior domains. Here, u is replaced by a solution to

$$\square u = F, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \Omega$$

where Ω is a bounded set with smooth boundary. The localized energy estimates have played a key role in proving a number of long time existence results for nonlinear wave equations in exterior domains. See, e.g., [7] and

¹Here and throughout, we shall use ∇ to denote a space-time gradient unless otherwise specified with subscripts.

[11, 12] for their proof and application. Here, it is convenient to assume that the obstacle Ω is star-shaped, though certain estimates are known (see e.g. [11], [3]) in more general settings. Exterior to star-shaped obstacles, the estimates for small perturbations of \square continue to hold (see [11]). This, however, only works for $n \geq 3$, and the bound which results is not strong enough in order to prove the Strichartz estimates which we desire. As such, we shall, in the sequel, couple this bound with certain frequency localized versions of the estimate in order to prove the Strichartz estimates. For time-independent perturbations, one may permit more general geometries. See, e.g., [3].

Certain global-in-time Strichartz estimates are also known in exterior domains, but, except for certain very special cases (see [4], [2], which are closely based on [21]), require that the obstacle be strictly convex. Local-in-time estimates were shown in [19] for convex obstacles, and using these estimates, global estimates were constructed in [20] for n odd and [3] and [14] for general n . See also [6].

In the present article, we explore variable coefficient cases of these estimates. Here, \square is replaced by the second order hyperbolic operator

$$P(t, x, D) = D_i a^{ij}(t, x) D_j + b^i(t, x) D_i + c(t, x),$$

where $D_0 = D_t$ is understood. We assume that (a^{ij}) has signature $(n, 1)$ and that $a^{00} < 0$, i.e., that time slices are space-like. We shall then consider the initial value boundary value problem

$$Pu = f, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \Omega. \quad (1.3)$$

When $\Omega = \emptyset$ and $b^i \equiv c \equiv 0$, the problem of proving Strichartz estimates is understood locally, and of course, localized energy estimates are trivial locally-in-time. For smooth coefficients, Strichartz estimates were first proved in [15] using Fourier integral operators. Using a wave packet decomposition, Strichartz estimates were obtained in [17] for $C^{1,1}$ coefficients in spatial dimensions $n = 2, 3$. Using instead an approach based on the FBI transform, these estimates were extended to all dimensions in [27, 28, 29]. For rougher coefficients, the Strichartz estimates as stated above are lost (see [18], [22]) and only certain estimates with losses are available [28, 29]. When the boundary is nonempty, far less is known, and we can only refer to the results of [19] for smooth time independent coefficients, $b^i \equiv c \equiv 0$, and Ω strictly geodesically convex. The proof of these estimates is quite involved and uses a Melrose–Taylor parametrix to approximate the reflected solution.

For the boundaryless problem, global-in-time localized energy estimates and Strichartz estimates were recently shown in [13] for small, C^2 , long-range perturbations. The former follow from a positive commutator argument with a multiplier which is akin to what we present in the sequel. For the latter, an outgoing parametrix is constructed using a time-dependent FBI transform in a fashion which is reminiscent to that of the preceding work [26] on Schrödinger

equations. Upon conjugating the half-wave equation by the FBI transform, one obtains a degenerate parabolic equation due to a nontrivial second-order term in the asymptotic expansion. Here, the bounds from [26], which are based on the maximum principle, may be cited. The errors in this parametrix construction are small in the localized energy spaces, which again are similar to those below, and it is shown that the global Strichartz estimates follow from the localized energy estimates.

The aim of the present article is to combine the approach of [13] with analogs of those from [20], [3], and [14] to show that global-in-time Strichartz estimates in exterior domains follow from the localized energy estimates and local-in-time Strichartz estimates for the boundary value problem. As we shall show the localized energy estimates for small perturbations outside of star-shaped obstacles, the global Strichartz estimates shall then follow for convex obstacles from the estimates of [19].

Let us now more precisely describe our assumptions. We shall look at certain long-range perturbations of Minkowski space. To state this, we set

$$D_0 = \{|x| \leq 2\}, \quad D_j = \{2^j \leq |x| \leq 2^{j+1}\}, \quad j = 1, 2, \dots$$

and

$$A_j = \mathbb{R} \times D_j, \quad A_{<j} = \mathbb{R} \times \{|x| \leq 2^j\}.$$

We shall then assume that

$$\sum_{j \in \mathbb{N}} \sup_{A_j \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega)} \langle x \rangle^2 |\nabla^2 a(t, x)| + \langle x \rangle |\nabla a(t, x)| + |a(t, x) - I_n| \leq \varepsilon \quad (1.4)$$

and, for the lower order terms,

$$\sum_{j \in \mathbb{N}} \sup_{A_j \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega)} \langle x \rangle^2 |\nabla b(t, x)| + \langle x \rangle |b(t, x)| \leq \varepsilon, \quad (1.5)$$

$$\sum_{j \in \mathbb{N}} \sup_{A_j \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega)} \langle x \rangle^2 |c(t, x)| \leq \varepsilon. \quad (1.6)$$

If ε is small enough, then (1.4) precludes the existence of trapped rays, while for arbitrary ε it restricts the trapped rays to finitely many dyadic regions.

We now define the localized energy spaces that we shall use. We begin with an initial choice which is convenient for the local energy estimates but not so much for the Strichartz estimates. Precisely, we define the localized energy space LE_0 as

$$\|\varphi\|_{LE_0} = \sup_{j \geq 0} \left(2^{-j/2} \|\nabla \varphi\|_{L^2(A_j \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega))} + 2^{-3j/2} \|\varphi\|_{L^2(A_j \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega))} \right),$$

while for the forcing term we set

$$\|f\|_{LE_0^*} = \sum_{k \geq 0} 2^{k/2} \|f\|_{L^2(A_k \cap (\mathbb{R} \times \mathbb{R}^n \setminus \Omega))}.$$

The local energy bounds in these spaces shall follow from the arguments in [12].

On the other hand, for the Strichartz estimates, we shall introduce frequency localized spaces as in [13], as well as the earlier work [26]. We use a Littlewood–Paley decomposition in frequency,

$$1 = \sum_{k=-\infty}^{\infty} S_k(D), \quad \text{supp } s_k(\xi) \subset \{2^{k-1} < |\xi| < 2^{k+1}\}$$

and for each $k \in \mathbb{Z}$, we use

$$\|\varphi\|_{X_k} = 2^{-k^-/2} \|\varphi\|_{L^2(A_{<k^-})} + \sup_{j \geq k^-} \||x|^{-1/2} \varphi\|_{L^2(A_j)}$$

to measure functions of frequency 2^k . Here $k^- = \frac{|k|-k}{2}$. We then define the global norm

$$\|\varphi\|_X^2 = \sum_{k=-\infty}^{\infty} \|S_k \varphi\|_{X_k}^2.$$

Then for the local energy norm we use

$$\|\varphi\|_{LE_\infty}^2 = \|\nabla \varphi\|_X^2.$$

For the inhomogeneous term we introduce the dual space $Y = X'$ with norm defined by

$$\|f\|_Y^2 = \sum_{k=-\infty}^{\infty} \|S_k f\|_{X'_k}^2.$$

To relate these spaces to the LE_0 respectively LE_0^* we use Hardy-type inequalities which are summarized in the following proposition:

Proposition 1.1 *We have*

$$\sup_j \||x|^{-1/2} u\|_{L^2(A_j)} \lesssim \|u\|_X \tag{1.7}$$

and

$$\|u\|_Y \lesssim \sum_j \||x|^{1/2} u\|_{L^2(A_j)}. \tag{1.8}$$

In addition,

$$\||x|^{-3/2} \varphi\|_{L^2} \lesssim \|\nabla_x \varphi\|_X, \quad n \geq 4. \tag{1.9}$$

The first bound (1.7) is a variant of a Hardy inequality, see [13, (16), Lemma 1], and also [26]. The second (1.8) is its dual. The bound (1.9), proved in [13, Lemma 1], fails in dimension three.

Now we turn our attention to the obstacle problem. For R fixed so that $\Omega \subset \{|x| < R\}$, we select a smooth cutoff χ with $\chi \equiv 1$ for $|x| < 2R$ and $\text{supp } \chi \subset \{|x| < 4R\}$. We shall use χ to partition the analysis into a portion near the obstacle and a portion away from the obstacle. In particular, we define the localized energy space $LE \subset LE_0$ as

$$\|\varphi\|_{LE}^2 = \|\varphi\|_{LE_0}^2 + \|(1 - \chi)\varphi\|_{LE_\infty}^2.$$

For the forcing term, we will respectively construct $LE^* \supset LE_0^*$ by

$$\|f\|_{LE^*}^2 = \|\chi f\|_{LE_0^*}^2 + \|(1 - \chi)f\|_Y^2, \quad n \geq 4.$$

This choice is no longer appropriate in dimension $n = 3$, as otherwise the local L^2 control of the solution is lost. Instead we simply set

$$\|f\|_{LE^*}^2 = \|f\|_{LE_0^*}^2, \quad n = 3.$$

Using these space, we now define what it means for a solution to satisfy our stronger localized energy estimates.

Definition 1.1 *We say that the operator P satisfies the localized energy estimates if for each initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ and each inhomogeneous term $f \in LE^*$, there exists a unique solution u to (1.3) with $u \in LE$ which satisfies the bound*

$$\|u\|_{LE} + \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial\Omega)} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{LE^*}. \tag{1.10}$$

We prove that the localized energy estimates hold under the assumption that P is a small perturbation of the d'Alembertian:

Theorem 1.1 *Let Ω be a star-shaped domain. Assume that the coefficients a^{ij} , b^i , and c satisfy (1.4), (1.5), and (1.6) with an ε which is sufficiently small. Then the operator P satisfies the localized energy estimates globally-in-time for $n \geq 3$.*

These results correspond to the $s = 0$ results of [13]. Some more general results are also available by permitting $s \neq 0$, but for simplicity we shall not provide these details.

Once we have the local energy estimates, the next step is to prove the Strichartz estimates. To do so, we shall assume that the corresponding Strichartz estimate holds locally-in-time.

Definition 1.2 For a given operator P and domain Ω , we say that the local Strichartz estimate holds if

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}([0,1] \times \mathbb{R}^n \setminus \Omega)} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p'_2} L^{q'_2}([0,1] \times \mathbb{R}^n \setminus \Omega)} \tag{1.11}$$

for any solution u to (1.3).

As mentioned previously, (1.11) is only known under some fairly restrictive hypotheses. We show a conditional result which says that the global-in-time Strichartz estimates follow from the local-in-time estimates as well as the localized energy estimates.

Theorem 1.2 Let Ω be a domain such that P satisfies both the localized energy estimates and the local Strichartz estimate. Let a^{ij}, b^i, c satisfy (1.4), (1.5), and (1.6). Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be two Strichartz pairs. Then the solution u to (1.3) satisfies

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p'_2} L^{q'_2}}. \tag{1.12}$$

Notice that this conditional result does not require the ε in (1.4), (1.5), and (1.6) to be small. We do, however, require this for our proof of the localized energy estimates which are assumed in Theorem 1.2.

As an example of an immediate corollary of the localized energy estimates of Theorem 1.1 and the local Strichartz estimates of [19], we have:

Corollary 1.1 Let $n \geq 3$, and let Ω be a strictly convex domain. Assume that the coefficients a^{ij}, b^i , and c are time-independent in a neighborhood of Ω and satisfy (1.4), (1.5), and (1.6) with an ε which is sufficiently small. Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be two Strichartz pairs which satisfy

$$\frac{1}{p_1} = \left(\frac{n-1}{2}\right)\left(\frac{1}{2} - \frac{1}{q_1}\right), \quad \frac{1}{p'_2} = \left(\frac{n-1}{2}\right)\left(\frac{1}{2} - \frac{1}{q'_2}\right).$$

Then the solution u to (1.3) satisfies

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p'_2} L^{q'_2}}. \tag{1.13}$$

This paper is organized as follows. In the next section, we prove the localized energy estimates for small perturbations of the d'Alembertian exterior to a star-shaped obstacle. In the last section, we prove Theorem 1.2 which says that global-in-time Strichartz estimates follow from the localized energy estimates as well as the local Strichartz estimates.

2 The localized energy estimates

In this section, we shall prove Theorem 1.1.

By combining the inclusions $LE \subset LE_0$, $LE_0^* \subset LE^*$ and the bounds (1.9), (1.5), and (1.6), one can easily prove the following which permits us to treat the lower order terms perturbatively. See also [13, Lemma 3].

Proposition 2.1 *Let b^i , c be as in (1.5) and (1.6), respectively. Then,*

$$\|b\nabla u\|_{LE^*} \lesssim \varepsilon \|u\|_{LE}, \tag{2.1}$$

$$\|cu\|_{LE^*} \lesssim \varepsilon \|u\|_{LE}. \tag{2.2}$$

We now look at the proof of the localized energy estimates. Due to Proposition 2.1 we can assume that $b = 0, c = 0$. To prove the theorems, we use positive commutator arguments. We first do the analysis separately in the two regions.

2.1 Analysis near Ω and classical Morawetz-type estimates

Here we sketch the proof from [12] which gives an estimate which is similar to (1.2) for small perturbations of the d'Alembertian. This estimate shall allow us to gain control of the solution near the boundary. It also permits local L^2 control of the solution, not just the gradient in three dimensions. The latter is necessary as the required Hardy inequality which can be utilized in higher dimensions corresponds to a false endpoint estimate in three dimensions.

The main estimate is the following:

Proposition 2.2 *Let Ω be a star-shaped domain. Assume that the coefficients a^{ij} , b^i , and c satisfy (1.4), (1.5), and (1.6), respectively, with an ε which is sufficiently small. Suppose that φ satisfies $P\varphi = F$, $\varphi|_{\partial\Omega} = 0$. Then*

$$\|\varphi\|_{LE_0} + \|\nabla\varphi\|_{L^\infty L^2} + \|\partial_\nu\varphi\|_{L^2(\partial\Omega)} \lesssim \|\nabla\varphi(0)\|_2 + \|F\|_{LE_0^*}. \tag{2.3}$$

Proof We provide only a terse proof. The interested reader can refer to [12] for a more detailed proof. For $f = \frac{r}{r+\rho}$, where ρ is a fixed positive constant, we use a multiplier of the form

$$\partial_t\varphi + f(r)\partial_r\varphi + \frac{n-1}{2}\frac{f(r)}{r}\varphi.$$

By multiplying $P\varphi$ and integrating by parts, one obtains

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{2} f'(r) (\partial_r \varphi)^2 + \left(\frac{f(r)}{r} - \frac{1}{2} f'(r) \right) |\nabla \varphi|^2 \\
& \quad + \frac{1}{2} f'(r) (\partial_t \varphi)^2 - \frac{n-1}{4} \Delta \left(\frac{f(r)}{r} \right) \varphi^2 \, dx dt \\
& \quad - \frac{1}{2} \int_0^T \int_{\partial \Omega} \frac{f(r)}{r} (\partial_\nu \varphi)^2 \langle x, \nu \rangle (a^{ij} \nu_i \nu_j) \, d\sigma dt + (1 + O(\varepsilon)) \|\nabla \varphi(T)\|_2^2 \\
& \lesssim \|\nabla \varphi(0)\|_2^2 + \int_0^T \int_{\mathbb{R}^n \setminus \Omega} |F| \left(|\partial_t \varphi| + |f(r) \partial_r \varphi| + \left| \frac{f(r)}{r} \varphi \right| \right) \, dx \, dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n \setminus \Omega} O \left(\frac{|a-I|}{r} + |\nabla a| \right) |\nabla \varphi| \left(|\nabla \varphi| + \left| \frac{\varphi}{r} \right| \right) \, dx \, dt \\
& \lesssim \|\nabla \varphi(0)\|_2^2 + \|F\|_{LE_0^*(0,T)} \|\varphi\|_{LE_0(0,T)} + \varepsilon \|\varphi\|_{LE_0(0,T)}^2.
\end{aligned} \tag{2.4}$$

Here, we have used the Hardy inequality $\||x|^{-1} \varphi\|_2 \lesssim \|\nabla \varphi\|_2$, $n \geq 3$, as well as (1.4).

All terms on the left are nonnegative. By direct computation, the first term controls

$$\rho^{-1} \|\nabla \varphi\|_{L^2([0,T] \times \{|x| \approx \rho\})}^2 + \rho^{-3} \|\varphi\|_{L^2([0,T] \times \{|x| \approx \rho\})}^2.$$

Taking a supremum over dyadic ρ provides a bound for the $\|\varphi\|_{LE_0(0,T)}$. In the second term we have $-\langle x, \nu \rangle \gtrsim 1$, which follows from the assumption that Ω is star-shaped, and also $a^{ij} \nu_i \nu_j \gtrsim 1$ which follows from (1.4). By simply taking $\rho = 1$, one can bound the third term on the left of (2.3) by the right side of (2.4). Thus we obtain

$$\begin{aligned}
& \|\varphi\|_{LE_0(0,T)} + \|\nabla \varphi(T)\|_{L^\infty L^2} + \|\partial_\nu \varphi\|_{L^2(\partial \Omega)} \\
& \lesssim \|\nabla \varphi(0)\|_2^2 + \|F\|_{LE_0^*(0,T)} \|\varphi\|_{LE_0(0,T)} + \varepsilon \|\varphi\|_{LE_0(0,T)}^2.
\end{aligned}$$

The LE_0 terms on the right can be bootstrapped for ε small which yields (2.3).

2.2 Analysis near ∞ and frequency localized estimates

In this section, we briefly sketch the proof from [13] for some frequency localized versions of the localized energy estimates for the boundaryless equation. The main estimate here, which is from [13], is the following.

Proposition 2.3 *Suppose that a^{ij} are as in Theorem 1.1 and $b = 0$, $c = 0$. Then for each initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ and each inhomogeneous term $f \in Y \cap L^1 L^2$, there exists a unique solution u to the boundaryless equation*

$$Pu = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

satisfying

$$\|\nabla u\|_{L^\infty L^2 \cap X} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{L^1 L^2 + Y}. \tag{2.5}$$

The proof here uses a multiplier of the form

$$D_t + \delta_0 Q + i\delta_1 B.$$

Here the parameters are chosen so that

$$\varepsilon \ll \delta_1 \ll \delta \ll \delta_0 \ll 1.$$

The multiplier Q is given by

$$Q = \sum_k S_k Q_k S_k$$

where Q_k are differential operators of the form

$$Q_k = (D_x x \varphi_k(|x|) + \varphi_k(|x|) x D_x).$$

The φ_k are functions of the form

$$\varphi_k(x) = 2^{-k^-} \psi_k(2^{-k^-} \delta x)$$

where for each k the functions ψ_k have the following properties:

- (i) $\psi_k(s) \approx (1+s)^{-1}$ for $s > 0$ and $|\partial^j \psi_k(s)| \lesssim (1+s)^{-j-1}$ for $j \leq 4$,
- (ii) $\psi_k(s) + s\psi'_k(s) \approx (1+s)^{-1} \alpha_k(s)$ for $s > 0$,
- (iii) $\psi_k(|x|)$ is localized at frequency $\ll 1$.

The α_k are slowly varying functions that are related to the bounds of the individual summands in (1.4). This construction is reminiscent of those in [26], [10], and [13].

For the Lagrangian term B , we fix a function b satisfying

$$b(s) \approx \frac{\alpha(s)}{1+s}, \quad |b'(s)| \ll b(s).$$

Then, we set $B = \sum_k S_k 2^{-k^-} b(2^{-k^-} x) S_k$.

The computations, which are carried out in detail in [13], are akin to those outlined in the previous section.

2.3 Proof of Theorem 1.1

Consider first the three-dimensional case. For $f \in LE^* = LE_0^*$ we can use Proposition 2.2 to obtain

$$\|u\|_{LE_0} + \|\nabla u\|_{L^\infty L^2} + \|\partial_\nu u\|_{L^2(\partial\Omega)} \lesssim \|\nabla u(0)\|_2 + \|f\|_{LE_0^*}.$$

It remains to estimate $\|(1 - \chi)u\|_{LE_\infty}$ with χ as in the definition of LE . By (2.5) we have

$$\begin{aligned} \|(1 - \chi)u\|_{LE_\infty} &\lesssim \|\nabla(1 - \chi)u(0)\|_{L^2} + \|P[(1 - \chi)u]\|_Y \\ &\lesssim \|\nabla u(0)\|_{L^2} + \|P[(1 - \chi)u]\|_{LE_0^*}. \end{aligned}$$

Finally, to bound the last term we write

$$P[(1 - \chi)u] = -[P, \chi]u + (1 - \chi)f.$$

The commutator has compact spatial support; therefore

$$\|P[(1 - \chi)u]\|_{LE_0^*} \lesssim \|u\|_{LE_0} + \|f\|_{LE_0^*}$$

and the proof is concluded.

Consider now higher dimensions $n \geq 4$. For fixed $f \in LE^*$, we first solve the boundaryless problem

$$Pu_\infty = (1 - \chi)f \in Y, \quad u_\infty(0) = 0, \quad \partial_t u_\infty(0) = 0$$

using Proposition 2.3. We consider χ_∞ which is identically 1 in a neighborhood of infinity and vanishes on $\text{supp } \chi$. For the function $\chi_\infty u_\infty$ we use the Hardy inequalities in Proposition 1.1 to write

$$\|\chi_\infty u_\infty\|_{LE} \approx \|\nabla(\chi_\infty u_\infty)\|_X \lesssim \|\nabla u_\infty\|_X \lesssim \|(1 - \chi_\infty)f\|_Y.$$

The remaining part $\psi = u - \psi_\infty u_\infty$ solves

$$P\psi = \chi_\infty f + [P, \chi_\infty]u_\infty;$$

therefore

$$\|P\psi\|_{LE_0^*} \lesssim \|f\|_{LE^*} + \|u_\infty\|_{LE_0} \lesssim \|f\|_{LE^*} + \|\nabla u_\infty\|_X \lesssim \|f\|_{LE^*}.$$

Then we estimate ψ as in the three dimensional case. The proof is concluded.

3 The Strichartz estimates

In this final section, we prove Theorem 1.2, the global Strichartz estimates. We use fairly standard arguments to accomplish this. In a compact region about the obstacle, we prove the global estimates using the local Strichartz estimates and the localized energy estimates. Near infinity, we use [13]. The two regions can then be glued together using the localized energy estimates.

We shall utilize the following two propositions. The first gives the result when the forcing term is in the dual localized energy space.

Proposition 3.1 *Let (ρ, p, q) be a Strichartz pair. Let Ω be a domain such that P satisfies both the localized energy estimates and the homogeneous local Strichartz estimate with exponents (ρ, p, q) . Then for each $\varphi \in LE$ with $P\varphi \in LE^*$, we have*

$$\| |D_x|^{-\rho} \nabla \varphi \|_{L^p L^q}^2 \lesssim \| \nabla \varphi(0) \|_{L^2}^2 + \| \varphi \|_{LE}^2 + \| P\varphi \|_{LE^*}^2. \tag{3.1}$$

The second proposition allows us to gain control when the forcing term is in a dual Strichartz space.

Proposition 3.2 *Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be Strichartz pairs. Let Ω be a domain such that P satisfies both the localized energy estimates and the local Strichartz estimate with exponents (ρ_1, p_1, q_1) , (ρ_2, p_2, q_2) . Then there is a parametrix K for P with*

$$\| \nabla K f \|_{L^\infty L^2}^2 + \| K f \|_{LE}^2 + \| |D_x|^{-\rho_1} \nabla K f \|_{L^{p_1} L^{q_1}}^2 \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}^2 \tag{3.2}$$

and

$$\| PK f - f \|_{LE^*} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}. \tag{3.3}$$

We briefly delay the proofs and first apply the propositions to prove Theorem 1.2.

Proof (Proof of Theorem 1.2) For

$$Pu = f + g, \quad f \in |D_x|^{-\rho_2} L^{p'_2} L^{q'_2}, \quad g \in LE^*,$$

we write

$$u = K f + v.$$

The bound for $\nabla K f$ follows immediately from (3.2).

To bound v , we note that

$$Pv = (1 - PK)f + g.$$

Applying (3.1) and the localized energy estimate, we have

$$\| |D_x|^{-\rho_1} \nabla v \|_{L^{p_1} L^{q_1}} \lesssim \| \nabla u(0) \|_{L^2} + \| \nabla K f \|_{L^\infty L^2} + \| (1 - PK)f \|_{LE^*} + \| g \|_{LE^*}.$$

The Strichartz estimates (1.12) then follow from (3.2) and (3.3).

Proof (Proof of Proposition 3.1) We assume $P\varphi \in Y$, and we write

$$\varphi = \chi\varphi + (1 - \chi)\varphi$$

with χ as in the definition of the LE norm. Since, using (1.8), the fundamental theorem of calculus, and (1.7), we have

$$\|[P, \chi]\varphi\|_{LE^*} \lesssim \|\varphi\|_{LE},$$

it suffices to show the estimate for $\varphi_1 = \chi\varphi$, $\varphi_2 = (1 - \chi)\varphi$ separately.

To show (3.1) for φ_1 , we need only assume that φ_1 and $P\varphi_1$ are compactly supported, and we write

$$\varphi_1 = \sum_{j \in \mathbb{Z}} \beta(t - j)\varphi_1$$

for an appropriately chosen, smooth, compactly supported function β . By commuting P and $\beta(t - j)$, we easily obtain

$$\sum_{j \in \mathbb{N}} \|\beta(t - j)\varphi_1\|_{LE}^2 + \|P(\beta(t - j)\varphi_1)\|_{L^1 L^2}^2 \lesssim \|\varphi_1\|_{LE}^2 + \|P\varphi_1\|_{LE^*}^2.$$

Here, as above, we have also used (1.8), the fundamental theorem of calculus, and (1.7). Applying the homogeneous local Strichartz estimate to each piece $\beta(t - j)\varphi_1$ and using Duhamel’s formula, the bound (3.1) for φ_1 follows immediately from the square summability above.

On the other hand, φ_2 solves a boundaryless equation, and the estimate (3.1) is just a restatement of [13, Theorem 7] with $s = 0$. This follows directly when $n \geq 4$ and easily from (1.8) when $n = 3$.

Proof (Proof of Proposition 3.2) We split f in a fashion similar to the above:

$$f = \chi f + (1 - \chi)f = f_1 + f_2.$$

For f_1 , we write

$$f_1 = \sum_j \beta(t - j)f_1$$

where β is supported in $[-1, 1]$. Let ψ_j be the solution to

$$P\psi_j = \beta(t - j)f_1.$$

By the local Strichartz estimate, we have

$$\||D_x|^{-\rho_1} \nabla \psi_j\|_{L^{p_1} L^{q_1}(E_j)} + \|\nabla \psi_j\|_{L^\infty L^2(E_j)} \lesssim \|\beta(t - j)|D_x|^{\rho_2} f_1\|_{L^{p'_2} L^{q'_2}}$$

where $E_j = [j - 2, j + 2] \times (\{|x| < 2\} \cap \mathbb{R}^n \setminus \Omega)$. Letting $\tilde{\beta}(t - j, r)$ be a cutoff which is supported in E_j and is identically one on the support of $\beta(t - j)\chi$, set $\varphi_j = \tilde{\beta}(t - j, r)\psi_j$. Then,

$$\| |D_x|^{-\rho_1} \nabla \varphi_j \|_{L^{p_1} L^{q_1}} + \| \nabla \varphi_j \|_{L^\infty L^2} \lesssim \| \beta(t-j) |D_x|^{\rho_2} f_1 \|_{L^{p'_2} L^{q'_2}}. \tag{3.4}$$

Moreover,

$$P\varphi_j - \beta(t-j)f_1 = [P, \tilde{\beta}(t-j, r)]\psi_j,$$

and thus,

$$\| P\varphi_j - \beta(t-j)f_1 \|_{L^2} \lesssim \| \beta(t-j) |D_x|^{-\rho_2} f_1 \|_{L^{p'_2} L^{q'_2}}. \tag{3.5}$$

Setting

$$Kf_1 = \sum_j \varphi_j$$

and summing the bounds (3.4) and (3.5) yields the desired result for f_1 .

For f_2 , we solve the boundaryless equation

$$P\psi = f_2.$$

For a second cutoff $\tilde{\chi}$ which is 1 on the support of $1 - \chi$ and vanishes for $\{r < R\}$, we set

$$Kf_2 = \tilde{\chi}\psi.$$

The following lemma, which is in essence from [13, Theorem 6], applied to ψ then easily yields the desired bounds.

Lemma 3.1 *Let $f \in |D_x|^{-\rho_2} L^{p'_2} L^{q'_2}$. Then the forward solution ψ to the boundaryless equation $P\psi = f$ satisfies the bound*

$$\| \nabla \psi \|_{L^\infty L^2}^2 + \| \psi \|_{LE}^2 + \| |D_x|^{-\rho_1} \nabla \psi \|_{L^{p_1} L^{q_1}}^2 \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}^2. \tag{3.6}$$

It remains to prove the lemma. From [13, Theorem 6], we have that

$$\| \nabla \psi \|_X^2 + \| |D_x|^{-\rho_1} \nabla \psi \|_{L^{p_1} L^{q_1}}^2 \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}^2. \tag{3.7}$$

By (1.7) we have

$$\sup_{j \geq 0} 2^{-j/2} \| \nabla \psi \|_{L^2(A_j)} \lesssim \| \nabla \psi \|_X.$$

It remains only to show the uniform bound

$$2^{-\frac{3j}{2}} \| \psi \|_{L^2(A_j)} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}} \tag{3.8}$$

when $n = 3$. Let $H(t, s)$ be the forward fundamental solution to P . Then

$$\psi(t) = \int_{-\infty}^t H(t, s) f(s) ds.$$

Therefore (3.8) can be rewritten as

$$2^{-\frac{3j}{2}} \left\| \int_{-\infty}^t H(t, s) f(s) ds \right\|_{L^2(A_j)} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}.$$

Since $p'_2 < 2$ for Strichartz pairs in $n = 3$, by the Christ-Kiselev lemma [5] (see also [20]) it suffices to show that

$$2^{-\frac{3j}{2}} \left\| \int_{-\infty}^{\infty} H(t, s) f(s) ds \right\|_{L^2(A_j)} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}. \quad (3.9)$$

The function

$$\psi_1(t) = \int_{-\infty}^{\infty} H(t, s) f(s) ds$$

solves $P\psi_1 = 0$, and from (3.7) we have

$$\|\nabla \psi_1\|_{L^\infty L^2} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p'_2} L^{q'_2}}.$$

On the other hand, from (2.3) with $P\psi_1 = 0$ and $\Omega = \emptyset$, we obtain

$$2^{-\frac{3j}{2}} \|\psi_1\|_{L^2(A_j)} \lesssim \|\nabla \psi_1(0)\|_2^2.$$

Hence (3.9) follows, and the proof is concluded.

References

1. SERGE ALINHAC, *On the Morawetz-Keel-Smith-Sogge inequality for the wave equation on a curved background*, Publ. Res. Inst. Math. Sci. **42**, no. 3 (2006), 705–720.
2. MATTHEW BLAIR, HART F. SMITH AND CHRISTOPHER D. SOGGE, *Strichartz estimates for the wave equation on manifolds with boundary*, preprint.
3. N. BURQ, *Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: “Global Strichartz estimates for nontrapping perturbations of the Laplacian”* [Commun. Partial Differential Equations **25**, no. 11-12 (2000), 2171–2183; MR1789924 (2001j:35180)], Commun. Partial Differential Equations **28**, no. 9-10 (2003), 1675–1683.
4. NICOLAS BURQ, GILLES LEBEAU AND FABRICE PLANCHON, *Global existence for energy critical waves in 3-d domains*, J. Amer. Math. Soc. **21** (2008), 831–845.
5. MICHAEL CHRIST AND ALEXANDER KISELEV, *Maximal functions associated to filtrations*, J. Funct. Anal. **179**, no. 2 (2001), 409–425.
6. KUNIO HIDANO, JASON METCALFE, HART F. SMITH, CHRISTOPHER D. SOGGE AND YI ZHOU, *On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles*, preprint.
7. MARKUS KEEL, HART F. SMITH AND CHRISTOPHER D. SOGGE, *Almost global existence for some semilinear wave equations*, J. Anal. Math. **87** (2002), 265–279. Dedicated to the memory of Thomas H. Wolff.
8. MARKUS KEEL AND TERENCE TAO, *Endpoint Strichartz estimates*, Amer. J. Math. **120**, no. 5 (1998), 955–980.
9. CARLOS E. KENIG, GUSTAVO PONCE AND LUIS VEGA, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal. **127**, no. 1 (1995), 204–234.

10. JEREMY MARZUOLA, JASON METCALFE AND DANIEL TATARU, *Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations*, J. Funct. Anal. **255** (2008), 1497–1553.
11. JASON METCALFE AND CHRISTOPHER D. SOGGE, *Hyperbolic trapped rays and global existence of quasilinear wave equations*, Invent. Math. **159**, no. 1 (2005), 75–117.
12. JASON METCALFE AND CHRISTOPHER D. SOGGE, *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal. **38**, no. 1 (2006), 188–209 (electronic).
13. JASON METCALFE AND DANIEL TATARU, *Global parametrices and dispersive estimates for variable coefficient wave equations*, preprint.
14. JASON L. METCALFE, *Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle*, Trans. Amer. Math. Soc. **356**, no. 12 (2004), 4839–4855 (electronic).
15. GERD MOCKENHAUPT, ANDREAS SEEGER AND CHRISTOPHER D. SOGGE, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. **6**, no. 1 (1993), 65–130.
16. CATHLEEN S. MORAWETZ, *Time decay for the nonlinear Klein-Gordon equations*, Proc. R. Soc. London Ser. A **306** (1968), 291–296.
17. HART F. SMITH, *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Ann. Inst. Fourier (Grenoble) **48**, no. 3 (1998), 797–835.
18. HART F. SMITH AND CHRISTOPHER D. SOGGE, *On Strichartz and eigenfunction estimates for low regularity metrics*, Math. Res. Lett. **1**, no. 6 (1994), 729–737.
19. HART F. SMITH AND CHRISTOPHER D. SOGGE, *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. **8**, no. 4 (1995), 879–916.
20. HART F. SMITH AND CHRISTOPHER D. SOGGE, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Commun. Partial Differential Equations **25**, no. 11-12 (2000), 2171–2183.
21. HART F. SMITH AND CHRISTOPHER D. SOGGE, *On the L^p norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198**, no. 1 (2007), 107–153.
22. HART F. SMITH AND DANIEL TATARU, *Sharp counterexamples for Strichartz estimates for low regularity metrics*, Math. Res. Lett. **9**, no. 2-3 (2002), 199–204.
23. JACOB STERBENZ, *Angular regularity and Strichartz estimates for the wave equation*, Int. Math. Res. Not. no. 4 (2005), 187–231. With an appendix by Igor Rodnianski.
24. WALTER A. STRAUSS, *Dispersal of waves vanishing on the boundary of an exterior domain*, Commun. Pure Appl. Math. **28** (1975), 265–278.
25. ROBERT S. STRICHARTZ, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44**, no. 3 (1977), 705–714.
26. DANIEL TATARU, *Parametrices and dispersive estimates for Schrödinger operators with variable coefficients*, Amer. J. Math. **130** (2008), 571–634.
27. DANIEL TATARU, *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, Amer. J. Math. **122**, no. 2 (2000), 349–376.
28. DANIEL TATARU, *Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. II*, Amer. J. Math. **123**, no. 3 (2001), 385–423.
29. DANIEL TATARU, *Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III*, J. Amer. Math. Soc. **15**, no. 2 (2002), 419–442 (electronic).