

Strichartz estimates on Schwarzschild space-times

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Two of the more robust ways of measuring dispersion for solutions to the wave equation are the localized energy estimates and the Strichartz estimates. For the wave equation on Minkowski space-time $\mathbb{R} \times \mathbb{R}^n$, these say, respectively, that solutions u to $\square u = (\partial_t^2 - \Delta)u = 0$ satisfy

$$(1) \quad \sup_j \|\langle x \rangle^{-1/2} \nabla u\|_{L_{t,x}^2(\mathbb{R} \times \{|x| \approx 2^j\})} \lesssim \|\nabla u(0, \cdot)\|_2, \quad n \geq 3$$

and

$$(2) \quad \||D|^{-\rho} \nabla u\|_{L_t^p L_x^q(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|\nabla u(0, \cdot)\|_2.$$

For the latter, we require that (p, q, ρ) be Strichartz admissible which we define to mean that $2 \leq p, q \leq \infty$,

$$(3) \quad \rho = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \quad \frac{2}{p} \leq \frac{n-1}{2} \left(1 - \frac{2}{q}\right),$$

and $(p, q, \rho) \neq (2, \infty, 1)$ when $n = 3$. We say that the Strichartz exponents are sharp if equality holds in the second equation in (3).

For simplicity of exposition, we shall only explore estimates for solutions to homogeneous equations here. In the flat case, there are also well known estimates for the inhomogeneous equation where the forcing term is in a dual space. The estimates below also have similar analogs.

We wish to explore to what extent these estimates carry over to variable coefficient settings and, in particular, to Schwarzschild space-times. While Strichartz estimates for variable coefficient wave equations are known to hold locally in time, there is still relatively little known concerning global estimates.

An outgoing paramatrix was constructed in [3] which allows one to roughly say
 global-in-time localized energy \implies global-in-time Strichartz.

This proof is based on the earlier construction [5] for Schrödinger equations. After reducing to a half-wave equation, we conjugate by a time-dependent FBI transform. A second order term in the asymptotic expansion is nontrivial, and we are left with a degenerate parabolic equation. The bounds from [5], which are based on the maximum principle, can then be referenced. Moreover, if the perturbation is small, the necessary frequency-localized versions of the localized energy estimates are shown using a positive commutator argument, which in turn yields the Strichartz estimates.

As an example of an asymptotically flat perturbation which is not small, we examine the wave equation on Schwarzschild space-times. The Schwarzschild space-time is the simplest solution to Einstein's equations in vacuum which contains a black hole. We restrict our analysis to the exterior of the black hole

$(t, r, \omega) \in \mathbb{R} \times (2M, \infty) \times S^2$, and the line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2.$$

A key obstacle, which is not encountered for small perturbations, is the existence of trapped rays. For the Schwarzschild space-time, trapping occurs at the event horizon $r = 2M$ and the photon sphere $r = 3M$.

Solutions to the homogeneous wave equation $\square_g \phi = 0$, where

$$\square_g = -\left(1 - \frac{2M}{r}\right)^{-1} \partial_t^2 + r^{-2} \partial_r \left(1 - \frac{2M}{r}\right) r^2 \partial_r + r^{-2} \Delta_{S^2},$$

have a conserved energy

$$E[\phi](t) = \int \left[\left(1 - \frac{2M}{r}\right)^{-1} (\partial_t \phi)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r \phi)^2 + |\nabla \phi|^2 \right] dx,$$

where ∇ denotes the angular derivatives. In analogy to (1), we prove

(4)

$$\int_{t>0} \int \left[c_r^0 \left(1 - \frac{2M}{r}\right) (\partial_r \phi)^2 + c_t^0 \left(1 - \frac{2M}{r}\right)^{-1} (\partial_t \phi)^2 + c_\omega^0 |\nabla \phi|^2 + c^0 \phi^2 \right] dx dt \lesssim E[\phi](0),$$

where $dx = r^2 dr d\omega$. Here

$$c_r^0 = \frac{1}{r^2 \left(1 - \ln \left(1 - \frac{2M}{r}\right)\right)^2}, \quad c_t^0 = \frac{\left(1 - \frac{3M}{r}\right)^2}{r^2 \left(1 - \ln \left(1 - \frac{2M}{r}\right)\right)^2}$$

$$c_\omega^0 = \frac{1}{r} \left(1 - \frac{3M}{r}\right)^2, \quad c^0 = \frac{1}{r^4 \left(1 - \frac{2M}{r}\right) \left(1 - \ln \left(1 - \frac{2M}{r}\right)\right)^2}.$$

Estimates of this form have been previously shown in [1] and [2]. These works proceed by expanding into spherical harmonics and choosing a different multiplier on each harmonic. We instead focus on finding a single multiplier which permits us to avoid the spherical harmonic decomposition. Our multiplier roughly looks like $X = f(r) \left(1 - \frac{2M}{r}\right) \partial_r$ where

$$f(r) = \begin{cases} \frac{1}{r^2} \left[\left(\frac{r^2}{3} + 2Mr + 10M^2\right)(r - 3M) + 8M^3 \ln \left(\frac{r-2M}{M}\right) \right], & r \in (2M, 3M), \\ \frac{9M}{2} \frac{r-3M}{r-\frac{3M}{2}}, & r \in [3M, \infty). \end{cases}$$

This is not quite sufficient as it is not bounded near $r = 2M$, where one needs to “smooth it out”. While relatively elementary, the estimate (4) plays an important role in the analysis. Since it is lossless away from $r = 2M$, $r = 3M$, and $r = \infty$, it allows us to glue together analyses which are performed separately in the regions surrounding these points.

In the estimate (4), notice, in particular, the vanishing of the coefficients at $r = 2M$ and $r = 3M$ which is a result of the trapping. We hope to be able to take

the losses at these points to be logarithmic in both cases. Roughly, we prove that the coefficients c_ω^0 and c_t^0 can be replaced by

$$c_\omega = \frac{1}{r} \left(1 - \ln \left|1 - \frac{3M}{r}\right|\right)^2, \quad c_t = \frac{\left(1 - \ln \left|1 - \frac{3M}{r}\right|\right)^2}{r^2 \left(1 - \ln \left(1 - \frac{2M}{r}\right)\right)^2}$$

respectively. To prove this, we may localize to a neighborhood of $r = 3M$, take the Fourier transform in t , and expand in spherical harmonics (indexed with λ). The estimates are easy unless λ and the time frequency variable have a delicate balance. In this case, the following serves as a fairly representative model problem:

$$u'' + \lambda^2(x^2 \pm \varepsilon)u = f, \quad \text{near } r = 0.$$

Estimates for solutions to this equation are proved using a WKB approximation.

Finally, we prove global Strichartz estimates which are roughly of the form

$$\left\| \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{2}\right) |D|^{1-\rho} \phi \right\|_{L^p L^q} \lesssim E[\phi](0)$$

for nonsharp Strichartz exponents. Here $|D|$ is a pseudodifferential operator which looks roughly like the derivatives occurring in the energy. There are also estimates available for sharp Strichartz exponents, but these necessitate a logarithmic loss.

Using (4), we may analyze the regions near $r = 2M, 3M, \infty$ separately. In the bounded regions near $r = 2M$ and $r = 3M$, the global Strichartz estimates follow, via fairly standard arguments, from the local-in-time Strichartz estimates (see, e.g., [4]) and the localized energy estimates (with the improved coefficients near $r = 3M$). Outside of a sufficiently large ball, the perturbation becomes small, and thus, in a region near ∞ , the analysis from [3] can be referenced.

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