A Survey of Existence Results for Nonlinear Wave Equations in Exterior Domains

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The Wave Equation

$$\Box u(t,x) = \partial_t^2 u(t,x) - \Delta u(t,x)$$

$$\Delta u(t,x) = \partial_{x_1}^2 u(t,x) + \partial_{x_2}^2 u(t,x) + \ldots + \partial_{x_n}^2 u(t,x)$$

The Cauchy Problem

$$(1) \quad \begin{array}{c} \Box u(t,x) = Q(u', u'') \\ (t,x) \in R \times R^n \end{array}$$

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

The Forcing Term

- Vanishes to Second Order
- Semilinear

$$Q(u', u'') = Q(u') \approx \partial_j u \, \partial_k u$$

- Quasilinear

$$Q(u', u'') \approx c_{jk} \partial_j u \, \partial_k u + d_{ij,k} \partial_k u \, \partial_i \partial_j u$$

Global Existence

- Well established local existence theory.
- When does (1) admit a global solution for “small” f,g?

If n≥ 4, the Cauchy problem (1) has a global solution. (~1985 – Klainerman, Hormander)

Sobolev Inequality

- What happens when n=2,3?

$$\sup_{\frac{R}{2} \le |x| \le R} |h(x)| \le CR^{-\frac{n-1}{2}} \sum_{|\alpha| \le \frac{n+2}{2}} \|Z^\alpha h\|_{L^2}$$

Notice the power of R… what happens as we sum (integrate) over R?

Almost Global Existence

- In n=3, there is almost global existence. That is, as f,g get smaller, the lifespan of the solutions grows exponentially. [1984 - John and Klainerman]

Examples

- The following example always blows up in finite time in n=3. [1981 – John]

$$\Box u = (\partial_t u)^2$$

- This seemingly similar example, however, has a global solution in n=3. [Nirenberg]

$$\Box u = (\partial_t u)^2 - \sum_{j=1}^{3} (\partial_j u)^2$$

Null Condition

- An extra condition on the quadratic part of the nonlinearity.
- If we are in the scalar valued, semilinear case, then the null condition says that the nonlinearity must be a constant multiple of the one in Nirenberg's example.

Global Existence in n=3

- In n=3, if the nonlinearity Q satisfies the null condition, then (1) has a global solution for "small" f,g. [1986 – Klainerman, Christodolou]

Existence when n=2

- In n=2, if Q satisfies the null condition, then (1) has almost global existence.
- In n=2, if Q also satisfies a second null condition (on the cubic terms), then (1) has a global solution for "small" f,g.
- Semilinear case – 1993, Godin.
- Quasilinear case – 2001, Alinhac.

Existence in Exterior Domains

- We want to look at existence exterior to some (compact) set with smooth boundary [an obstacle].

$$K \subset \{|x| < D\} \subset R^n$$

$$\Omega = R^n \setminus K$$

The Dirichlet Problem

$$(2)\quad \begin{aligned} \Box u(t,x) &= Q(u', u'') \\ (t,x) &\in R\times\Omega \end{aligned}$$

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

$$u(t,x) = 0 \quad when\ x \in \partial\Omega$$

Types of Obstacles

- Free space – no obstacle
- Convex
- Star-shaped
- Nontrapping
- Some Hyperbolic Trapped Rays
- Schwarzschild Space-Time
- Elliptic Trapped Rays

New Challenges

- The invariant vector fields do not preserve the boundary condition.
- You no longer have strong Huygen's principle.
- Problems related to local energy decay.

Local Energy Decay

K nontrapping, $\Box u = 0,$

$$u(0,x) = \partial_t u(0,x) = 0 \ \ for\ |x| > D.$$

n odd

$$\int_{|x|<D} |\nabla u(t,x)|^2 + (\partial_t u(t,x))^2 dx \le Ce^{-\alpha t} \int |\nabla u(0,x)|^2 + (\partial_t u(0,x))^2 dx$$

n even

$$\int_{|x|<D} |\nabla u(t,x)|^2 + (\partial_t u(t,x))^2 dx \le Ct^{-n} \int |\nabla u(0,x)|^2 + (\partial_t u(0,x))^2 dx$$

Summary of Known Results

n=3	Almost GE		GE with null cond.	
	Semi	Quasi	Semi	Quasi
Free	'84, J-K		'86, K-C	
Convex	'02, Keel-Smith-Sogge	'03 (prepr), K-S-S	'00, K-S-S	'02, Keel-Smith-Sogge
Star				
Nontrap			2 weeks ago, Metcalfe - Sogge	
HypTrap				
Schwa				!!!!!!!!!!!

Summary of Known Results

GE	n=4,5		n=6,7,8,…	
	Semi	Quasi	Semi	Quasi
Free	'85, Klainerman, Hormander			
Convex	'95, Hayashi (K=ball)		'86, Shibata - Tsutsumi	
Star	'03 (prepr), Metcalfe			
Nontrap				
HypTrap				
Schwa				

Summary of Known Results

n=2	Semilinear		Quasilinear	
	AGE	GE	AGE	GE
Free	'93, Godin		'01, Alinhac	
Convex	?			
Star				
Nontrap				
HypTrap				
Schwa				