Localized Energy Estimates on High Dimensional Schwarzschild Blackhole Backgrounds

Jason Metcalfe
(joint work with Parul Laul)

University of North Carolina

August 17, 2010

On Minkowski space... Morawetz estimates

- $ds^2 = -dt^2 + dr^2 + r^2 d\omega^2$
- $\Box = \partial_t^2 - \Delta$
- Conserved energy: If $\Box u = 0$, then $\|u'(t,\,\cdot\,)\|_2^2 = \|u'(0,\,\cdot\,)\|_2^2$.
- Morawetz estimate, $n \geq 4$: If $\Box u = 0$, then

$$\int_0^T \int \frac{1}{|x|} |\nabla\!\!\!/ u|^2 \, dx \, dt + \int_0^T \int \frac{1}{|x|^3} u^2 \, dx \, dt \lesssim \|u'(0,\,\cdot\,)\|_2^2.$$

Localized energy estimates

- More generally, $n \geq 4$: If $\Box u = 0$, then

$$\|\langle x\rangle^{-1/2-}u'\|_{L^2_t L^2_x} + \|\langle x\rangle^{-3/2}u\|_{L^2_t L^2_x} \lesssim \|u'(0,\,\cdot\,)\|_2.$$

- Prove via a positive commutator argument...

$$\int_0^T \int \Box u\Big(f(r)\partial_r u + \frac{n-1}{2}\frac{f(r)}{r}u\Big)\,dx\,dt.$$

- Commutator yields

$$f'(r)(\partial_r u)^2 + \frac{f(r)}{r}|\not\nabla u|^2 - \frac{1}{4}\Big(r^{-(n-1)}\partial_r r^{n-1}\partial_r \frac{f(r)}{r}\Big)u^2.$$

Localized energy estimates

- Want $f'(r) > 0$, $f(r) > 0$, $-\frac{1}{4}\Delta\frac{f(r)}{r} > 0$, and $f(r)$ bounded.
- An estimate then follows from Cauchy-Schwarz, a Hardy inequality, and conservation of energy.
- E.g. choose $f(r) = r/(r+2^j)$, $j \geq 0$.
- Similar estimates available for small, asymptotically flat perturbations of Minkowski space and for asymptotically flat, time-independent perturbations.

Hyperspherical Schwarzschild space-times

- $1+n$ dimensions. Set $d=n-3$.
- $(1+n)$-dimensional hyperspherical Schwarzschild space-times:

$$ds^2 = -\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)dt^2 + \Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^{-1} dr^2 + r^2 d\omega^2.$$

- $\Box_g = \nabla^\alpha \partial_\alpha$
- In these coordinates,

$$\Box_g = -\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^{-1}\partial_t^2 + r^{-(d+2)}\partial_r\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)r^{d+2}\partial_r + \not\nabla\cdot\not\nabla.$$

Conserved energy and photon sphere

- Conserved energy, coming from the Killing vector field ∂_t:

$$E[\phi] = \int \Big[\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)^{-1} (\partial_t \phi)^2 + \Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)(\partial_r \phi)^2 + |\not\nabla \phi|^2 \Big] r^{d+2}\, dr\, d\omega.$$

- Trapping: At the event horizon $r = r_s$. Also at the photon sphere $r^{d+1} = \frac{d+3}{2} r_s^{d+1}$.

Localized energy estimates

If $\Box_g \phi = 0$, then

$$\int \left\{ c_r \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r \phi)^2 + c_\omega |\nabla\!\!\!/ \phi|^2 + c_0 \phi^2 \right\} r^{d+2}\, dr\, d\omega\, dt \lesssim E[\phi](0).$$

Here

$$c_r = \frac{1}{r^{d+3}\left(1 - \log \frac{r-r_s}{r}\right)^2}$$

$$c_\omega = \frac{1}{r}\left(\frac{r - r_{ps}}{r}\right)^2$$

$$c_0 = \frac{r}{r - r_s} \frac{1}{\left(1 - \log \frac{r-r_s}{r}\right)^4} \frac{1}{r^3}.$$

Commutator

- Multiplier

$$f(r)\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)\partial_r\phi+\frac{1}{2}\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\phi$$

- Commutator

$$f'(r)\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^2(\partial_r\phi)^2+\Big(\frac{r^{d+1}-r_{ps}^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\,\phi|^2 \\ +l(f)\phi^2.$$

Here

$$l(f)=-\frac{1}{4}r^{-(d+2)} \\ \times\Big\{\partial_r\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)r^{d+2}\partial_r\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big\}$$

Multiplier construction

- $g(r) = \frac{r^{d+2} - r_{ps}^{d+2}}{r^{d+2}}$, $h(r) = \log \frac{r^{d+1} - r_s^{d+1}}{\frac{d+1}{2} r_s^{d+1}}$
- First attempt
$$g(r) + \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} h(r).$$
- Smoothing function, $\alpha = 5-$
$$a(x) = \begin{cases} -\frac{1}{\varepsilon} \frac{\varepsilon x + 1}{\delta(\varepsilon x + 1) - 1} - \frac{1}{\varepsilon}, & x \le -1/\varepsilon, \\ x, & -1/\varepsilon \le x \le 0, \\ x - \frac{2}{3\alpha^2} x^3 + \frac{1}{5\alpha^4} x^5, & 0 \le x \le \alpha, \\ \frac{8\alpha}{15}, & x \ge \alpha. \end{cases}$$

Multiplier construction

- $f(r) = g(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}a(h(r))$.
- f is not C^2 across $h(r) = -1/\varepsilon$. The resulting boundary term has a coefficient which is $O(\delta\varepsilon)$.
- The desired properties when $-1/\varepsilon \le h(r) \le 0$ and $h(r) \ge \alpha$ are straightforward to verify.
- Checking the sign of f and f' is straightforward everywhere.

Bounding $l(f)$

$$
\begin{aligned}
l(f) = \frac{d+2}{4r^{2d+5}} &\Big(d r^{2d+2} + (d+3) r_s^{d+1} r^{d+1} - (d+2)^2 r_s^{2d+2} \Big) \\
&+ \frac{(d+1)(d+2)}{2} \frac{r_{ps} r_s^{d+1}}{r^{2d+6}} (r_{ps}^{d+1} - r^{d+1}) a'(h(r)) \\
&\quad + \frac{(d+1)^2(d+2)(d+5)}{4(d+3)} \frac{r_{ps} r_s^{d+1}}{r^{d+5}} a''(h(r)) \\
&\qquad - \frac{(d+1)^3(d+2)}{4(d+3)} \frac{r_{ps} r_s^{d+1}}{r^4} \frac{1}{r^{d+1} - r_s^{d+1}} a'''(h(r)).
\end{aligned}
$$

Near the event horizon

- Key term $-\frac{1}{r^4}\frac{r^{d+1}-r_s^{d+1}}{a}'''(h(r))$
- Set $r_{-1/\varepsilon}$ so that $h(r_{-1/\varepsilon}) = -1/\varepsilon$.
- Use the Fundamental Theorem of Calculus...

$$\int_{r_s}^{r_{-1/\varepsilon}} \partial_r\Big(\frac{(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2}a''(r)\phi^2\Big)\,dr$$
$$= O(\delta\varepsilon) \text{ bdy term.}$$

- Computing derivative and using Cauchy-Schwarz the unfavorably signed a''' term can be controlled by the a'' term, the $(\partial_r\phi)^2$ term coming from $f'(r)$, and a $O(\delta\varepsilon)$ boundary term.

$r_{-1/\varepsilon}$ boundary term

- $\phi(r) = -\int_r^{r_{ps}} \partial_r(\beta\phi)\,dr$
- Using Cauchy-Schwarz...

$$\varepsilon(\phi(r_{-1/\varepsilon}))^2 \lesssim \int_{r_{-1/\varepsilon}}^{r_{ps}} l(f)\phi^2 + f'(r)\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)^2 (\partial_r\phi^2)\,dr.$$