Long time existence for quasilinear wave equations in exterior domains

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Dirichlet problem

- $\Box = \partial_t^2 - \Delta$
- $\mathcal{K} \subset \mathbb{R}^n$ bounded, smooth boundary, star-shaped
- $$\begin{cases} \Box u = Q(u, u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = \varepsilon f, \quad \partial_t u(0,\,\cdot\,) = \varepsilon g \end{cases}$$
- Say $f, g \in C_c^\infty(\mathbb{R}^n)$, satisfying compatibility conditions.
- Q vanishes to second order at 0
- Question: Long-time existence...

Keel-Smith-Sogge method

- Set
  $Z = \{\partial_i, \Omega_{jk} = x_j\partial_k - x_k\partial_j \,:\, 0 \le i \le n,\, 1 \le j < k \le n\}.$
- Energy estimates
$$\|u'(t,\,\cdot\,)\|_2 \le \|u'(0,\,\cdot\,)\|_2 + \int_0^t \|\Box u(s,\,\cdot\,)\|_2\, ds$$
- Localized energy estimates
$$\log(2+T)^{-1/2}\|\langle x\rangle^{-1/2}u'\|_{L^2_{t,x}} + \|\langle x\rangle^{-1/2-}u'\|_{L^2_{t,x}} \lesssim \|u'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box u(s,\,\cdot\,)\|_2\, ds$$
- Weighted Sobolev estimates
$$\|h\|_{L^\infty(R<|x|<2R)} \lesssim R^{-(n-1)/2} \sum_{|\alpha|\le\frac{n+2}{2}} \|Z^\alpha h\|_{L^2(R/2<|x|<4R)}.$$

Keel-Smith-Sogge method

- Similar estimates have been proved for small perturbations of $\Box$ which allow one to similarly handle quasilinear equations.
- Idea of proof, $n = 4$, $\Box u = (\partial_t u)^2$:
  - Set $M(T) = \sum_{|\alpha|\le 10} \sup_{0\le t\le T} \|(Z^\alpha u)'(t,\,\cdot\,)\|_2 + \|\langle x\rangle^{-3/4}(Z^\alpha u)'\|_{L^2_{t,x}}$.
  - By the estimates above,

$$M(T) \lesssim \varepsilon + \int_0^T \sum_{|\beta|\le 5, |\alpha|\le 10} \|\partial_t Z^\beta u \partial_t Z^\alpha u\|_2 \, dt$$
$$\lesssim \varepsilon + \sum_{|\alpha|\le 10} \|\langle x\rangle^{-3/4}(Z^\alpha u)'\|^2_{L^2_{t,x}} = \varepsilon + (M(T))^2$$

With u dependence

- Shorter lifespan is expected, based on boundaryless results of Lindblad, Hörmander
- $n = 3$, Du and Zhou showed $T_\varepsilon \gtrsim 1/\varepsilon^2$ outside star-shaped obstacles.
- $n = 4$, Du, M., Sogge, and Zhou showed almost global existence ($T_\varepsilon \gtrsim \exp(c/\varepsilon)$) for star-shaped obstacles.

Variant of the localized energy estimate

Roughly, for the $n = 4$ proof,

$$(\log(2+T))^{-1/2} \|\langle x\rangle^{-1/2} w\|_{L^2_{t,x}} \lesssim \varepsilon + \int_0^T \|\langle x\rangle^{-(n-2)/2} \Box w(t,\,\cdot\,)\|_{L^1_r L^2_\omega}\, dt$$

which follows from

$$\|h\|_{\dot{H}^{-1}(\mathbb{R}^n)} \lesssim \|h\|_{L^{2n/(n+2)}(|x|<1)} + \||x|^{-(n-2)/2} h\|_{L^1_r L^2_\omega(|x|>1)}.$$

Improved lifespan

With the additional assumption that $\partial_u^2 Q(0,0,0) = 0$, the works of Lindblad $(n = 3)$ and Hörmander $(n = 4)$ suggest that a longer lifespan should be possible.

- $n = 4$, M.-Sogge, Global existence for star-shaped obstacles.

Generalized estimate

Roughly, for $n = 4$ and $0 < \gamma < 1/2$,

$$\||x|^{-\frac{1}{2}-\gamma} w\|_{L^2_{t,x}} \lesssim \varepsilon + \||x|^{-1-\gamma} \Box w\|_{L^1_{t,r} L^2_\omega}.$$

- A special case of the weighted Strichartz estimates of Hidano-M.-Smith-Sogge-Zhou and Fang-Wang.

$$\Big\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \Big\|_{L^p_{t,r} L^2_\omega} \lesssim \|u'(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} \Box u \Big\|_{L^1_{t,r} L^2_\omega}$$

for $2 \le p \le \infty$, $\frac{1}{2} - \frac{1}{p} < \gamma < \frac{n}{2} - \frac{1}{p}$, $\frac{1}{2} < 1 - \gamma < \frac{n}{2}$.

Global existence

- If $\Box v = \sum_0^n a_j \partial_j G$ with vanishing initial data, then

$$\|\langle x\rangle^{-1/2-\delta} v\|_{L^2_{t,x}} \lesssim \|G(0,\,\cdot\,)\|_{\dot H^{\gamma-1}} + \int_0^T \|G(t,\,\cdot\,)\|_2 \, dt.$$

- Use a similar argument for long-time existence with

$$M(T) = \sum_{|\alpha|\le 20} \sup_t \|(Z^\alpha u)'(t,\,\cdot\,)\|_2 + \|\langle x\rangle^{-1/2-\delta} (Z^\alpha u)'\|_{L^2_{t,x}} + \|\langle x\rangle^{-1/2-2\delta} Z^\alpha u\|_{L^2_{t,x}}$$

for $0 < \delta < 1/8$.