Abstract Strichartz estimates and the Strauss conjecture for the wave equation in exterior domains

Jason Metcalfe

University of North Carolina

Joint work with K. Hidano, H. Smith, C. Sogge, and Y. Zhou

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The Strauss conjecture

- $\Box = \partial_t^2 - \Delta$
- $$\begin{cases} \Box u = |u|^p, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,\,\cdot\,) = \varepsilon f, & \partial_t u(0,\,\cdot\,) = \varepsilon g \end{cases}$$
- Say $f, g \in C_c^\infty(\mathbb{R}^n)$.
- Question: For which p is there a global solution for ε sufficiently small?
- Answer: There are global solutions for ε sufficiently small provided $p > p_c$ where $p_c > 1$ is the positive root of

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0.$$

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Exterior domain problem

- Let $\mathcal{K} \subset \{|x| < R\} \subset \mathbb{R}^n$ be a bounded set with smooth boundary. Set $\Omega = \mathbb{R}^n \backslash \mathcal{K}$.
- Look at

$$\begin{cases} \Box u = |u|^p, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ (Bu)|_{\partial\Omega} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g \end{cases}$$

- Either $B = I$ (Dirichlet boundary conditions) or $B = \partial_\nu$ (Neumann boundary conditions)
- Is there similar small data global existence?

Geometric assumption

- We assume that a localized energy estimates holds for Ω.
- If

$$\begin{cases} \Box u = F, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ (Bu)|_{\partial\Omega} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g, \\ \operatorname{supp} f, \operatorname{supp} g, \operatorname{supp} F(t,\,\cdot\,) \subset \{|x| < R_0\}, \end{cases}$$

then

$$\int_0^\infty \Big(\|u(t,\,\cdot\,)\|^2_{H^1(\{|x|<R_0\})} + \|\partial_t u(t,\,\cdot\,)\|^2_{L^2(\{|x|<R_0\})} \Big)\, dt \lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2} + \int_0^\infty \|F(s,\,\cdot\,)\|^2_{L^2}\, ds$$

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Main Theorem

Theorem (Hidano-M.-Smith-Sogge-Zhou)

Let $n = 3, 4$, and suppose that Ω satisfies the localized energy estimates. Then for $p_c < p < \frac{n+3}{n-1}$, u exists globally if f, g satisfy the relevant compatibility conditions and

$$\sum_{|\alpha|\le 2} \|Z^\alpha f\|_{\dot{H}^\gamma} + \|Z^\alpha g\|_{\dot{H}^{\gamma-1}} < \varepsilon, \quad \gamma = \frac{n}{2} - \frac{2}{p-1}$$

for ε sufficiently small.

- Here $\{Z\} = \{\partial_i, \Omega_{jk} = x_j\partial_k - x_k\partial_j \,:\, 1 \le i \le n, \quad 1 \le j < k \le n\}$

Weighted Strichartz estimate

For

$$\begin{cases} \Box u = F, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,\,\cdot\,) = f, & \partial_t u(0,\,\cdot\,) = g, \end{cases}$$

and for $2 \leq p \leq \infty$, and γ satisfying

$$\frac{1}{2} - \frac{1}{p} < \gamma < \frac{n}{2} - \frac{1}{p}, \quad \frac{1}{2} < 1 - \gamma < \frac{n}{2},$$

we have

$$\Big\| |x|^{\frac{n}{2} - \frac{n+1}{p} - \gamma} u \Big\|_{L^p_t L^p_r L^2_\omega} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} F \Big\|_{L^1_t L^1_r L^2_\omega}.$$

Weighted Strichartz estimates - cont.

$$\Big\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \Big\|_{L^p_t L^p_r L^2_\omega} \lesssim \|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} F \Big\|_{L^1_t L^1_r L^2_\omega}$$

- Notice that $\| |x|^{-\frac{n}{2}+1-\gamma} |u|^p \|_{L^1} = \| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \|^p_{L^p}$.
- Proof:
  - $\| |x|^{-\alpha} e^{it|D|}\varphi \|_{L^\infty_r L^2_\omega} \lesssim \|\varphi\|_{\dot H^{\frac{n}{2}+\alpha}}, \quad -\frac{n-1}{2} < \alpha < 0.$
  - $\| |x|^{-s} e^{it|D|}\varphi \|_{L^2_{t,x}} \lesssim \| |D|^{s-\frac{1}{2}}\varphi \|_{L^2}, \quad \frac{1}{2} < s < \frac{n}{2}.$

Weighted Strichartz estimates - cont.

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Abstract Strichartz estimates

Theorem (Hidano-M.-Smith-Sogge-Zhou)

Let (X,γ,q) be such that X is a localizable normed function space, $q>2$, $\gamma\in\left[-\frac{n-3}{2},\frac{n-1}{2}\right]$, and

1. *The localized energy estimate holds*
2. *If $\Box v=0$ on $\mathbb{R}_+\times\mathbb{R}^n$, then*

$$\|v\|_{L^q_t X(\mathbb{R}_+\times\mathbb{R}^n)}\lesssim\|v(0,\,\cdot\,)\|_{\dot H^\gamma}+\|\partial_t v(0,\,\cdot\,)\|_{\dot H^{\gamma-1}}.$$

3. *If $\Box u=0$, $(Bu)|_{\partial\Omega}=0$ in $R_+\times\Omega$, then*

$$\|u\|_{L^q_t X([0,1]\times\Omega)}\lesssim\|u(0,\,\cdot\,)\|_{\dot H^\gamma}+\|\partial_t u(0,\,\cdot\,)\|_{\dot H^{\gamma-1}}.$$

then $\|u\|_{L^q_t X(\mathbb{R}_+\times\Omega)}\lesssim\|u(0,\,\cdot\,)\|_{\dot H^\gamma}+\|\partial_t u(0,\,\cdot\,)\|_{\dot H^{\gamma-1}}.$

Abstract Strichartz estimates - cont.

Corollary

Let (X, γ, q), $(Y, 1-\gamma, \tilde{q})$ *be admissible. Then,*

$$\|u\|_{L^q_t X(\mathbb{R}_+\times\Omega)} \lesssim \|u(0,\,\cdot\,)\|_{\dot{H}^\gamma} + \|\partial_t u(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}} + \|\Box u\|_{L^{\tilde{q}'} Y'(\mathbb{R}_+\times\Omega)}.$$

- (Strichartz estimates) $X = L^r_x$ with $\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2$, $\frac{2}{q} + \frac{n-1}{r} \le \frac{n-1}{2}$, $\frac{2}{\tilde{q}} + \frac{n-1}{\tilde{r}} \le \frac{n-1}{2}$.

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Main exterior domain estimate

$$\Big\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \Big\|_{L^p_t L^p_r L^2_\omega} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} F \Big\|_{L^1_t L^1_r L^2_\omega}$$

- Still need a local-in-time exterior domain analog.
- We shall use a modification that makes this nearly trivial.
- For $\gamma = n\Big(\frac{1}{2} - \frac{1}{s_\gamma}\Big)$, set

$$\|h\|_X = \|h\|_{L^{s_\gamma}(\{|x|<2R\})} + \Big\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} h \Big\|_{L^p_r L^2_\omega(\{|x|>2R\})}.$$

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