Elastic waves in exterior domains

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Equations of elasticity

- Linearized equations

$$Lu = \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\nabla(\nabla \cdot u)$$

- $c_1 > c_2 > 0$
- For homogeneous, isotropic, hyperelastic mediums, instead

$$(Lu)^I = \partial_l (B_{lmn}^{IJK} \partial_m u^J \partial_n u^K)$$

- This is truncated at the quadratic level, but the higher order terms won't have adverse affects on the long-time existence

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Exterior domain problem

- Fix a compact obstacle $\mathcal{K}$ with smooth boundary that is star-shaped (with respect to the origin)
- WLOG $\mathcal{K} \subset \{|x| \leq 1\}$
- Fix Dirichlet boundary conditions
- Solve the following in $\mathbb{R}^3 \setminus \mathcal{K}$

$$(Lu)^I = \partial_l (B_{lmn}^{IJK} \partial_m u^J \partial_n u^K)$$

(1) $$u\big|_{\partial\mathcal{K}} = 0$$

$$u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g$$

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Theorem

- Initial data smooth, satisfying the relevant compatibility conditions and small in a weighted Sobolev norm (size $\approx \varepsilon$).
- $\mathcal{K}$ as above
- $B_{lmn}^{IJK} = B_{nlm}^{KIJ} = B_{mnl}^{JKI}$
- Then, there is a unique solution

$$u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^3 \setminus \mathcal{K})$$

to (1), where

$$T_\varepsilon = \exp(\kappa / \varepsilon)$$

for some small, fixed κ

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Related results in $\mathbb{R}^3$

- John ['88]: Almost global existence
- John ['84]: Counterexamples to general, small-data global existence
- Klainerman-Sideris ['96]: Another proof of almost global existence
- Agemi ['00], Sideris ['96, '00]: Small-data global existence with a null condition

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Related results for multiplied speed wave equations in $\mathbb{R}^3 \setminus \mathcal{K}$

- Keel-Smith-Sogge ['02, '04]: Almost global existence
- M.-Sogge [preprint]: Almost global existence using only energy methods
- M.-Sogge ['05], M.-Nakamura-Sogge ['05, preprint]: Small data global existence with null-condition
- M.-Sogge [preprint]: Small data global existence with null condition using only energy methods.

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Outline of Proof

- Our methods most closely resemble those of Klainerman-Sideris ['96] and Keel-Smith-Sogge ['04].
- Key estimates:
  - Decay of local energy
  - Higher order energy estimates
    - Elliptic regularity
    - Star-shapedness
  - Weighted mixed-norm Keel-Smith-Sogge (KSS) estimates
  - Lower order pointwise decay estimates

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Notation / Vector fields

- Vector fields:
  - Translations:

$$\partial = (\partial_t = \partial_0, \partial_1, \partial_2, \partial_3)$$

  - Scaling:

$$S = t\partial_t + r\partial_r$$

  - Spatial rotations:

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \times \nabla_x$$

  - Simultaneous rotations:

$$\tilde{\Omega}_l = \Omega_l I + U_l$$

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Notation / Vector fields

– With

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad U_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$U_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $Z = \{\partial, \tilde{\Omega}\}$
  $\Gamma = \{\partial, \tilde{\Omega}, S\}$

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Decay of local energy

- Suppose that $Lv = 0$, $v|_{\partial\mathcal{K}} = 0$ and $(v(0,\cdot), \partial_t v(0,\cdot))$ vanish for $|x| \geq 4$
- Suppose that $\mathcal{K}$ is smooth and nontrapping.
- Then

$$\|v'(t,\cdot)\|_{L^2(|x|\leq 4)} \leq Ce^{-ct}\|v'(0,\cdot)\|_2$$

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Decay of local energy

- Due to: Yamamoto ['89]. See also Dassios ['83], Kapitonov ['87] (star-shaped case).
- Especially useful for establishing exterior domain estimates based on estimates in the boundaryless case (using arguments of Smith-Sogge ['00]).
- Analog of the well-known estimates of Morawetz-Ralston-Strauss ['77] for the wave equation.

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Energy estimates

- For γ satisfying

$$\gamma^{IJ,jk} = \gamma^{JI,kj}$$

$$|\gamma^{IJ,jk}(t,x)| \leq \frac{\delta}{1+t}$$

$$\|\partial\gamma\|_{L_t^1 L_x^\infty(S_T)} \leq C_0$$

we have

$$\|u'(T)\|_2 \leq C\|u'(0)\|_2 + C\int_0^T \|L_\gamma u(t)\|_2 \, dt$$

where

$$(L_\gamma u)^I = \partial_t^2 u^I - c_2^2 \Delta u^I - (c_1^2 - c_2^2)\nabla(\nabla\cdot u) + \gamma^{IJ,jk}\partial_j\partial_k u^J$$

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Energy estimates

- Since ∂_t preserves the Dirichlet boundary condition, the above estimate also holds for u replaced by $\partial_t^j u$
- To get estimates involving spatial translations, we use elliptic regularity.
  - For the wave equation, this is as follows:

$$\|\partial^\alpha \partial_x^2 u(t)\|_2 \le C\|\partial^\alpha \Delta u(t)\|_2 + C\|\partial^\alpha u'(t)\|_2$$
$$\le C\|\partial^\alpha \partial_t^2 u(t)\| + C\|\partial^\alpha \Box u(t)\|_2 + \|\partial^\alpha u'(t)\|_2$$

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Energy estimates

- Argue recursively to turn all of the spatial derivatives into time derivatives modulo lower order terms.
- For elasticity, the elliptic regularity follows similarly. Either by noticing that

$$\mathcal{W}u = c_2^2 \Delta u + (c_1^2 - c_2^2)\nabla(\nabla \cdot u)$$

is strongly elliptic, or by using a decomposition into curl-free and divergence free components.

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Energy estimates

- For energy estimates involving the scaling vector field, we rely on the assumption that the obstacle is star-shaped. Using arguments similar to those of Morawetz ['61], certain boundary terms have a favorable sign.

$$\|(Su)'(T)\|_2 \le \varepsilon + \int_0^T \|L_\gamma Su(t)\|_2 \, dt + \sum_{|\alpha|\le 2} \|\partial^\alpha u'\|_{L_t^2 L_x^2([0,T]\times\{|x|\le 2\})}$$

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Energy estimates

- The last term above will be handled using the KSS estimates.
- Energy estimates involving a single occurrence of the scaling vector field and translations can be obtained using elliptic regularity as above.

- To get energy estimates involving rotations (and possibly one scaling vector field), we replace u by $S^\mu Z^\alpha u$ in the energy integral method. The resulting boundary terms are controlled by

$$\sum_{\substack{|\beta|+\nu\le|\alpha|+\mu+1 \\ \nu\le\mu}} \|S^\nu \partial^\beta u'\|_{L_t^2 L_x^2([0,T]\times\{|x|\le 2\})}$$

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KSS estimates for wave equations

- For the wave equation, developed and first used by Keel-Smith-Sogge ['02] to study nonlinear problems. Earlier related estimates Strauss ['75] and Mochizuki ['84].
- New proof of Rodnianski ['05], M.-Sogge [preprint] yields

$$(\log(2+T))^{-1/2}\left\|\langle x\rangle^{-1/2} w'\right\|_{L_t^2 L_x^2(S_T)} \le C\|w'(0)\|_2 + C\int_0^T\int |\partial w \Box w|\,dx\,dt + C\int_0^T\int \left|\langle x\rangle^{-1} w \Box w\right| dx\,dt$$

KSS estimates for boundaryless elasticity

- Follows from the KSS estimates for wave equations and a Hodge decomposition.
- Suppose $Lv = F + G$ and that G vanishes for $|x| > 4$ for each time. Then,

$$(\log(2+T))^{-1/2}\left\|\langle x\rangle^{-1/2} v'\right\|_{L_t^2 L_x^2(S_T)} \le C\|v'(0)\|_2 + C\int_0^T \|F(s)\|_2\,ds + C\|G\|_{L_t^2 L_x^2(S_T)}$$

KSS estimates for elasticity

- Will use the boundaryless estimate and techniques of Smith-Sogge ['00] to get the following exterior domain estimate:

$$(\log(2+T))^{-1/2}\left\|\langle x\rangle^{-1/2} S^{\mu}\partial^{\alpha} u'\right\|_{L_t^2 L_x^2(S_T)} \le C\varepsilon + C\int_0^T \sum_{\substack{|\beta|+\nu\le|\alpha|+\mu \\ \nu\le\mu}} \left\|S^{\nu}\partial^{\beta} Lu(t)\right\|_2 dt + C\sum_{\substack{|\beta|+\nu\le|\alpha|+\mu-1 \\ \nu\le\mu}} \left\|S^{\nu}\partial^{\beta} Lu\right\|_{L_t^2 L_x^2(S_T)}$$

- A similar estimate holds for ∂ replaced by Z.

Sketch of the proof

- Near the obstacle. I.e. when the norm in the left is over $L_t^2 L_x^2([0,T]\times\{|x|<2\})$
  - When the data and forcing term are compactly supported, apply the decay of local energy.
  - Else, write $u = v + u_r$ where v is the solution to the associated boundaryless problem. Set $\tilde{u} = \rho(x)v + u_r$ where ρ is a smooth cutoff which is identically 1 for $|x| \le 2$ and vanishes for $|x| > 3$.

Sketch of the proof

– Notice that $u = \tilde{u}$ for $|x| < 2$, and that
$$L\tilde{u} \approx \tilde{\rho} \sum_{|\alpha| \le 1} \partial^\alpha v$$
– Thus, by local energy decay, it suffices to control
$$\sum_{\substack{|\beta| + \nu \le |\alpha| + \mu \\ \nu \le \mu}} \| S^\nu \partial^\beta v' \|_{L_t^2 L_x^2([0,T] \times \{|x| \le 3\})}$$
which can be obtained from the boundaryless estimates.

Sketch of the proof

- Away from the obstacle. I.e. when the spatial norm is over $|x| > 3/2$
  – Multiply by a cutoff function which is identically 1 in this region and vanishes inside the unit ball.
  – This function $\beta(x)u$ satisfies a boundaryless equation.
  – Using the boundaryless estimate (where we set G to equal the commutator terms), it suffices to control
$$\sum_{\substack{|\beta| + \nu \le |\alpha| + \mu \\ \nu \le \mu}} \| S^\nu \partial^\beta u' \|_{L_t^2 L_x^2([0,T] \times \{|x| \le 3/2\})}$$
  which follows from the previous case.

Decay estimates

- Decay in $|x|$ follows from well-known weighted Sobolev estimates (see Klainerman [‘86]):

$$\|h\|_{L^\infty(R/2 < |x| \le R)} \le CR^{-1} \sum_{|\alpha| + |\beta| \le 2} \|\tilde{\Omega}^\alpha \partial_x^\beta h\|_{L^2(R/4 < |x| \le 2R)}$$

- Decay in “ $t - |x|$ “ will be obtained from estimates of Klainerman-Sideris [’96].

Klainerman-Sideris estimates

- For the boundaryless case:

$$\frac{\langle c_1 t - r \rangle \langle c_2 t - r \rangle}{\langle c_1 t - r \rangle + \langle c_2 t - r \rangle} |\mathcal{W} v(t,x)| \le \sum_{|\alpha| \le 1} |\partial \Gamma^\alpha v(t,x)| + t\,|Lv(t,x)|$$

$$\left\| \frac{\langle c_1 t - r \rangle \langle c_2 t - r \rangle}{\langle c_1 t - r \rangle + \langle c_2 t - r \rangle} \nabla^2 v(t) \right\|_2 \le \sum_{|\alpha| \le 1} \|\partial \Gamma^\alpha v(t)\|_2 + t \|Lv(t)\|_2$$

- When there is a boundary, the first estimate holds, but the second takes more care

Klainerman-Sideris estimates

- Applying the Sobolev type estimate (Sideris ['96])

$$r^{1/2}h \le C\sum_{|\alpha|\le 1}\left\|\Omega^{\alpha}\nabla h\right\|_2$$

followed by the second estimate above to u with appropriate cutoffs, we can easily prove

$$r^{1/2}(1+t)\,|Z^{\alpha}u'(t,x)| \le C\sum_{\substack{|\beta|+\mu\le|\alpha|+2\\ \mu\le 1}}\left\|S^{\mu}Z^{\beta}u'(t)\right\|_2$$

$$+C\sum_{|\beta|\le|\alpha|+1} t\left\|Z^{\beta}Lu(t)\right\|_2 + C(1+t)\sum_{|\beta|\le|\alpha|+2}\left\|\partial^{\beta}u'(t)\right\|_{L^2(|x|\le 2)}$$

for $r \le c_2 t/2$

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Klainerman-Sideris estimates

- We can reduce the last term above to the case of the boundaryless wave equation. The necessary decay result then follows from the estimates for the boundaryless case and the energy inequality.

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Sketch of the proof of almost global existence

- Use an iteration to show a bound like

$$\sum_{|\alpha|\le 30}\left\|\partial^{\alpha}u'(t)\right\|_2 + (\log(2+T))^{-1/2}\sum_{|\alpha|\le 29}\left\|\langle x\rangle^{-1/2}\partial^{\alpha}u'\right\|_{L_t^2L_x^2(S_T)}$$

$$+\sum_{|\alpha|\le 28}\left\|Z^{\alpha}u'(t)\right\|_2 + (\log(2+T))^{-1/2}\sum_{|\alpha|\le 27}\left\|\langle x\rangle^{-1/2}Z^{\alpha}u'\right\|_{L_t^2L_x^2(S_T)}$$

$$+\sum_{\substack{|\alpha|+\mu\le 27\\ \mu\le 1}}\left\|S^{\mu}\partial^{\alpha}u'(t)\right\|_2 + (\log(2+T))^{-1/2}\sum_{\substack{|\alpha|+\mu\le 26\\ \mu\le 1}}\left\|\langle x\rangle^{-1/2}S^{\mu}\partial^{\alpha}u'\right\|_{L_t^2L_x^2(S_T)}$$

$$+\sum_{\substack{|\alpha|+\mu\le 25\\ \mu\le 1}}\left\|S^{\mu}Z^{\alpha}u'(t)\right\|_2 + (\log(2+T))^{-1/2}\sum_{\substack{|\alpha|+\mu\le 24\\ \mu\le 1}}\left\|\langle x\rangle^{-1/2}S^{\mu}Z^{\alpha}u'\right\|_{L_t^2L_x^2(S_T)}$$

$$+(1+t)\sum_{|\alpha|\le 20}\left\|Z^{\alpha}u'(t)\right\|_{\infty} \le C\varepsilon$$

for all $0\le t\le T_{\varepsilon}$

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Sketch of the proof

- Look at the third line in more detail. Recall that the boundary terms in the energy estimate can be controlled by the second term on the first line.
- Thus, the main terms to control are

$$\sum_{\substack{|\alpha|+\mu\le 27\\ \mu\le 1}}\int_0^{T_{\varepsilon}}\left\|L_{\gamma}(S^{\mu}\partial^{\alpha}u)(t)\right\|_2\,dt$$

$$+\sum_{\substack{|\alpha|+\mu\le 26\\ \mu\le 1}}\int_0^{T_{\varepsilon}}\left\|S^{\mu}\partial^{\alpha}Lu(t)\right\|_2\,dt$$

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Sketch of the proof

- These are controlled by

$$\int_0^{T_\varepsilon} \sum_j \left[\sum_{\substack{|\alpha|+\mu\le 14 \\ \mu\le 1}} \left\| S^\mu \partial^\alpha u'(t) \right\|_{L^\infty(|x|\in[2^{j-1},2^j))} \sum_{|\alpha|\le 27} \left\| \partial^\alpha u'(t) \right\|_{L^2(|x|\in[2^{j-1},2^j))} \right] dt$$

$$+ \int_0^{T_\varepsilon} \sum_{|\alpha|\le 14} \left\| \partial^\alpha u'(t) \right\|_2 \sum_{\substack{|\alpha|+\mu\le 27 \\ \mu\le 1}} \left\| S^\mu \partial^\alpha u'(t) \right\|_2 dt$$

  - To handle the first term, we apply the weighted Sobolev estimate and bound using the KSS terms.
  - The second term is controlled easily using the temporal decay.
- Similar arguments work for the other pieces.

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