Global Existence of Solutions to Multiple Speed Systems of Quasilinear Wave Equations in Exterior Domains

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Outline

- The Beginnings… the results of Klainerman and Christodolou.
- Multiple Speed Wave Equations in Free Space.
- Wave Equations in Exterior Domains.
- Two New Results: M.-Sogge, M.-Nakamura-Sogge

The Wave Equation

$$\Box u^I(t,x) = \left(\partial_t^2 - \Delta\right) u^I(t,x) = Q^I(du, d^2u)$$

(1) $u(0,x) = f(x); \quad \partial_t u(0,x) = g(x)$

$$(t,x) \in R_+ \times R^3 \qquad I = 1,2,\ldots,D$$

$$u = (u^1, u^2, \ldots, u^D)$$

The Nonlinearity

- Vanishes to Second Order
- Linear in d^2u

$$Q^I(du, d^2u) = \sum_{\substack{0\le j,k,l\le 3\\ 1\le J,K\le D}} B_{K,l}^{IJ,jk} \partial_l u^K \partial_j \partial_k u^J$$

- Symmetry: $B_{K,l}^{IJ,jk} = B_{K,l}^{JI,jk} = B_{K,l}^{IJ,kj}$
- We can also allow similar semilinear terms.

Applications

- Such wave equations come up in a number of applications. For example, the previously stated wave equation is closely related to geometric wave maps.
- In the multiple speed case that we will discuss soon, they are related to the equations of elasticity, charged plasma, and magneto-hydrodynamics.

Long Time Solutions

- Well established local existence theory.
- When does (1) admit a global solution for "small" f,g?
- If n≥4, the Cauchy problem (1) has a global solution. [Klainerman, Hörmander (~1985)]

Klainerman-Sobolev Inequality

- What happens when n=3?
- Klainerman-Sobolev Inequality:

$$(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{1/2}|u(t,x)| \le C \sum_{|\alpha|\le\frac{n+2}{2}} \left\|\Gamma^\alpha u(t,\cdot)\right\|_2$$

- The solution has $O(t^{-1})$ decay. Not quite integrable.

Almost Global Existence

- In n=3, there is almost global existence. That is, as f,g get smaller, the lifespan of the solution grows exponentially. [John-Klainerman (1984)]
- The following example always blows up in finite time in n=3. [John (1981)]

$$\Box u = (\partial_t u)^2$$

Null Condition

- The following example has a global solution in n=3 for sufficiently small data. [Nirenberg]

$$\Box u = (\partial_t u)^2 - \sum_{j=1}^{3} (\partial_j u)^2$$

- We want to put an extra condition on the quadratic part of the nonlinearity, the null condition.

Null Condition

- We want to limit the amount of the nonlinearity in the "bad" direction.
- In the scalar valued, semilinear case, the null condition says that the nonlinearity must be a constant multiple of the one in Nirenberg's example.

Null Condition

- More generally, in the current case, we require

$$\sum_{J,K=1}^{D} \sum_{j,k,l=0}^{3} B_{K,l}^{IJ,jk} \xi_j \xi_k \xi_l = 0$$

whenever

$$\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$$

Global Existence in n=3

- Suppose n=3 and suppose further that the nonlinearity Q satisfies the null condition. Then, (1) has a global solution for "small" f,g. [Klainerman, Christodolou (1986)]

Christodolou's Method

- Uses the Penrose conformal compactification of Minkowski space.
- A hyperbolic space analog of stereographic projection. Also, related to the Kelvin transformation (inversion about a sphere).

Christodolou's Method (2)

- Map space-time conformally to the Einstein diamond,

$$\{(T,X)\in(-\pi,\pi)\times S^3 : |T|+R<\pi\}$$
$$R=\arctan(t+r)-\arctan(t-r)$$
$$T=\arctan(t+r)+\arctan(t-r)$$

- Use local existence theory to get a solution on this compact space.

Klainerman's Method

- Method of commuting vector fields.
- Continuity argument (a continuous version of induction).
- The Invariant Vector Fields

$$\Gamma=\{\partial_t,\partial_k,\Omega_{0k},\Omega_{ij},L\}$$
$$1\le k\le 3, 1\le i<j\le 3$$

Invariant Vector Fields

- Spatial Rotations

$$\Omega_{ij}=x_i\partial_j-x_j\partial_i$$

- Hyperbolic Rotations

$$\Omega_{0k}=t\partial_k+x_k\partial_t$$

- Scaling

$$L=t\partial_t+r\partial_r$$

Invariant Vector Fields (2)

- All of the vector fields preserve the equation $\Box u=0$
- Generators of the Poincaré group.
- Using these, we want to set up a continuity argument. In this case, it is a coupling between a dispersive estimate and an energy estimate.

Continuity Argument

$$\sum_{|\alpha|\le k}|\Gamma^\alpha u(t,x)|\le\frac{A\varepsilon}{(1+t+|x|)}$$

$$\sum_{|\alpha|\le k+3}\left\|\Gamma^\alpha u'(t,\cdot)\right\|_2\le C_1(1+t)^{C_0\varepsilon}\sum_{|\alpha|\le k+3}\left\|\Gamma^\alpha u'(0,\cdot)\right\|_2$$

The Continuity Argument (2)

- There are two basic steps:
  - Show that the energy estimate follows from the pointwise estimate.
  - Show that the pointwise estimate with A replaced by A/2 under the assumption of the energy estimate and the pointwise estimate.
- In order to accomplish the latter, we must use the extra decay that follows from the null condition.

Decay from the Null Condition

$$\sum_{|\alpha|\le M}\left|\Gamma^\alpha Q(v,w)\right| \le \frac{C_M}{(1+t+|x|)}\sum_{1\le|\alpha|\le M+1}\left|\Gamma^\alpha v\right|\cdot\sum_{1\le|\alpha|\le(M+2)/2}\left|\Gamma^\alpha w\right|$$
$$+\frac{C_M}{(1+t+|x|)}\sum_{1\le|\alpha|\le(M+2)/2}\left|\Gamma^\alpha v\right|\cdot\sum_{1\le|\alpha|\le M+1}\left|\Gamma^\alpha w\right|$$

Nonrelativistic Multiple Speed Systems

$$\Box_{c_I} u^I(t,x) = \left(\partial_t^2 - c_I^2\Delta\right) u^I(t,x) = Q^I(du, d^2u)$$

(2) $u(0,x) = f(x); \quad \partial_t u(0,x) = g(x)$

$$(t,x)\in R_+\times R^3 \qquad I = 1,2,\dots,D$$
$$u = (u^1, u^2, \dots, u^D)$$
$$0 < c_D < \dots < c_1$$

Two Null Conditions

(N1) $\sum_{J=1}^{D}\sum_{j,k,l=0}^{3} B_{J,l}^{IJ,jk}\xi_j\xi_k\xi_l = 0$ whenever $\frac{\xi_0^2}{c_J^2} = \xi_1^2+\xi_2^2+\xi_3^2$

(N2) $\sum_{J=1}^{D}\sum_{j,k,l=0}^{3} B_{J,l}^{JJ,jk}\xi_j\xi_k\xi_l = 0$ whenever $\frac{\xi_0^2}{c_J^2} = \xi_1^2+\xi_2^2+\xi_3^2$

New Challenge

- The generators of the hyperbolic rotations Ω_{0k} have an associated wave speed, and thus do not seem appropriate for this setting.
- Denote, now,

$$Z = \{\partial_t, \partial_k, \Omega_{ij}\},\ 1\le k\le 3,\ 1\le i<j\le 3$$
$$\Gamma = \{Z, L\}$$

Decay from the Null Condition

- In case of (N1) or (N2), we get the following additional decay:

$$\left|\sum_{0\le j,k,l\le 3} B_{J,l}^{IJ,jk}\partial_l u\partial_j\partial_k u\right| \le C\langle r\rangle^{-1}\left(|\Gamma u||\partial^2 v| + |\partial u||\partial\Gamma v|\right)$$
$$+ C\frac{\langle c_J t - r\rangle}{\langle t+r\rangle}|\partial u||\partial^2 v|$$

Existence Results for (2)

- Almost Global Existence [Klainerman-Sideris 1996]
- Global Existence with Null Condition (N1) [Sideris 2000, Sogge 2001]
- Global Existence with Null Condition (N2) [Agemi-Yokoyama 1998, Kubota-Yokoyama 2001, Sideris-Tu 2001]

Sideris-Tu Argument

- Rather than using a coupling of a pointwise estimate with a higher order energy estimate, they couple two energy inequalities.
- By doing so, they are able to avoid direct estimation of the fundamental solution for the linear wave equation.

The Exterior Domain

- We want to look at existence exterior to a compact set with smooth boundary (an obstacle).

$$K \subset \{|x| < D\} \subset R^n$$

$$\Omega = R^n \setminus K$$

The Dirichlet Problem

$$\Box u^I(t,x) = \left(\partial_t^2 - \Delta\right) u^I(t,x) = Q^I(du, d^2u)$$

(3) $u(0,x) = f(x); \quad \partial_t u(0,x) = g(x)$

$$u(t,x) = 0 \quad when\ x \in \partial K$$

$$(t,x) \in R_+ \times \Omega \qquad I = 1,2,\ldots,D$$

$$u = (u^1, u^2, \ldots, u^D)$$

Existence results for (3)

- If we assume the null condition, there is a global solution to (3) if the obstacle is star-shaped and f,g are sufficiently small. [Keel-Smith-Sogge 2002]
- They also establish the local existence theory in the exterior domain.
- As in Christodolou, they conformally map $R_+ \times \Omega$ to the Einstein diamond.

The Nonrelativistic Dirichlet Problem

$$\Box u^I(t,x) = \left(\partial_t^2 - c_I^2\Delta\right) u^I(t,x) = Q^I(du, d^2u)$$

(4) $u(0,x) = f(x); \quad \partial_t u(0,x) = g(x)$

$$u(t,x) = 0 \quad when\ x \in \partial K$$

$$(t,x) \in R_+ \times \Omega \qquad I = 1,2,\ldots,D$$

$$u = (u^1, u^2, \ldots, u^D) \qquad 0 < c_D < \ldots < c_1$$

New Challenges

- The invariant vector fields do not preserve the Dirichlet conditions.
- The coefficients of Z remain small in a neighborhood of our (compact) obstacle. The coefficients of the scaling vector field however can grow large in a small neighborhood of the obstacle.
- No strong Huygen's principle.

Almost Global Existence to (4)

- There are almost global solutions to (4) for sufficiently small data f,g if the obstacle is star-shaped. [Keel-Smith-Sogge 2003]

Keel-Smith-Sogge Argument

- Rather than relying on the $O(t^{-1})$ decay that one obtains from the Klainerman-Sobolev inequality, they exploit the decay that follows from

$$\|h\|_{L^\infty(R/2<|x|<R)} \le CR^{-1} \sum_{|\alpha|+|\beta|\le 2} \|\Omega^\alpha \partial^\beta h\|_{L^2(R/4<|x|<2R)}$$

Keel-Smith-Sogge Argument

- To utilize this decay and to handle certain boundary terms that necessary arise when working with obstacles, KSS use a class of weighted $L_t^2 L_x^2(\langle x\rangle^{-1/2} dtdx)$ estimates. They are exterior domains analogs of

$$\left\|\langle x\rangle^{-1/2} v'\right\|_{L_t^2 L_x^2([0,T]\times R^3)} \le C\big(\ln(2+t)\big)^{1/2} \int_0^T \|\Box v(s,\cdot)\|\, ds$$

More on Exterior Domains...

- We want to look at global existence results for multiple speed systems in more general exterior domains.
- We want to look at domains with a smooth, compact (but not necessarily connected) obstacle.
- Our only other assumption is...

Exponential Local Energy Decay

- If u is a solution to $\Box u = 0$ with data f,g supported in $\{|x|<D\}$, then

(E) $$\|u'(t,\cdot)\|_{L^2(B_D)} \le Ce^{-ct} \sum_{|\alpha|\le 1} \|\partial_x^\alpha u'(0,\cdot)\|_2$$

where

$$B_D = \{|x|<D\} \cap \Omega$$

- We could allow for more "loss" on the right. By interpolation, (E) holds WLOG

Exponential Local Energy Decay

- An inequality stronger than (E) holds for any nontrapping obstacle. [e.g. Morawetz-Ralston-Strauss (1977)]
- If there are trapped rays, you must allow for some loss of derivative. [Ralston (1969)]
- (E) holds in the exterior of several convex bodies with separation [Ikawa (1998)]

Exponential Local Energy Decay

- In any exterior domain, if you allow a loss of k derivatives on the right side, the local energy is $O((\log(2+t))^{-k})$. [Burq (1998)]
  - This does not seem sufficient for our results. Since this result allows elliptic trapped rays, it seems doubtful that global existence results could hold in this setting.
- We don't actually need exponential decay.

Global Existence for (4)

- Suppose that the local energy decay estimate (E) holds and that the null condition (N1) holds. Then, there is a global solution to (4) for sufficiently small data f,g. [M.-Sogge (2003)]

Simplifying Assumptions

- By scaling, we may assume D=1.
- Using the local existence theory, we can assume that f,g vanish and that Q is supported $|x| < 10t$
  - This assumption also allows us to ignore the necessary compatibility conditions on the data f,g.

The Continuity Argument

$$(1+t+|x|)\sum_{|\alpha|\le 40}|Z^\alpha u(t,x)| \le A_0\varepsilon$$

$$(1+t+|x|)\sum_{|\alpha|+\nu\le 55,\nu\le 2}\left|L^\nu Z^\alpha u(t,x)\right| \le B_1\varepsilon(1+t)^{1/5}\log(2+t)$$

$$\sum_{|\alpha|\le 100}\left\|\partial^\alpha u'(t,\cdot)\right\|_2 \le B_2\varepsilon(1+t)^{1/20}$$

$$\sum_{|\alpha|+\nu\le 70,\nu\le 3}\left\|L^\nu Z^\alpha u'(t,\cdot)\right\|_2 \le B_3\varepsilon(1+t)^{1/10}$$

$$\sum_{|\alpha|+\nu\le 68,\nu\le 3}\left\|\langle x\rangle^{-1/2}L^\nu Z^\alpha u'\right\|_{L^2_tL^2_x(S_t)} \le B_4\varepsilon(1+t)^{1/10}(\log(2+t))^{1/2}$$

The Continuity Argument

- Again, two steps:
  - Show that the last 4 estimates follow from the first.
  - Show that the first estimate with the constant halved follows from the 5 estimates.
- For the second step, we must use the extra decay that follows from the null condition.

Key Innovation

- Establishing the energy estimates without the convenient assumption of star-shapedness.
- Allowing for the loss of derivative in the local energy decay.

Global Existence for (4)

- Suppose that the local energy decay estimate (E) holds and that the null condition (N2) holds. Then, there is a global solution to (4) for sufficiently small data f,g. [M.-Nakamura-Sogge (2003)]

Another Continuity Argument

- Three tiered argument… (1) low order energy, (2) pointwise bound on the gradient, (3) higher order energy estimate and weighted mixed norm estimates.

The Continuity Argument

$$\sum_{\mu+|\alpha|\le 52,\mu\le 2} \left\| L^\mu Z^\alpha u'(t,\cdot)\right\|_2 \le A_0\varepsilon$$

$$(1+t+|x|)\sum_{|\alpha|\le 40} |Z^\alpha u'(t,x)| \le A_1\varepsilon$$

$$(1+t+|x|)\sum_{|\alpha|+\nu\le 55,\nu\le 3} \left| L^\nu Z^\alpha u(t,x)\right| \le B_1\varepsilon^2(1+t)^{1/10}\log(2+t)$$

$$\sum_{|\alpha|\le 100} \left\| \partial^\alpha u'(t,\cdot)\right\|_2 \le B_2\varepsilon(1+t)^{1/40}$$

$$\sum_{|\alpha|+\nu\le 65,\nu\le 4} \left\| L^\nu Z^\alpha u'(t,\cdot)\right\|_2 \le B_3\varepsilon(1+t)^{1/20}$$

$$\sum_{|\alpha|+\nu\le 62,\nu\le 4} \left\| \langle x\rangle^{-1/2} L^\nu Z^\alpha u'\right\|_{L^2_t L^2_x(S_t)} \le B_4\varepsilon(1+t)^{1/20}(\log(2+t))^{1/2}$$

The Pointwise Estimate

- In order to establish the pointwise estimate in MS, we use an exterior domain analog of

$$(1+t+|x|)|v(t,x)| \le C \sum_{\mu+|\beta|\le 3,\mu\le 1} \int_0^t\int \left| L^\mu Z^\beta \Box v(s,y)\right| \frac{dy\,ds}{|y|}$$

- It is not clear how to use (N2) here.

Energy Estimate Instead

- Standard Energy Integral Method…

$$\frac{1}{2}\partial_t |u'|^2 - div_x(\partial_t u \nabla u) = \partial_t u \Box u$$

- Integrate both sides in both x and t, and apply the divergence theorem.
- Use the mixed norm estimate to handle terms on the boundary of the obstacle.

Weighted Sobolev Estimates

- To handle the multi-speed terms, we use some weighted Sobolev estimates. These are exterior domain analogs of the following [Klainerman-Sideris, Sideris, Sideris-Tu, Hidano]

$$\left\| \langle t-|x| \rangle \partial^2 u(t,\cdot) \right\|_2 \le C \sum_{|\alpha|\le 1} \left\| \Gamma^\alpha u'(t,\cdot) \right\|_2 + C \left\| \langle t+|x| \rangle \Box u(t,\cdot) \right\|_2$$

$$|x|^{1/2} \langle t-|x| \rangle |\partial u(t,x)| \le C \sum_{|\alpha|\le 2} \left\| \Gamma^\alpha u'(t,\cdot) \right\|_2 + C \sum_{|\alpha|\le 1} \left\| \langle t+|x| \rangle \Gamma^\alpha \Box u(t,\cdot) \right\|_2$$

Pointwise Gradient Estimate

- Trivial in free space since derivatives commute with the d'Alembertian.
- Not at all trivial in the exterior domain… the spatial derivatives do not preserve the Dirichlet boundary condition.
- Here we use the assumption that the forcing term is supported in $|x| < 10t$
- Then, on $|x| < t/10$, we have...

Pointwise Gradient Estimate

$$(1+t+|x|)\,|L^\nu Z^\alpha u'(t,x)| \le C \sum_{\mu+|\beta|\le \nu+|\alpha|+3,\, \mu\le\nu+1} \int_0^t \int \left| L^\mu Z^\beta \partial \Box u(s,y) \right| \frac{dy\,ds}{|y|}$$

$$+ C \sup_{0\le s\le t} (1+s) \sum_{\mu+|\beta|\le\nu+|\alpha|+4,\, \mu\le\nu} \left\| L^\mu Z^\beta \Box u(s,\cdot) \right\|_\infty$$

$$+ C \sup_{0\le s\le t} (1+s) \sum_{\mu+|\beta|\le\nu+|\alpha|+6,\, \mu\le\nu} \int_0^s \int_{\substack{||y|-(s-\tau)|\le 10 \\ |y|\le(1000+\tau)/2}} \left| L^\mu Z^\beta \Box u(\tau,y) \right| \frac{dy\,d\tau}{|y|}$$

$$+ C \sup_{0\le s\le t} (1+s) \sum_{\mu+|\beta|\le\nu+|\alpha|+7,\, \mu\le\nu+1} \int_{s/100}^s \int_{|y|\ge(1+\tau)/10} \left| L^\mu Z^\beta \Box u(\tau,y) \right| \frac{dy\,d\tau}{|y|}$$

Further Studies...

- Allow the forcing term to contain cubic terms. [In early preparation… M.-Nakamura-Sogge]
- Global Existence in Higher Dimensions. [In early preparation… M.]
- Equations of Elasticity in Exterior Domains ???