Price's law on nonstationary space-times

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Schwarzschild space-time

- $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2$
- $\Box_g = -\left(1 - \frac{2M}{r}\right)^{-1}\partial_t^2 + r^{-2}\partial_r\left(1 - \frac{2M}{r}\right)r^2\partial_r + r^{-2}\partial_\omega \cdot \partial_\omega$
- Unique spherically symmetric solution to Einstein's equations (Birkhoff)
- Trapping at $r = 2M$ (event horizon), $r = 3M$ (photon sphere)
- ∂_t Killing vector field, timelike for $r > 2M$.

Kerr space-time

- Axially symmetric, rotating black holes
- $ds^2 = g_{tt}\ dt^2 + g_{t\phi}\ dt\ d\phi + g_{rr}\ dr^2 + g_{\phi\phi}\ d\phi^2 + g_{\theta\theta}\ d\theta^2$
- $g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}, \quad g_{t\phi} = -2a\frac{2Mr\sin^2\theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}$

  $$g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \quad g_{\theta\theta} = \rho^2.$$
- $\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.$
- Domain of outer communication $r > r_+$,
  $r_+ = M + \sqrt{M^2 - a^2}$.

Price's law

- Decay of linear waves? $\Box u = 0$, compactly supported Cauchy data
- On Minkowski space ($1+3$ dimensional), $|u| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\infty}$
- On Schwarzschild, e.g., Price's conjecture: $|u| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-2}$.
- Recent proofs: Donninger-Soffer-Schlag, Tataru
- Rely heavily on the background being stationary

Permissible metrics

- g Lorentzian metric, $\Box_g = |g|^{-1/2}\partial_i g^{ij}|g|^{1/2}\partial_j$
- $t = const$ space-like
- $g = m + g_{lr} + g_{sr}$, m Minkowski metric, g_{lr} spherically symmetric and $\sim 1/r$, $g_{sr} \sim 1/r^2$.
- We also allow exteriors $\mathbb{R} \times \mathbb{R}^3 \backslash \{|x| < R_0\}$ where $\mathbb{R} \times \{|x| = R_0\}$ is outgoing space-like.
- Vector fields: $Z = \{\partial_t, \partial_i, \Omega = x_i\partial_j - x_j\partial_i, S = t\partial_t + r\partial_r\}$. We use $u_{\leq m}$ to denote the collection $Z^\alpha u$ where $|\alpha| \leq m$.
- Change of coordinates:

$$\Box_g u \approx \Box u + \frac{1}{r^3}(\Omega^2 u + u) + \frac{1}{r^2}(\nabla^2 u + \nabla u)$$

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Localized Energy Estimates (Minkowski)

$$\sup_j \Big(\|\langle r\rangle^{-1/2} u'\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^j\})} + \|\langle r\rangle^{-3/2} u\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^j\})}\Big) \lesssim \|u'(0,\,\cdot\,)\|_{L^2} + \sum_k \|\langle r\rangle^{1/2}\Box u\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^k\})}.$$

- $\|u\|_{LE} = \sup_j \|\langle r\rangle^{-1/2} u\|_{L^2_{t,x}(\mathbb{R}\times\{|x|\approx 2^j\})}$
- $\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE}$
- $\|f\|_{LE^*} = \sum_j \|\langle r\rangle^{1/2} f\|_{L^2_{t,x}(\mathbb{R}\times\{|x|\approx 2^j\})}$
- Minkowski space estimate:

$$\|u\|_{LE^1} \lesssim \|\nabla u(0)\|_{L^2} + \|\Box u\|_{LE^*}$$

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Weak Localized Energy Estimates

$$\|u\|_{LE^1_w} = \|(1-\chi)\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE}$$
$$\|f\|_{LE^*_w} = \|\chi f\|_{L^2H^1} + \|(1-\chi) f\|_{LE^*}$$

- We say that the weak local energy decay property holds if

$$\|u\|_{LE^1_w} \lesssim \|\nabla u(0)\|_{L^2} + \|\Box_g u\|_{LE^*_w}.$$

- More precisely,

$$\|u\|_{LE^{1,k}_w} \lesssim_k \|\nabla u(0)\|_{H^k} + \|\Box_g u\|_{LE^{*,k}_w}.$$

- Corollary:

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Stationary local energy decay

- We say that the stationary local energy decay property holds if on any time interval $[t_0, t_1]$ and $k \geq 0$

$$\|u\|_{LE^{1,k}[t_0,t_1]} \lesssim_k \|\nabla u(t_0)\|_{H^k} + \|\nabla u(t_1)\|_{H^k} + \|\Box_g u\|_{LE^{*,k}[t_0,t_1]} + \|\partial_t u\|_{LE^{0,k}[t_0,t_1]}.$$

- The loss (due to possible trapping) is replaced by the $\partial_t u$ term in the right.
- This is a more robust estimate. It holds much more generally than the weak local energy decay provided ∂_t is timelike near the trapped set.

Main theorem

Theorem (M.-Tataru-Tohaneanu)

For a permissible metric g which satisfies the stationary local energy decay property, assume

$$\Box_g u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

where u_0, u_1 are compactly supported. Then

$$|u(t,x)| \lesssim \kappa \frac{1}{\langle t\rangle \langle t-r\rangle^2}, \quad |\nabla u(t,x)| \lesssim \kappa \frac{1}{\langle r\rangle \langle t-r\rangle^3}$$

where $\kappa = \|u_{\leq m}\|_{LE^1}$.

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If additionally g satisfies the weak local energy decay property, then κ can be replaced by $\|\nabla u(0)\|_{H^k}$ for some k, and thus Price's law holds.

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Corollary

Price's law holds for Kerr space-times with $a \ll M$.

Applications to nonstationary spacetimes

Theorem (M.-Tataru-Tohaneanu)

Let g be a permissible metric. Let χ_{ps} be a smooth cutoff function which selects a small neighborhood of $\{r = 3M\}$. If

$$|\partial^\alpha [g_{\mu\nu} - (g_{\mathbf{S}})_{\mu\nu}]| \leq \varepsilon \langle r \rangle^{-|\alpha|-}, \quad 0 \leq |\alpha| \leq 1,$$

for ε sufficiently small, then the $k = 0$ stationary local energy decay property holds.
If additionally

$$|\partial^\alpha g^{\mu\nu}| \lesssim r^{-1-}, \quad 1 \leq |\alpha| \leq k + 1$$

then the full stationary local energy decay property holds.

Theorem (M.-Tataru-Tohaneanu)

If, in addition to the above,

$$\chi_{ps}|\partial^\alpha[g_{\mu\nu} - (g_{\mathbf{K}})_{\mu\nu}]| \le c_\alpha(t), \quad 0 \le |\alpha| \le 1,$$

where $c_\alpha \in L^1_t$, *then the weak local energy decay holds.*

Price's law for perturbations of Kerr

Theorem (M.-Tataru-Tohaneanu)

If, in addition to the above,

$$\chi_{ps}|\partial^\alpha[g_{\mu\nu} - (g_{\mathbf{K}})_{\mu\nu}]| \le c_\alpha(t), \quad 0 \le |\alpha| \le 1,$$

where $c_\alpha \in L^1_t$, then the weak local energy decay holds.

Corollary

Price's law holds for a class of time-dependent perturbations of the Kerr metric with $a \ll M$.

Initial decay bound

$$|u(t,x)| \lesssim \frac{\log \langle t-r\rangle}{\langle r\rangle \langle t-r\rangle^{1/2}} \|u_{\le n}\|_{LE^1}$$

- Recall: $\Box u \approx \frac{1}{r^2}\nabla u_{\le 1} + \frac{1}{r^3} u_{\le 2} =: F.$
- Let w be the radial solution to $\Box w = \sum_{|\alpha|\le 2} \|\Omega^\alpha F\|_{L^2(\mathbb{S}^2)} =: H.$
- It suffices to estimate $|w|$, which we can do via the one dimensional reduction:

$$rw(t,r) = \frac{1}{2}\int_0^t \int_{|r-t+s|}^{r+t-s} \rho H(s,\rho)\, d\rho\, ds.$$

Initial decay bound

- We need

$$\int_{D_{tr}} \rho H(s,\rho)\,d\rho\,ds \lesssim \frac{\log\langle t-r\rangle}{\langle t-r\rangle^{1/2}}(\|\langle r\rangle^2 H\|_{LE} + \|\langle r\rangle^2 SH\|_{LE}).$$

- We dyadically decompose the region of integration: $\rho \approx R$.
- $(t-r)/8 < R < t$: The bound follows using only Cauchy-Schwarz.
- $0 < R < (t-r)/8$: We apply the Fundamental Theorem of Calculus along the integral curves for S.

Initial decay bound

- More precisely, if $\gamma_{s,\rho}(\tau)$ is the integral curve corresponding to S (parametrized by time) with $\gamma_{s,\rho}(0) = (s,\rho)$, then

$$|H(\gamma_{s,\rho}(0))|^2 \leq \frac{1}{s}\int_0^s |H(\gamma_{s,\rho}(\tau))|^2\,d\tau + \frac{1}{s}\int_0^s |(SH)(\gamma_{s,\rho}(\tau))|^2\,d\tau.$$

- Integrating over the dyadic region, we obtain

$$\int_{D_{tr}\cap\{\rho\approx R\}} |H|^2\,ds\,d\rho \lesssim \frac{R}{t-r}\int_{\rho\approx R} |H|^2 + |SH|^2\,ds\,d\rho.$$

- From this, the first decay result follows from Cauchy-Schwarz and summation over R.

Iteration 1

$$|u| \lesssim \frac{\log \langle t-r\rangle}{t\langle t-r\rangle^{1/2}} \|u_{\le n}\|_{LE^1}, \quad |\nabla u| \lesssim \frac{\log \langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^{3/2}} \|u_{\le n}\|_{LE^1}$$

Key tools for the improvement away from the light cone:

- Sobolev embedding: With $C_T^{<T/2} = [T, 2T] \times \{|x| < T/2\}$,

$$\|w\|_{L^\infty(C_T^{<T/2})} \lesssim T^{-1/2} \sum_{i+j\le 2} \|S^i \Omega^j w\|_{LE^1(\tilde{C}_T^{<T/2})}.$$

- Stationary local energy decay.
  - For the time boundary terms, the fundamental theorem of calculus is again applied along the integral curves of S.
  - For the ∂_t error term, one can obtain decay by passing to S and utilizing that supports are away from the light cone.

Iteration 1

Key tools for the improved gradient bound:

- Off the light cone, e.g.,

$$|\nabla w|^2 \leq M\frac{1}{(t-r)^2}|Sw|^2 + \frac{t}{t-r}(|\nabla_x w|^2 - |\partial_t w|^2)$$

- Integration by parts:

$$\int \beta(|\nabla_x w|^2 - |\partial_t w|^2)\, dx\, dt = \int \Box w \cdot \beta w\, dx\, dt - \frac{1}{2}\int (\Box\beta) w^2\, dx\, dt.$$

Iteration 2

$$|u| \lesssim \frac{\log \langle t-r\rangle}{t\langle t-r\rangle}\|u_{\le n}\|_{LE^1}, \quad |\nabla u| \lesssim \frac{\log \langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^2}\|u_{\le n}\|_{LE^1}.$$

- Using the previous bounds in the one dimensional reduction,

$$|u| \lesssim \frac{\log \langle t-r\rangle}{\langle r\rangle\langle t-r\rangle}\|u_{\le n}\|_{LE^1}.$$

- Argue as in Iteration 1 to improve the bound for r small, and to obtain an improved gradient bound.

Iteration 3

$$|u| \lesssim \frac{\log^3 \langle t-r\rangle}{t\langle t-r\rangle^2}\|u_{\le n}\|_{LE^1}, \quad |\nabla u| \lesssim \frac{\log^3 \langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^3}\|u_{\le n}\|_{LE^1}.$$

- As above, it suffices to prove the first bound with the t decay replaced by r decay.
- One dimensional reduction does not suffice. It misses a cancellation.
- The worst contribution to the one dimensional reduction contains a derivative which we can exploit via the following lemma.

Iteration 3

Lemma

For a smooth f supported in $\{\frac{t}{2} \leq r \leq t\}$ with

$$|f| + |Sf| + |\Omega f| + \langle t-r\rangle |\partial_r f| \lesssim \frac{1}{t^3 \langle t-r\rangle \log^2 \langle t-r\rangle},$$

the forward solution u to $\Box u = \partial_t f$ satisfies

$$|u| \lesssim \frac{1}{t\langle t-r\rangle^2}.$$

Iteration 4

$$|u| \lesssim \frac{1}{t\langle t-r\rangle^2}\|u_{\le n}\|_{LE^1}, \quad |\nabla u| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^3}\|u_{\le n}\|_{LE^1}.$$

- In essence, the arguments of Iteration 3 are repeated.

This completes the proof of the main theorem.

Localized energy estimates on Schwarzschild

$$\Box_g = -\Big(1-\frac{2M}{r}\Big)^{-1}\partial_t^2 + r^{-2}\partial_r\Big(1-\frac{2M}{r}\Big)r^2\partial_r + \nabla\!\!\!/ \cdot \nabla\!\!\!/$$

$$(-\Box_g u)\Big[a(r)\Big(1-\frac{2M}{r}\Big)\partial_r u + \frac{1}{2}\Big(1-\frac{2M}{r}\Big)r^{-2}\partial_r(r^2 a(r))u\Big]$$

Integration by parts:

$$\int a(r)\partial_r u\partial_t u\;dx\Big|_0^T + \frac{1}{2}\int r^{-2}\partial_r(r^2a(r))(\partial_t u)u\;dx\Big|_0^T$$
$$+\int\int\Big(1-\frac{2M}{r}\Big)^2 a'(r)(\partial_r u)^2\;dV + \int\int a(r)\frac{r-3M}{r^2}|\nabla\!\!\!/ u|^2\;dV$$
$$-\frac{1}{4}\int\int r^{-2}\partial_r\Big\{\Big(1-\frac{2M}{r}\Big)r^2\Big[\Big(1-\frac{2M}{r}\Big)r^{-2}\partial_r\Big(r^2a(r)\Big)\Big]\Big\}u^2\,dV$$

Localized energy estimates on Schwarzschild

$$a(r) = r^{-2}\Big((r-3M)(r+2M) + 6M^2\log\Big(\frac{r-2M}{M}\Big)\Big) =: r^{-2}R$$

Replace by $a_\varepsilon(r) = \frac{1}{r^2}\varepsilon^{-1}f(\varepsilon R)$

This gives estimates for norm

$$\int\Big(\frac{1}{r^2}(\partial_r u)^2 + \Big(1-\frac{3M}{r}\Big)^2\Big(\frac{1}{r^2}(\partial_{\tilde{t}}u)^2 + \frac{1}{r}|\not\nabla u|^2\Big) + \frac{1}{r^4}u^2\Big)\, dV.$$