Dispersive Estimates for Wave Equations on Black Hole Backgrounds

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Dispersive estimates

- Integrated localized energy estimates

$$\sup_j \Big(\|\langle r\rangle^{-1/2} u'\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^j\})} + \|\langle r\rangle^{-3/2} u\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^j\})}\Big) \lesssim \|u'(0,\,\cdot\,)\|_{L^2} + \sum_k \|\langle r\rangle^{1/2}\Box u\|_{L^2_{t,x}(\mathbb{R}_+\times\{\langle r\rangle\approx 2^k\})}.$$

- Strichartz estimates

$$\||D|^{-\rho}\nabla u\|_{L^pL^q} \lesssim \|\nabla u(0,\,\cdot\,)\|_2 + \||D|^{\tilde\rho}\Box u\|_{L^{\tilde p'}L^{\tilde q'}}.$$

$$2\le p,q\le\infty,\quad \rho=\frac{n}{2}-\frac{n}{q}-\frac{1}{p},\quad \frac{2}{p}\le\frac{n-1}{2}\Big(1-\frac{2}{q}\Big)$$

- Pointwise decay estimates: $|u|\lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-2}$.

Schwarzschild space-time

- $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2$
- $\Box_g = -\left(1 - \frac{2M}{r}\right)^{-1}\partial_t^2 + r^{-2}\partial_r\left(1 - \frac{2M}{r}\right)r^2\partial_r + r^{-2}\partial_\omega \cdot \partial_\omega$
- Unique spherically symmetric solution to Einstein's equations (Birkhoff)
- Trapping at $r = 2M$ (event horizon), $r = 3M$ (photon sphere)
- ∂_t Killing vector field, timelike for $r > 2M$.

Kerr space-time

- Axially symmetric, rotating black holes
- $ds^2 = g_{tt}\ dt^2 + g_{t\phi}\ dt\ d\phi + g_{rr}\ dr^2 + g_{\phi\phi}\ d\phi^2 + g_{\theta\theta}\ d\theta^2$
- $g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}, \quad g_{t\phi} = -2a\frac{2Mr\sin^2\theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}$

  $$g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \quad g_{\theta\theta} = \rho^2.$$
- $\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.$
- Domain of outer communication $r > r_+$,
  $r_+ = M + \sqrt{M^2 - a^2}$.

Localized energy estimates on Schwarzschild

$$\Box_g = -\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2 + r^{-2}\partial_r\left(1-\frac{2M}{r}\right)r^2\partial_r + \nabla\!\!\!/ \cdot \nabla\!\!\!/$$

Multiplier: $a(r)\left(1-\frac{2M}{r}\right)\partial_r u + \frac{1}{2}\left(1-\frac{2M}{r}\right)r^{-2}\partial_r(r^2 a(r))u$

$$a(r) = r^{-2}\left((r-3M)(r+2M) + 6M^2\log\left(\frac{r-2M}{M}\right)\right) =: r^{-2}R$$

Replace by $a_\varepsilon(r) = \frac{1}{r^2}\varepsilon^{-1}f(\varepsilon R)$

This gives estimates for norm

$$\int\left(\frac{1}{r^2}(\partial_r u)^2 + \left(1-\frac{3M}{r}\right)^2\left(\frac{1}{r^2}(\partial_{\tilde{t}}u)^2 + \frac{1}{r}|\nabla\!\!\!/ u|^2\right) + \frac{1}{r^4}u^2\right)\,dV.$$

Logloss localized energy estimates

Theorem (Marzuola-M.-Tataru-Tohaneanu)

The quadratic loss $\left(1-\frac{3M}{r}\right)^2$ *can be replaced by* $\left(1-\log\left|1-\frac{3M}{r}\right|\right)^{-2}$.

Ideas of proof:

- After localization, expanding in terms of spherical harmonics, applying the Fourier transform in time, the most essential piece is modeled by

$$u'' - \lambda^2(x^2 \pm \varepsilon)u = f$$

near $x = 0$. A WKB expansion and an elliptic estimate are then used.

Strichartz estimates on Schwarzschild

Theorem (Marzuola-M.-Tataru-Tohaneanu)

The non-sharp Strichartz estimates, $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, *hold on Schwarzschild.*

Ideas of proof:

- Near the event horizon, one can apply local-in-time Strichartz estimates and sum them using the localized energy estimates.
- Near the photon sphere, one argues similarly, but the loss due to trapping must be addressed. The lack of estimates in the case of equality is due to this.
- Near infinity, the parametrix construction of M.-Tataru can be applied.

Price's law

- Decay of linear waves? $\Box u = 0$, compactly supported Cauchy data
- On Minkowski space ($1+3$ dimensional), $|u| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\infty}$
- On Schwarzschild, e.g., Price's conjecture: $|u| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-2}$.
- Recent proofs: Donninger-Soffer-Schlag, Tataru
- Rely heavily on the background being stationary