

Global Strichartz Estimates for Solutions of the Wave Equation in Exterior Domains

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The Exterior Domain

- ◆ Fix a smooth, compact obstacle $\kappa \subset \{x \mid |x| \leq R\} \subset \mathbb{R}^n$
- ◆ Let $\Omega = \mathbb{R}^n \setminus \kappa$
- ◆ We will assume that our exterior domain is nontrapping. That is, there is a T_R so that no geodesic of length T_R is completely contained in $\{x \mid |x| \leq R\} \cap \Omega$

The Wave Equation

$$n \geq 2$$

$$\square u(t, x) = (\partial_t^2 - \Delta) u(t, x) = F(t, x)$$

$$(*) \quad u(0, x) = f(x); \quad \partial_t u(0, x) = g(x)$$

$$u(t, x)|_{\partial\Omega} = 0 \quad (t, x) \in \mathbb{R} \times \Omega$$

Admissibility

- (1) Assume $1 \leq r, s \leq 2 \leq p, q \leq \infty; \quad r < 2$
- (2) [Local Strichartz Estimates] For $f, g, F(t, \cdot)$ supported in $\{x \mid |x| \leq R\}$, solutions of (*) satisfy

$$\|u\|_{L_t^p L_x^q((0,1) \times \Omega)} + \sup_{0 \leq t \leq 1} \|u(t, \cdot)\|_{\dot{H}_B^s(\Omega)} + \sup_{0 \leq t \leq 1} \|\partial_t u(t, \cdot)\|_{\dot{H}_B^{s-1}(\Omega)} \leq C \left[\|f\|_{\dot{H}_B^s(\Omega)} + \|g\|_{\dot{H}_B^{s-1}(\Omega)} + \|F\|_{L_t^r L_x^q((0,1) \times \Omega)} \right]$$

Admissibility, continued

- (3) [Global Minkowski Strichartz Estimates] In the case that $\Omega = \mathbb{R}^n$, solutions to (*) satisfy

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^{1+n})} + \sup_t \|u(t, \cdot)\|_{\dot{H}_B^s(\mathbb{R}^n)} + \sup_t \|\partial_t u(t, \cdot)\|_{\dot{H}_B^{s-1}(\mathbb{R}^n)} \leq C \left[\|f\|_{\dot{H}_B^s(\mathbb{R}^n)} + \|g\|_{\dot{H}_B^{s-1}(\mathbb{R}^n)} + \|F\|_{L_t^r L_x^q(\mathbb{R}^{1+n})} \right]$$

- (4) If $n = 2$, then $q > 2$

Global Strichartz Estimates

- ◆ Theorem [Metcalfe, Burq independently]: For admissible (p, q, r, s, γ) , solutions u to (*) satisfy

$$\|u\|_{L_t^p L_x^q(\mathbb{R} \times \Omega)} \leq C \left[\|f\|_{\dot{H}_B^s(\Omega)} + \|g\|_{\dot{H}_B^{s-1}(\Omega)} + \|F\|_{L_t^r L_x^q(\mathbb{R} \times \Omega)} \right]$$

Global Minkowski Strichartz Estimate

- ◆ The Minkowski Strichartz Estimate (3) is a generalized version of the work of Strichartz [early 70's] in the conformal case:

$$p = q = \frac{2(n+1)}{n-1}, r = s = \frac{2(n+1)}{n+3}, \gamma = 1/2$$

- ◆ The generalization is based on the works of: Genibre-Velo, Pecher, Kapitanski, Lindblad-Sogge, Mockenhaupt-Seeger-Sogge, Keel-Tao

Global Minkowski Strichartz Estimates

- ◆ The estimate (3) is a Minkowski analog of the Hardy-Littlewood-Sobolev inequality.
- ◆ It is also related to certain restriction problems for the Fourier transform.
- ◆ Typical application: Show that nonlinear equations of the form (in $n=3$)
 $\square u = |u|^3$
 have a global solution for small data.

Necessary Condition for Admissibility

- ◆ It is well-known that in the homogeneous case ($F=0$), (3) holds if and only if

$$n \geq 2, 2 \leq p \leq \infty, 2 \leq q < \infty,$$

$$(5) \quad \gamma = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \quad \frac{2}{p} \leq \frac{n-1}{2} \left(1 - \frac{2}{q}\right)$$

Local Strichartz Estimates

- ◆ The local estimates (2) were first proved in the homogeneous case ($F=0$) when the obstacle is convex by Smith-Sogge ['95].
- ◆ Application: Studied the critical wave equation in $n=3$, $\square u = -|u|^5$
- ◆ Smith-Sogge later showed that inhomogeneous estimates follow from the homogeneous estimates using a lemma of Christ-Kiselev.

Admissibility

- ◆ It is known that we have admissibility for a range of coefficients if the obstacle is convex:

$$n \geq 3, \quad q, s' < \frac{2(n-1)}{n-3}, \quad \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{r} + \frac{n}{s} - 2$$

$$\frac{1}{p} = \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), \quad \frac{1}{r'} = \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{s'} \right)$$

- ◆ In particular, we have admissibility in the conformal case.
- ◆ Admissibility is not known, however, if the obstacle is not convex.

Global Strichartz Estimates

- ◆ Smith-Sogge ['00] proved the main theorem in odd spatial dimensions.
- ◆ Here we extend their result to all spatial dimensions $n \geq 2$.
- ◆ For simplicity, let's stick to the case $n \geq 4$.
- ◆ By scaling, we may also assume $R = 1/2$.

Local Energy Decay

- ◆ Let β be a smooth function supported in $\{|x| \leq 1\}$. For $n \geq 2$ even, $0 \leq \gamma \leq n/2$, $F = 0$, and f, g supported in $\{|x| \leq 1\}$, then solutions of (*) satisfy

$$(6) \quad \|\beta(\cdot)u(t, \cdot)\|_{\dot{H}_b^\gamma(\Omega)} + \|\beta(\cdot)\partial_t u(t, \cdot)\|_{\dot{H}_b^{\gamma-1}(\Omega)} \leq C(1+|t|)^{-\frac{n}{2}} (\|f\|_{\dot{H}_b^\gamma(\Omega)} + \|g\|_{\dot{H}_b^{\gamma-1}(\Omega)})$$

Local Energy Decay

- ◆ The local energy decay (6) is due to Melrose [79] $n \geq 4$ and Morawetz [75] $n = 2$.
- ◆ It is here that we require the nontrapping assumption on the exterior domain.
- ◆ In odd spatial dimensions, the local energy decay is exponential.

Huygen's Principle

- ◆ In Minkowski space, a solution to the wave equation in odd spatial dimensions satisfies strong Huygen's principle.
- ◆ In even spatial dimensions or in the exterior domain, we only have weak Huygen's principle.
- ◆ We will thus look at "weighted" estimates.

Weighted Strichartz Estimates in Free Space

- ◆ We want to look at the homogeneous wave equation in free space:

$$(**) \quad \begin{cases} \square w(t, x) = 0 & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ w(0, x) = f(x); \quad \partial_t w(0, x) = g(x) \end{cases}$$

- ◆ For admissible p, q, γ and f, g supported in $\{|x| \leq 1\}$, we have that solutions w to (**) satisfy

$$(7) \quad \|(t-|x|)w\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n)} \leq C [\|f\|_{\dot{H}_b^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}_b^{\gamma-1}(\mathbb{R}^n)}]$$

Sketch of proof...

- ◆ Interpolate between a weighted energy estimate
- $$(8) \quad \|(t-|x|)\partial_t w\|_{L_t^2 L_x^\infty(\mathbb{R} \times \mathbb{R}^n)} \leq C [\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}]$$
- and a weighted dispersive estimate.

$$(9) \quad \|(t-|x|)\partial_t w\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^n)} \leq C t^{-(n-1)/2} [\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}]$$

- ◆ "Divide" through by a time derivative.
- ◆ Use the necessary condition for admissibility (5) to show that there is sufficient integrability in t .

Weighted Energy Estimate

- ◆ The weighted energy estimate (8) follows from slight modifications of the arguments of Morawetz, Hormander.

- ◆ Basic idea: Standard energy integral method

$$\frac{1}{2} \partial_t |u|^2 - \sum_{j=1}^n \partial_j (\partial_t u \partial_j u) = \partial_t u \square u$$

Then integrate over a cylinder and apply the divergence theorem.

Weighted Dispersive Estimate

- ◆ Apply Littlewood-Paley dyadic decomposition.
- ◆ There are then two cases to consider:
 - In the case where $|x| < t/2$, the estimate follows from integration by parts.
 - In the case where $|x| \geq t/2$, the estimate follows from the theory of oscillatory integrals. In particular, we apply the following lemma...

Oscillatory Integrals...

- ◆ Theorem [see, e.g., Sogge]: Let $d\sigma(\omega)$ be the induced Lebesgue measure on the sphere S^{n-1} . Then

$$d\hat{\sigma}(\xi) = \int e^{i\xi\omega} d\sigma(\omega) = a_+(\xi)e^{i|\xi|} + a_-(\xi)e^{-i|\xi|}$$

where $\left(\frac{\partial}{\partial \xi}\right)^\alpha a_\pm(\xi) \leq C_\alpha (1+|\xi|)^{\frac{n-1}{2}-|\alpha|}$

- ◆ In $n=3$,

$$d\hat{\sigma}(\xi) = 4\pi \frac{\sin|\xi|}{|\xi|}$$

Weighted Strichartz Estimates in the Exterior Domain

- ◆ Let u be a solution to (*) with forcing term F replaced by $F+G$. Assume that F, G are supported in $\{t \leq 1\} \times \{|x| \leq 1\}$ and the initial data f, g are supported in $\{|x| \leq 1\}$. Then, for admissible (p, q, r, s, γ) there is a $\theta > 1/2$ so that

$$(10) \quad \left\| (t-|x|+2)^\theta u(t, x) \right\|_{L_t^p L_x^q(\mathbb{R}^3 \times \mathbb{R})} \leq C \left(\|f\|_{\dot{H}_B^s(\Omega)} + \|g\|_{\dot{H}_B^{s-1}(\Omega)} + \|F\|_{L_t^r L_x^q(\mathbb{R}^3 \times \mathbb{R})} + \int \|G(t, \cdot)\|_{\dot{H}_B^{s-1}(\Omega)} dt \right)$$

Sketch of the proof...

- ◆ On $t \in [0, 1]$, the result follows from (2) and Duhamel's principle. Moreover, we may assume then that $F=G=0$.
- ◆ For a fixed, smooth β with $\beta(x) \equiv 1$ for $|x| \leq 1/2$ and $\beta(x) \equiv 0$ for $|x| \geq 1$, write

$$u(t, x) = \beta(x)u(t, x) + (1-\beta(x))u(t, x)$$

Sketch of the proof, cont...

- ◆ For the piece near the obstacle, we apply the local Strichartz estimate (2) and local energy decay (6).
- ◆ The piece away from the obstacle solves a Minkowski wave equation. Thus the result for this piece follows from (7).

Proof of the Main Theorem...

- ◆ Write

$$u(t, x) = u_0(t, x) - u_r(t, x) = (1-\beta(x))u_0(t, x) + [\beta(x)u_0(t, x) - u_r(t, x)]$$
 where u_0 solves the free wave equation with the same data and forcing term.
- ◆ The result for the $(1-\beta)u_0$ piece, then follows from (3).

Proof of Main Theorem, cont.

- ◆ For the other piece, break it into solutions whose forcing terms are also compactly supported in time.
- ◆ Use the weighted Strichartz estimates (10) on each individual piece.
- ◆ Use the following lemma that follows from Plancherel's theorem and the global Strichartz estimates (3) to piece the sum back together...

Mixed Estimate

- ◆ Assume that the Minkowski Strichartz estimate (3) holds, that $\gamma \leq n/2$, and that $r < 2$. For solutions to (*) in the case that $\Omega = \mathbb{R}^n$, we have

$$\sup_{|t| \leq 1} \int_{-\infty}^{\infty} \left\| \beta(\cdot) \partial_x^\alpha w(t, \cdot) \right\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^n)}^2 dt \leq C \left(\|f\|_{\dot{H}_y^{\gamma}(\mathbb{R}^n)} + \|g\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^r L_x^q(\mathbb{R}^{1+n})} \right)^p$$