Nonlinear Hyperbolic Equations in Waveguides

Jason Metcalfe
Georgia Institute of Technology

November 17, 2004

Waveguide

- $\Omega \subset R^d$ denotes a nonempty, bounded domain with smooth boundary.
- Waveguide: $R^n \times \Omega$
- D'Alembertian on $(t,x,y) \in R_+ \times R^n \times \Omega$:

$$\Box = \partial_t^2 - (\Delta + \Delta_\Omega)$$

- Boundary conditions:
  - Dirichlet: $u(t,x,y)\big|_{y\in\partial\Omega} = 0$
  - Neumann: $\partial_\nu u(t,x,y)\big|_{y\in\partial\Omega} = 0$

Wave and Klein-Gordon Equation

$$\begin{aligned}(\Box + m^2)u &= Q(u,u',u'') \\ u(0,x,y) = f(x,y), &\quad \partial_t u(0,x,y) = g(x,y)\end{aligned} \tag{1}$$

Q is quadratic in its arguments. $u' = \nabla_{t,x,y} u$

Assume that

$$f(x,y) = g(x,y) = 0, \quad |x| > B$$

$$\|f\|_{H^N(R^n\times\Omega)} + \|g\|_{H^{N-1}(R^n\times\Omega)} \le \varepsilon$$

Global Existence for Dirichlet Boundary Conditions

Theorem 1 [M.-Sogge-Stewart]: Suppose that $m \ge 0$ and $n \ge 3$. Assume that the Cauchy data $(f,g) \in C^\infty(R^n \times \Omega)$ are as above and satisfy the relevant Dirichlet compatibility conditions. Then (1) with Dirichlet boundary conditions has a global smooth solution if N is sufficiently large and ε is sufficiently small.

Previous Work

- Lesky-Racke first obtained this result when $n \ge 5$
  - Relies on $L^p \to L^{p/(p-1)}$ decay estimates for the linear equation that are reminiscent to those of Marshall-Strauss-Wainger
  - Use the arguments of Shibata-Tsutsumi to show global existence.
- Here, we obtain results that are sharp in terms of the dimension.
- Let's restrict our discussion to the n=3 case.

Idea of Proof

- Prove linear estimates in the boundaryless case that are independent of the mass term.
- Project onto the eigenspaces to prove linear estimates in the waveguide. Since there are no zero eigenvalues in the Dirichlet case, these estimates enjoy $O(t^{-3/2})$ decay (vs. the usual $O(t^{-1})$ decay for boundaryless wave equations).
- Use these a priori estimates and energy estimates, as in Klainerman, to show global existence

Invariant Vector Fields

$$\{\Gamma_{t,x}\}=\{\partial_t,\partial_x,\Omega_{jk}:0\le j<k\le n\}$$
$$\Omega_{jk}=x_j\partial_k-x_k\partial_j;\quad 1\le j<k\le n$$
$$\Omega_{0k}=x_k\partial_t+t\partial_k;\quad 1\le k\le n$$

$$\{\Gamma\}=\{\Gamma_{t,x}\}\cup\{\partial_y\}$$

Linear Estimates (Boundaryless)

Proposition: Suppose that $u\in C^\infty(R\times R^3)$ satisfies $u(t,x)=0,\ t\le 2B$ where B is a fixed positive constant. Suppose $(\Box_{R^{1+3}}+\mu^2)u=0$ for $|x|>t-B$. Then there is a constant depending only on B so that when $\mu\ge 1$:

(See, e.g., Hörmander)

$$\sup_x t^{3/2}|u(t,x)|$$
$$\le C\sum_{|\alpha|\le 5}\sum_k \sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]} 2^k\left\|\Gamma_{t,x}^\alpha(\Box_{R^{1+3}}+\mu^2)u(\tau,\cdot)\right\|_2$$

Spectral Theory Review

- Since the domain is compact with smooth boundary, we know that the spectrum of $-\Delta_\Omega$ is discrete and nonnegative. Moreover, if $h|_{\partial\Omega}=0$, we have $0<\lambda_1^2\le\lambda_2^2\le\dots$
- Let $E_j:L^2(\Omega)\to L^2(\Omega)$ denote the projection onto the jth eigenspace. Thus,
$$-\Delta_\Omega E_jh(x)=\lambda_j^2E_jh(x),\quad h\in L^2(\Omega)$$
and
$$\|h\|_{L^2(\Omega)}^2=\sum_{j=1}^\infty\|E_jh\|_2^2$$

Spectral Theory Review

- Weyl formula gives $\lambda_j\approx j^{1/d},\ j=2,3,\dots$
- Thus, for $h\in C^\infty(\overline{\Omega})$,
$$(1+j)^{2/d}\|E_jh\|_{L^2(\Omega)}\le C\|(I-\Delta_\Omega)E_jh\|_{L^2(\Omega)}$$
$$=C\|E_j(I-\Delta_\Omega)h\|_{L^2(\Omega)}$$
- Elliptic Regularity:
$$\sum_{|\alpha|\le N}\|\partial_y^\alpha h\|_{L^2(\Omega)}\le C\sum_{|\alpha|\le N-2}\|\partial_y^\alpha\Delta_\Omega h\|_{L^2(\Omega)}+C\|h\|_{L^2(\Omega)}$$

Spectral Theory

- $E_j(\Box+m^2)u(t,x,y)=(\partial_t^2-\Delta_{R^n}+m^2+\lambda_j^2)E_ju(t,x,y)$
- $$\sum_j\left\|(\partial_t^2-\Delta_{R^n}+m^2+\lambda_j^2)E_ju(t,x,\cdot)\right\|_{L^2(\Omega)}^2$$
$$=\left\|(\Box+m^2)u(t,x,\cdot)\right\|_{L^2(\Omega)}^2$$
- $$\sum_{|\alpha|\le N}\|\partial_y^\alpha u(t,x,\cdot)\|_{L^2(\Omega)}\le C\sum_{\substack{|\alpha|+|\beta|\le N\\ |\beta|\le 1}}\|\partial_{t,x}^\alpha\partial_y^\beta u(t,x,\cdot)\|_{L^2(\Omega)}$$
$$+C\sum_{|\alpha|\le N-2}\|\partial_{t,x,y}^\alpha(\Box+m^2)u(t,x,\cdot)\|_{L^2(\Omega)}$$

Main Estimate

Proposition: Fix B and suppose $u\in C^\infty(R\times R^3\times\overline{\Omega})$ satisfies $u(t,x,y)=0,\ t\le 2B$ and $(\Box+m^2)u(t,x,y)=0$ if $|x|>t-B$. Suppose also that $m\ge 0$ and $u(t,x,y)=0,\ y\in\partial\Omega$. Then,

$$t^{3/2}\left|\Gamma^\beta u(t,x,y)\right|$$
$$\le C\sum_{|\alpha|\le|\beta|+5+(5d+4)/2}\sum_k\sup_{\tau\in[2^{k-1},2^{k+1}]\cap[2B,t]}2^k\left\|\Gamma^\alpha(\Box+m^2)u(\tau,\cdot)\right\|_2$$
$$+Ct^{3/2}\sum_{|\alpha|\le|\beta|+(5d+5)/2}\left\|\Gamma^\alpha(\Box+m^2)u(t,\cdot)\right\|_2$$

Proof of Main Estimate

- It suffices to consider $\Gamma^{\beta} = \partial_y^{\beta}$
- Use Sobolev estimates in the y-variable
- Project onto the eigenspaces
- Use the estimate for the boundaryless case which is uniform in the mass term

- When convenient, we write: $x_0 = t,\ x_{3+j} = y_j$

Energy Estimates

If w solves $\left(\Box + m^2\right)w + \sum_{j,k=0}^{3+d} \gamma^{jk}(t,x,y)\partial_j\partial_k w = F,$

$w(t,x,y) = 0, \quad y \in \partial\Omega; \qquad w(t,x,y) = 0, \quad t \le 2B$

vanishes for large $|x|$, and $\sum_{j,k=0}^{n+d} |\gamma^{jk}| \le 1/2$, then

$$\left\|\nabla_{t,x,y} w(t,\cdot)\right\|_2 + m\left\|w(t,\cdot)\right\|_2 \le C \exp\left(\int_0^t 2\sum_{i,j,k=0}^{3+d} \left\|\partial_i \gamma^{jk}(s,\cdot)\right\|_\infty ds\right) \int_0^t \left\|F(s,\cdot)\right\| ds$$

Energy Estimates

and, using elliptic regularity,

$$\left\|\Gamma^{\alpha}\nabla_{t,x,y} w(t,\cdot)\right\|_2 + m\left\|\Gamma^{\alpha} w(t,\cdot)\right\|_2 \le C \exp\left(\int_0^t 2\sum_{i,j,k=0}^{3+d} \left\|\partial_i \gamma^{jk}(s,\cdot)\right\|_\infty ds\right) \sum_{|\beta|\le|\alpha|} \int_0^t \left\|\Gamma_{t,x}^{\beta} F(s,\cdot)\right\| ds$$
$$+ C \exp\left(\int_0^t 2\sum_{i,j,k=0}^{3+d} \left\|\partial_i \gamma^{jk}(s,\cdot)\right\|_\infty ds\right) \sum_{|\beta|\le|\alpha|} \sum_{j,k=0}^{3+d} \int_0^t \left\|[\gamma^{jk}, \Gamma_{t,x}^{\beta}]\partial_j\partial_k w(s,\cdot)\right\|_2 ds$$
$$+ C \sum_{|\beta|\le N-1} \left\|\partial_{t,x,y}^{\beta}\left(\Box + m^2\right) w(t,\cdot)\right\|_2$$

Proof of Theorem 1

- Using local existence theory, one can reduce to the case of vanishing initial data.
- The proof of Theorem 1 follows using a standard continuity argument (of Klainerman) which uses a coupling between the pointwise estimate and the energy estimate.
- When m=0, we use Poincaré's lemma to get

$$\left\|w(t,\cdot)\right\|_2 \le C\left\|\nabla_y w(t,\cdot)\right\|_2$$

Neumann boundary conditions

- Two new challenges:
  - Since $-\Delta_{\Omega}$ has zero eigenmodes for Neumann boundary conditions, the previous arguments do not work for wave equations ($m = 0$).
  - Without further assumption, the energy integral method fails. We, thus, introduce a natural Neumann nonlinear compatibility condition on the quasilinear terms.

Neumann nonlinear compatibility condition

- We first expand our nonlinear term in (1) as follows:

$$Q(u,u',u'') = \sum_{0\le j,k,l\le 3+d} A_l^{jk}\partial_l u \partial_j\partial_k u + u \sum_{0\le j,k\le 3+d} A^{jk}\partial_j\partial_k u + R(u,u')$$

- Here, $R(u,u')$ is a quadratic function.

Neumann nonlinear compatibility condition

- The necessary assumption is:

$$\sum_{0\le j,k,l\le 3+d} A_l^{jk}\xi_l\eta_j\vartheta_k = 0 \quad \text{and} \quad \sum_{0\le j,k\le 3+d} A^{jk}\xi_j\vartheta_k = 0$$

if $(\vartheta,\xi,\eta)\in X$

where

$$X = \left\{ \begin{array}{r} (\vartheta,\xi,\eta) : \vartheta = (0,0,0,0,\nu_1,\nu_2,\ldots\nu_d), \\ \xi\cdot\vartheta = 0, \eta\cdot\vartheta = 0,\, y\in\partial\Omega \end{array} \right\}$$

Neumann nonlinear compatibility condition

- In order to prove the energy estimate mentioned previously in the Neumann case, we require $\sum_{0\le j,k\le 3+d}\gamma^{jk}(t,x,y)\xi_j\vartheta_k = 0, \text{if } \ y\in\partial\Omega,$

$$\vartheta = (0,0,0,0,\nu_1(y),\ldots,\nu_d(y)), \quad \xi\cdot\nu(y) = 0$$

- When applying this with

$$\gamma^{jk}(t,x,y) = \sum_{l=0}^{3+d} A_l^{jk}\partial_l u + uA^{jk}$$

as we do, we see that the Neumann nonlinear compatibility conditions suffices.

Global Existence for Neumann Boundary Conditions

Theorem 1 [M.-Sogge-Stewart]: Suppose that $m>0$ and $n\ge 3$. Assume that the Cauchy data $(f,g)\in C^\infty(R^n\times\Omega)$ are as above and satisfy the relevant Neumann compatibility conditions. Moreover, assume that (1) satisfies the Neumann nonlinear compatibility condition. Then (1) with Neumann boundary conditions has a global smooth solution if N is sufficiently large and ε is sufficiently small.

Neumann-wave equations

- We can handle some simplified equations using techniques developed by Keel-Smith-Sogge and M. for wave equations in exterior domains.
- In particular, let's look at:

$$\Box u = Q(\nabla_{t,x}u) \qquad (3)$$

- Then, we have…

Existence for Neumann-wave equations

Theorem 3 [M.-Sogge-Stewart]: Assume that the initial data are as in Theorem 2. Then, if $n\ge 4$, (3) with Neumann boundary conditions has a global solution if ε is sufficiently small. If $n=3$ and ε is sufficiently small, then there is a constant c so that if $T_\varepsilon = \exp(c/\varepsilon)$ then (3) has a solution $u\in C^\infty([0,T_\varepsilon)\times R^3\times\Omega)$.

Proof of Theorem 3:

- The proof uses arguments similar to those of Keel-Smith-Sogge and M. for wave equations in exterior domains.
- The main linear estimate is the following weighted mixed norm estimate (due to Keel-Smith-Sogge). It follows easily from the local version which can be shown using arguments reminiscent to those of Smith-Sogge.

Linear (Boundaryless) Estimate

- If u is smooth, has vanishing initial data, and vanishes for large $|x|$ for any fixed time, then

$$(\ln(2+T))^{1/2}\left(\int_0^T\int_{R^n}\left|\langle x\rangle^{-1/2}u'(s,x)\right|^2 dxds\right)^{1/2}$$
$$\le C\int_0^T\left\|\left(\Box+\mu^2\right)u(s,\cdot)\right\|_2 ds$$

- For $n\ge 3$, the constant is independent of $\mu\ge 0$

Local (Boundaryless) Estimate

$$\int_{-\infty}^{\infty}\left\|\beta(\cdot)(e^{it\sqrt{\mu^2-\Delta}}f)(t,\cdot)\right\|^2_{H^1(R^n)}dt\le C\int_{R^n}\left|\hat f(\xi)\right|^2(\mu^2+\xi^2)d\xi$$

If β is a fixed smooth function supported in $\{|x|<1\}$

- The above follows from Plancherel's and Cauchy-Schwarz. (See Smith-Sogge)
- By Duhamel's, this implies that the previous estimate holds over $|x|<1$.
- The general estimate follows from scaling.

More remarks

- Using a similar argument involving projections onto the eigenspaces, one can extend the weighted mixed-norm estimate to waveguides.
- To illustrate the iteration argument of Keel-Smith-Sogge which uses this estimate, let's prove almost global existence for a similar nonlinear wave equation in Minkowski space.

Keel-Smith-Sogge Argument

$$\Box u=(\partial_t^2-\Delta)u=Q(u'),\quad (t,x)\in R\times R^3$$
$$u(0,\cdot)=f,\quad \partial_t u(0,\cdot)=g$$

Assume that we have small initial data.

$$\sum_{|\alpha|+j\le 10}\left\|\partial_x^j\Omega^\alpha f\right\|_2+\sum_{|\alpha|+j\le 9}\left\|\partial_x^j\Omega^\alpha g\right\|_2\le\varepsilon$$

We will show that there is a solution in $[0,T_\varepsilon]\times R^3$ for $T_\varepsilon=\exp(c/\varepsilon)$. Here

$$\{Z\}=\{\partial_t,\partial_x,\Omega=\Omega_{jk}:1\le j<k\le n\}$$

Main estimates

- Energy inequality:

$$\|u'(t,\cdot)\|_2\le\|u'(0,\cdot)\|_2+\int_0^t\|\Box u(s,\cdot)\|_2 ds$$

- KSS mixed norm estimate:

$$(\ln(2+t))^{1/2}\left\|\langle x\rangle^{-1/2}u'\right\|_{L^2([0,t]\times R^3)}\le C\|u'(0,\cdot)\|_2+C\int_0^t\|\Box u(s,\cdot)\|_2 ds$$

- Weighted Sobolev estimate:

$$\|h\|_{L^\infty(R/2\le|x|\le R)}\le CR^{-1}\sum_{|\alpha|\le 2}\|Z^\alpha h\|_{L^2(R/4\le|x|\le 2R)}$$