Better-Reply Dynamics and Global Convergence to Nash Equilibrium in Playing-Against-The-Sum Games

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Abstract

We consider n-person, generic, quasi-concave games with continuous action spaces and in which the payoff of a player depends on her own action and the sum of the actions of opponents. We study a discrete-time, stochastic adjustment process (the *better-reply* dynamics) in which players move towards better replies. Our main result is a sufficient condition for this process to converge globally to a Nash equilibrium of the game. This condition requires that actions be either locally strategic substitutes or locally strategic complements for all players at each Nash equilibrium that is locally asymptotically stable under an associated deterministic, adjusted best-reply dynamics. We provide an example of a 2-person game with a unique Nash equilibrium at which the derivatives of the best-reply functions have different signs and in which the better-reply dynamics does not converge to the equilibrium.

1 Introduction

In his study of duopoly, Cournot [4] introduced the noncooperative equilibrium later generalized by Nash [14], [15] and investigated its stability under a version of the best-reply dynamics in which firms alternate changing output from its current level to a best reply to the opponent's level. The more recent literature has studied several versions of the best-reply dynamics both in the framework of oligopoly models and in the more general setup of a noncooperative game. The focus of most of this literature has been on finding conditions that guarantee either the global or the local asymptotic stability of a Nash equilibrium. The general message is that these conditions are very strong, especially when global stability of a particular Nash equilibrium is required, since in this case they must imply uniqueness of the Nash equilibrium. (See Al-Nowaihi and Levine [1], Dastidar [5], and Vives [23] for results on convergence of the continuous-time version of the dynamics and Gabay and Moulin [9] and Moulin [18] for results on convergence of the discrete-time dynamics.)

In this paper we are interested in studying global convergence to Nash equilibrium, but we do not require that the equilibrium be unique. Instead, we study conditions under which the system eventually settles in an equilibrium, without imposing that all possible paths converge to the same equilibrium. This is only the first of several differences between our approach and the rest of the literature on the best-reply dynamics. A more fundamental difference is that we look at stochastic, rather than deterministic, adjustment processes. In our model players have status quo actions. Players are randomly selected, one at the time, to sample new actions. When a player is selected to sample, she randomly draws one of her available actions and only changes her status quo to the sampled action if this improves her payoff (i.e., if the move constitutes a *single-player improvement*). We call the stochastic process so generated the *better-reply dynamics*, because players move from their current actions to a better reply, not necessarily a best reply; even though players move in the direction of their best replies, they can overshoot or undershoot them.

Our better-reply dynamics can be viewed as a simple stimulus-response model of the behavior of boundedly rational players. It can be seen as a formalization of results from experimental research in economics and psychology showing that players' behavior gravitates towards actions that have been successful (see Roth and Erev

[21]). Our dynamics is related to the recent literature on learning in games (see Fudenberg and Levine [8] for a survey). However, the focus of this literature is on how players may learn to play a mixed strategy Nash equilibrium in a finite game (often with only two players), while we focus on convergence to a pure strategy Nash equilibrium in a game with continuous action spaces and several players. A distinguishing feature of our model is that the bound on players' rationality and knowledge is more severe than in most of the learning literature. The better-reply dynamics is consistent with a player not having precise knowledge, or memory, of her own and her opponents' payoff functions and past actions.

A standard criticism levied against the deterministic, best-reply dynamics first studied by Cournot is that when a player moves to a best reply to her opponents' current actions, she acts as if her opponents never changed their actions, in spite of collecting repeated evidence that actions do change. This criticism is less pertinent to our model, because our players need not know the actions of their opponents or their own best-reply functions. Our players simply experiment new actions and make definite changes after experiencing an increase in payoff.

A version of the better-reply dynamics studied in this paper was first introduced by Friedman and Mezzetti [7]. They noticed that in finite games having the *weak finite improvement property* (weak FIP), the better-reply dynamics globally converges to a Nash equilibrium. The weak FIP requires that starting from any action profile there exists a finite path of single-player improvements that leads to a Nash equilibrium of the game. Friedman and Mezzetti [7] showed that finite supermodular games and generic, continuous, two-person, quasi-concave games have the weak FIP.

The focus of our analysis is on a class of n-person, noncooperative games that we call *playing-against-the-sum* (*PAS*) *games*. In a PAS game the payoff of each player is a function of the player's own action and of the sum of the actions (or, equivalently, the mean action) of the other players. We take the players' action spaces to be closed intervals on the real line and assume that a player's payoff is a quasi-concave function of her own action. We restrict attention to games in which, for each player, the slope of the best-reply function is bounded below by -1.[1] The class of playing-against-the-

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[1] In most PAS games such an assumption is not very restrictive; for example, in the Cournot model it requires that the difference between price and marginal cost is a decreasing function of the firm's output.

sum games contains many interesting games from economics and political science. A wide class of oligopoly games, including Cournot's original model, models of the private provision of a public good, models of the joint exploitation of a common resource, collective actions models, and macroeconomic models with catching-up-with-the-Joneses are all examples of PAS games.

After describing the model in the next section, in Section 3 we introduce a deterministic, continuous-time, adjusted best-reply dynamics. While of limited independent interest, this dynamics will prove useful in the proofs of our main results. Then, in Section 4 we study the stochastic process generated by the better-reply dynamics. We provide a sufficient condition for the better-reply dynamics to globally converge to a Nash equilibrium of almost all playing-against-the-sum games. This condition is that actions be either strategic substitutes or strategic complements for all players (i.e., the derivatives of the best-reply functions have the same sign) at all Nash equilibria that are asymptotically stable under the adjusted best-reply dynamics defined in Section 3. Note that this sufficient condition is a local condition, actions need to be either strategic complements or strategic substitutes only at the Nash equilibria. Furthermore, actions are allowed to be strategic substitutes at some equilibria and strategic complements at other equilibria; we only need to rule out equilibria where actions are strategic complements for some players and strategic substitutes for other players.

In Section 5 we provide an example of a 2-person game with a unique Nash equilibrium at which the derivatives of the two best-reply functions have different signs; we show that in such a game the stochastic process generated by the better-reply dynamics does not converge to the equilibrium. This demonstrates that our condition on the derivatives of the best reply functions at the Nash equilibria cannot be easily relaxed; without it the better reply dynamics may fail to converge.

In Section 6 we prove that playing-against-the-sum games have the weak finite improvement property. This implies that any discretization of a game with a continuous action space also has the weak FIP and that in such a discretized game the better-reply dynamics converges to an action profile that is close to a Nash equilibrium of the original game. At first, it may seem puzzling that with a discrete state space global convergence requires less stringent conditions (actions need not be strategic complements or substitutes at a Nash equilibrium). The puzzle is easily resolved by noting that although a discretized version of the non-convergent example of a 2-person game described in Section 5 would converge to the Nash equilibrium, the average time that it takes to converge goes to infinity as the discretized version converges to the continuous game.

Section 7 contains some concluding remarks. There we argue that our convergence results for the stochastic better-reply dynamics are considerably stronger than existing results on the convergence of the deterministic best-reply dynamics. Proofs of several technical results are in the Appendix.

2 The Model

We study n-person games $g = \langle N, A, U \rangle$ where each player $i \in N = \{1, \dots, n\}$ has a one dimensional, compact, convex strategy set $A_i \subset \mathbb{R}$, and a payoff function $U_i : A \to \mathbb{R}$ that is twice continuously differentiable in $a \in A = A_1 \times \ldots \times A_n$ and strictly quasi-concave with respect to $a_i \in A_i$.[2] $U = (U_1, \dots, U_n)$ is the payoff profile of the game. We assume that for all $i \in N$ the payoff function U_i only depends on player i's own action and the sum of the actions of the opponents:

$$U_i(a) = U_i(a_i, \Sigma_{-i}), \qquad \text{where} \qquad \Sigma_{-i} = a_1 + \ldots + a_{i-1} + a_{i+1} + \ldots + a_n. \tag{1}$$

Note that if $n = 2$, any strictly quasi-concave game satisfies Condition (1). Let $\Sigma = a_1 + \ldots + a_n$ and denote by A_Σ the set of all admissible sums Σ, i.e.,

$$A_\Sigma = \{a_1 + a_2 + \ldots + a_n : a_i \in A_i \quad \text{for all} \quad i \in N\}.$$

The partial derivative of U_i with respect to a_i, $\partial U_i / \partial a_i$, must have the same form as U_i; that is, it must depend on a_i and Σ_{-i}. It is thus possible to define a function D_i

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[2] Note that U_i being twice continuously differentiable implies that the partial and cross partial derivatives of U_i are bounded on A. Furthermore, since A is compact, they obtain a maximum and a minimum; the maximum coincides with the least upper bound and the minimum with the greatest lower bound.

that depends on a_i and Σ as follows

$$D_i(a_i, \Sigma) = \frac{\partial U_i}{\partial a_i}(a_i, \Sigma - a_i). \tag{2}$$

Thus, for example, if $U_i(a_i, \Sigma_{-i}) = \alpha a_i + \beta a_i^2 + \gamma \Sigma_{-i} + \delta \Sigma_{-i}^2 + \eta a_i \Sigma_{-i}$ then $\partial U_i / \partial a_i = \alpha + 2\beta a_i + \eta \Sigma_{-i}$, and $D_i = \alpha_i + (2\beta - \eta) a_i + \eta \Sigma$.

Let $B : A \to A$, with $B(a) = (B_1(a_{-1}), \ldots, B_n(a_{-n}))$, denote the best-reply correspondence of g, where $a_{-i} \in A_{-i} = A_1 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n$. At an interior solution, for any given a_{-i} the best response function $B_i(a_{-i}) = B_i(\Sigma_{-i})$ is an implicit solution to the equation

$$D_i\left(B_i(\cdot), B_i(\cdot) + \Sigma_{-i}\right) = 0. \tag{3}$$

If for some a_{-i} we have $D_i(a_i, a_i + \Sigma_{-i}) \neq 0$ for all $a_i \in A_i$, then B_i is simply the right endpoint of A_i if $D_i > 0$, and the left one if $D_i < 0$. Strict quasi-concavity of U_i with respect to $a_i \in A_i$ implies that: (i) B_i is continuous and single-valued (i.e., the implicit equation (3) has, at most, a unique solution for all a_{-i}; at this solution U_i attains its maximum) and (ii) player i's payoff declines as a_i moves away from $B_i(a_{-i})$. This latter property and differentiability of U_i imply that for all $a_i \in A_i$ and all $\Sigma \in A_\Sigma$:

$$D_i(a_i, \Sigma) > 0 \quad \text{if} \;\; a_i < B_i(\Sigma_{-i}); \qquad D_i(a_i, \Sigma) < 0 \quad \text{if} \;\; a_i > B_i(\Sigma_{-i}) \quad \text{and} \tag{4}$$

$$\frac{dD_i(a_i, \Sigma)}{da_i} = \frac{\partial D_i(a_i, \Sigma)}{\partial a_i} + \frac{\partial D_i(a_i, \Sigma)}{\partial \Sigma} < 0 \quad \text{if} \;\; a_i = B_i(\Sigma_{-i}) \in A_i^0, \tag{5}$$

where A_i^0 is the interior of A_i. We also make the additional assumption that the partial derivative of D_i with respect to its first argument, a_i, is negative everywhere; that is, for all $a_i \in A_i$ and all $\Sigma \in A_\Sigma$:

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} = \frac{\partial^2 U_i(a_i, \Sigma_{-i})}{\partial a_i^2} - \frac{\partial^2 U_i(a_i, \Sigma_{-i})}{\partial a_i \partial \Sigma_{-i}} < 0. \tag{6}$$

Note that this does not imply, and is not implied by, the concavity of U_i with respect to a_i. For example, if $U_i(a_i, \Sigma_{-i}) = \alpha a_i + \beta a_i^2 + \gamma \Sigma_{-i} + \delta \Sigma_{-i}^2 + \eta a_i \Sigma_{-i}$, then this assumption requires $2\beta < \eta$, while concavity would require $\beta < 0$. In many games

Condition (6) is not very restrictive; together with Condition (5), it implies that the slope of the best-reply function of each player is bounded below by -1

$$\frac{dB_i(\Sigma_{-i})}{d\Sigma_{-i}} = -\frac{\partial D_i(a_i,\Sigma)/\partial\Sigma}{dD_i(a_i,\Sigma)/da_i} = \frac{\partial D_i(a_i,\Sigma)/\partial a_i}{dD_i(a_i,\Sigma)/da_i} - 1 > -1 \quad \text{for} \quad a_i = B_i(\Sigma_{-i}).$$

Note that $\partial D_i(a_i,\Sigma)/\partial a_i$ is a continuous function of a_i and Σ_{-i} (since U_i is twice continuously differentiable) and thus attains a maximum in the compact set A; since the function must be negative, it follows that the value at the maximum must be negative. Thus, $\partial D_i(a_i,\Sigma)/\partial a_i$ must be bounded away from zero. Note also that, if $U_i(a_i,\Sigma_{-i})$ is concave in a_i, supermodularity of $U_i(a_i,\Sigma_{-i})$ implies but is not implied by Condition (6).

Definition 1 *The game* $g = \langle N, A, U\rangle$ *is a* playing-against-the-sum (PAS) game *if for all* $i \in N$*: (a)* $A_i \subset \mathbb{R}$ *is compact and convex; (b)* $U_i : A \to \mathbb{R}$ *is twice continuously differentiable in* $a \in A$ *and strictly quasi-concave with respect to* $a_i \in A_i$*; (c)* U_i *satisfy Conditions (1) and (6).*

Here are some examples of playing-against-the-sum games.

Example 1. Consider a homogeneous product, Cournot oligopoly with n firms. Let a_i be the output level of firm i and Σ be total output. $P(\Sigma)$ is the inverse demand function and $C_i = C_i(a_i)$ is the cost function of firm i. Then the payoff, or profit, function of firm i can be written as

$$U_i(a) = P(\Sigma)a_i - C_i(a_i).$$

and its derivative with respect a_i is

$$\frac{dU_i(a)}{da_i} = D_i(a_i,\Sigma) = P'(\Sigma)a_i + P(\Sigma) - C_i'(a_i).$$

Condition (6) requires that the difference between price and marginal cost be a decreasing function of a firm's output

$$\frac{\partial D_i(a_i,\Sigma)}{\partial a_i} = P'(\Sigma) - C_i''(a_i) < 0.$$

Condition (5) implies that

$$P''(\Sigma)a_i + 2P'(\Sigma) - C_i''(a_i) < 0, \quad \text{for} \quad a_i = B_i(a_{-i}). \tag{7}$$

Note that concavity of U_i would require that Condition (7) hold for all a_i.

Example 2. Consider a differentiated product, price competition oligopoly. Assume that the demand q_i for the output of firm i is a decreasing function of i's price a_i and an increasing function of the average price of its competitors $\Sigma_{-i}/(n-1)$; that is, $q_i = q_i(a_i, \Sigma_{-i}/(n-1))$ with $\partial q_i/\partial a_i < 0$ and $\partial q_i/\partial \Sigma_{-i} > 0$. Let $C_i(q_i)$ be the cost function of firm i, with $C_i' > 0$ and $C_i'' \geq 0$ for all $i \in N$, and define the function $Q_i(a_i, \Sigma) = q_i(a_i, (\Sigma - a_i)/(n-1))$. Note that $\partial Q_i/\partial a_i = \partial q_i/\partial a_i - (\partial q_i/\partial \Sigma_{-i})/(n-1) < 0$ and $\partial Q_i/\partial \Sigma = \partial q_i/\partial \Sigma_{-i} > 0$.

The profit function of firm i can be written as

$$U_i(a) = a_i Q_i(a_i, \Sigma) - C_i(Q_i(a_i, \Sigma)).$$

and the function D_i is

$$\frac{dU_i(a)}{da_i} = D_i(a_i, \Sigma) = Q_i(a_i, \Sigma) + (a_i - C_i'(Q_i(a_i, \Sigma)))\frac{dQ_i(a_i, \Sigma)}{da_i},$$

where

$$\frac{dQ_i(a_i, \Sigma)}{da_i} = \frac{\partial Q_i(a_i, \Sigma)}{\partial a_i} + \frac{\partial Q_i(a_i, \Sigma)}{\partial \Sigma} = \frac{\partial q_i(a_i, (\Sigma - a_i)/(n-1))}{\partial a_i} < 0.$$

Concavity of U_i would require that for all a_i and all Σ

$$\frac{dD_i(a_i, \Sigma)}{da_i} = \left(2 - C_i''\frac{dQ_i}{da_i}\right)\frac{dQ_i}{da_i} + (a_i - C_i')\left(\frac{\partial^2 Q_i}{\partial a_i^2} + 2\frac{\partial^2 Q_i}{\partial \Sigma \partial a_i} + \frac{\partial^2 Q_i}{\partial \Sigma^2}\right) < 0,$$

while Condition (6) requires that

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} = \left(1 - C_i''\frac{dQ_i}{da_i}\right)\frac{\partial Q_i}{\partial a_i} + \frac{dQ_i}{da_i} + (a_i - C_i')\left(\frac{\partial^2 Q_i}{\partial a_i^2} + \frac{\partial^2 Q_i}{\partial \Sigma \partial a_i}\right) < 0.$$

If, for example, the demand functions are linear, $q_i = \alpha_i - \gamma_i a_i + \eta_i \Sigma_{-i}/(n-1)$ where

α_i, γ_i and η_i are positive constants, then

$$\left(\frac{\partial^2 Q_i}{\partial a_i^2} + \frac{\partial^2 Q_i}{\partial \Sigma \partial a_i}\right) = \left(\frac{\partial^2 Q_i}{\partial a_i^2} + 2\frac{\partial^2 Q_i}{\partial \Sigma \partial a_i} + \frac{\partial^2 Q_i}{\partial \Sigma^2}\right) = 0,$$

and (4), (5) and (6) hold.

Example 3. Consider a collective action problem. Each of n players privately chooses a_i at a cost $C_i(a_i)$ (a_i could be i's private provision of a public good, or her private use of a common resource). The sum of individual choices determines the benefit $V_i(\Sigma)$ to the player. Thus, player i's payoff function is

$$U_i(a) = V_i(\Sigma) - C_i(a_i).$$

and the function D_i is

$$\frac{dU_i(a)}{da_i} = D_i(a_i, \Sigma) = V_i'(\Sigma) - C_i'(a_i).$$

Then

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} = -C_i''(a_i) < 0,$$

and (6) is satisfied provided cost is a convex function of effort. Concavity of U_i would require

$$V_i''(\Sigma) - C_i''(a_i) < 0.$$

Example 4. Consider an economy with catching-up-with-the-Joneses. There are n consumers (players) and a single consumption good produced from labor with a linear technology (one unit of labor l_i produces one unit of the good). Letting a_i be i's consumption, player i payoff function is

$$U_i^*(a, l_i) = V_i((a_i - \Sigma/n)^2) - C_i(l_i).$$

with $V_i' < 0$ and $C_i' > 0$. Each of n players chooses consumption a_i and labor l_i subject to the budget constraint $a_i = l_i$. Replacing the budget constraint into U_i^* we can write i's payoff as

$$U_i(a) = V_i((a_i - \Sigma/n)^2) - C_i(a_i).$$

Then, the function D_i is

$$\frac{dU_i(a)}{da_i} = D_i(a_i, \Sigma) = 2(a_i - \Sigma/n)\frac{n-1}{n}V_i'((a_i - \Sigma/n)^2) - C_i'(a_i).$$

Hence,

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} = 2\frac{n-1}{n}V_i'((a_i - \Sigma/n)^2) + 4(a_i - \Sigma/n)^2\frac{n-1}{n}V_i''((a_i - \Sigma/n)^2) - C_i''(a_i),$$

and a sufficient condition for (6) to be satisfied is that V_i be a concave function, $V_i'' < 0$, and C_i be a convex function, $C_i'' > 0$.

3 A Globally Convergent, Deterministic Dynamics in Continuous Time

We are interested in studying the convergence properties of a stochastic, discrete time, adjustment process, called the *better-reply dynamics*, in which each player randomly samples among her available actions and only changes her status quo to the sampled action if this improves her payoff. Before proceeding to study the better-reply dynamics, in this section we take a small detour and study a deterministic, continuous-time dynamics. While of limited interest in its own right, this dynamics will prove very useful in the proofs of our main results.

The standard formulation of the continuous-time, deterministic, best-reply dynamics in a game $g = \langle N, A, U \rangle$ is the following system of n differential equations:

$$\dot{a}_i = B_i(a_{-i}) - a_i, \qquad i = 1, 2, \ldots, n. \tag{8}$$

Using Liouville's Theorem, Corchon and Mas-Colell [3] established global convergence of the dynamics to a Nash equilibrium of the game provided $n = 2$ (under the assumption that $B_i(a_{-i})$ is a Lipschitz function). They also pointed out that for $n \geq 3$ convergence need not occur; there are games with payoff functions that yield chaotic dynamics (e.g., if the differential equations in (8) are Lorenz's equations; see Guckenheimer and Holmes [11]). This simply follows from the fact that, without restrictions on the payoff functions, any dynamical system can be generated as the best-reply dynamics of some economic game.

In fact, Corchon and Mas-Colell considered the more general system:

$$\dot{a}_i = \mu_i(B_i(a_{-i}) - a_i), \qquad i = 1, 2, \ldots, n, \tag{9}$$

where $\mu_i > 0$ are positive constants.

Suppose, instead, that we allow μ_i to be a function that depends on the vector a, but still require that $\mu_i(a) > 0$ for all $a \in A$. If a_{-i} remains constant and $\mu_i(a)(B_i(a_{-i}) - a_i)$ is a Lipschitz function, then the equation

$$\dot{a}_i = \mu_i(a)(B_i(a_{-i}) - a_i) \tag{10}$$

still yields $a_i \to B_i(a_{-i})$ as $t \to \infty$; that is, player i moves towards his best reply. We will call any such dynamics a *continuous-time, adjusted best-reply dynamics.*

In this section, we show that for playing-against-the-sum games, there is a choice of functions $\mu_i(a)$ which yields global convergence to a Nash equilibrium of the game g for any number of players n.

If equation (6) holds, then for any $\Sigma \in A_\Sigma$ there is (at most) a unique solution $a_i = a_i(\Sigma)$ to the implicit equation:

$$D_i(a_i(\Sigma), \Sigma) = 0. \tag{11}$$

Denote this solution by $M_i(\Sigma)$; if equation (11) does not have a solution in A_i (i.e., $D_i(a_i, \Sigma) \neq 0$ for all $a_i \in A_i$) then let $M_i(\Sigma)$ be the right endpoint of the interval A_i if $D_i(a_i, \Sigma) > 0$ and the left endpoint otherwise.

We want to consider the following system of differential equations:

$$\dot{a}_i = M_i(\Sigma) - a_i, \qquad i = 1, 2, \ldots, n, \tag{12}$$

or, in vector notation:

$$\dot{a} = M(\Sigma) - a. \tag{13}$$

We claim that $M_i(\Sigma) - a_i$ and $B_i(a_{-i}) - a_i$ have always the same sign.

Lemma 1 *Let g be a playing-against-the-sum game. For all $a \in A$, we have: (a)*

$(M_i(\Sigma) - a_i)\,(B_i(a_{-i}) - a_i) \geq 0$, and (b) $B_i(a_{-i}) = a_i$, if and only if $M_i(\Sigma) = a_i$.

Proof. If $B_i(a_{-i}) = a_i$ then (a) holds. Consider any $a^0 \in A$ with $B_i(a^0_{-i}) > a^0_i$. Condition (4) guarantees that for any given $\Sigma^0_{-i} = \sum_{j \neq i} a^0_j$, if $B_i(a^0_{-i}) > a^0_i$, then $D_i(a^0_i, a^0_i + \Sigma^0_{-i}) = D_i(a^0_i, \Sigma^0) > 0$. Equation (6) implies that for any given Σ^0, the function $a_i \mapsto D_i(a_i, \Sigma^0)$ is decreasing. So, if $M_i(\Sigma^0) = M_i(a^0_i + \Sigma^0_{-i})$ were less than or equal to a^0_i, it would follows that $D_i(M_i(\Sigma^0), \Sigma^0) \geq D_i(a^0_i, \Sigma^0) > 0$. This implies that equation (11) does not have a solution, and that $M_i(\Sigma^0)$ is equal to the right endpoint of the interval A_i. Thus, $M_i(\Sigma^0) > a^0_i$, which is a contradiction (note that $B_i(a^0_{-i}) > a^0_i$ implies that a^0_i is to the left of the right endpoint of the interval A_i). It follows that $a^0_i < M_i(\Sigma^0)$. A similar argument can be made for the case $a^0_i > B_i(a^0_{-i})$. This concludes the proof of part (a) of the lemma.

If $B_i(a_{-i}) = a_i$, then it must also be $M_i(\Sigma) = a_i$, since by (11) for all a_{-i}:

$$M_i(B_i(a_{-i}) + \Sigma_{-i}) = B_i(a_{-i}),$$

while if $M_i(\Sigma) = a_i$, then $B_i(a_{-i}) = a_i$, since by (3) for all Σ

$$B_i(\Sigma - M_i(\Sigma)) = M_i(\Sigma).$$

This concludes the proof. ■

We can use this lemma to establish that equation (13) defines a continuous-time, adjusted best-reply dynamics for the game g. Define the function $\mu_i(a)$ by

$$\mu_i(a) = \frac{M_i(\Sigma) - a_i}{B_i(a_{-i}) - a_i} \qquad \text{for } a_i \neq B_i(a_{-i}) \tag{14}$$

$$\mu_i(a) = \lim_{a_i \to B_i(a_{-i})} \frac{M_i(\Sigma) - a_i}{B_i(a_{-i}) - a_i} = \frac{1 - dM_i/d\Sigma}{1} \qquad \text{for } a_i = B_i(a_{-i}), \tag{15}$$

where the second equality in equation (15) follows from L'Hôpital's rule. Lemma 1 implies that $\mu_i(a) > 0$ for all a such that $B_i(a_{-i}) \neq a_i$. To see that if $a_i = B_i(a_{-i})$ then $\mu_i(a) > 0$ also holds, note that the implicit function theorem and equations (4),

(6) and (11) imply[3]

$$\frac{dM_i(\Sigma)}{d\Sigma} = -\frac{\partial D_i(a_i, \Sigma)/\partial \Sigma}{\partial D_i(a_i, \Sigma)/\partial a_i} < 1.$$

Let $f(a)$ be a Lipschitz function and recall that, given a system of ordinary differential equations $\dot{a} = f(a)$ with initial condition a^0, we can think of the unique solution $a(t)$, with $a(0) = a^0$, as the trajectory of the system starting at a^0. Any point $a \in A$ with the property that there exists a sequence $t_1, t_2, \ldots$ such that $\lim_{m\to\infty} a(t_m) = a$ is called an *ω-limit point* of the trajectory $a(t)$; the set of all such points is called the *ω-limit set* of the trajectory $a(t)$. If the ω-limit set of the trajectory $a(t)$ contains a single element $a^* \in A$, then a^* is a stationary point (i.e., $f(a^*) = 0$) and if the system starts at a^0 then it will converge to a^*; $\lim_{t\to\infty} a(t) = a^*$. The system of ordinary differential equations $\dot{a} = f(a)$ *globally converges to an equilibrium*, if for all $a^0 \in A$ the ω-limit set of the system with initial condition a^0 is a singleton.

If it is globally convergent, then from any given initial state a^0 the system converges to an equilibrium a^*; $\lim_{t\to\infty} a(t) = a^*$. Note that global convergence of a system does not imply that the system has a unique equilibrium or stationary point. Rather, it means that starting from any initial position an equilibrium is eventually reached. Cycling or chaotic dynamics are ruled out. We now show that the continuous-time, adjusted best-reply dynamics defined by (13) globally converges to some Nash equilibrium of the game g.

Theorem 1 *Let $g = \langle N, A, U \rangle$ be an n-person, playing-against-the-sum game. Then the continuous-time, adjusted best-reply dynamics defined by equation (13) globally converges to a Nash equilibrium of g.*

Proof. If we sum the system (13) we obtain a differential equation for Σ. Namely:

$$\dot{\Sigma} = \sum_{i=1}^{n} M_i(\Sigma) - \Sigma.$$

This is a single ordinary differential equation satisfying a Lipschitz condition, since

$$\frac{dM_i}{d\Sigma} = -\frac{\partial D_i/\partial \Sigma}{\partial D_i/\partial a_i}$$

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[3] At a value of Σ at which equation (11) does not hold M_i equals one of the endpoints of the interval A_i and $dM_i/d\Sigma = 0$.

and the numerator is bounded (see fn.2) while the denominator is bounded away from zero by Condition (6). Because one dimensional autonomous equations cannot exhibit oscillations, it follows that given any initial condition Σ^0 the trajectory $\Sigma(t)$ is monotonic. Since $\Sigma(t)$ is also bounded, it follows that there is $\Sigma_\infty(\Sigma^0) \in A_\Sigma$ such that

$$\Sigma_\infty(\Sigma^0) = \lim_{t\to\infty} \Sigma(t).$$

This implies that for any $i \in N$: $M_i(\Sigma(t)) \to M_i(\Sigma_\infty(\Sigma^0))$ as $t \to \infty$. Hence, for large t the system (13) with initial condition $a^0 \in A$, such that $\sum_{i=1}^n a_i^0 = \Sigma^0$, becomes:

$$\dot{a}_i = M_i(\Sigma_\infty(\Sigma^0)) - a_i + h_i(t, \Sigma^0), \quad i = 1, 2, \ldots, n,$$

where the functions h_i satisfy $h_i(t, \Sigma^0) \to 0$, as $t \to \infty$. This immediately yields that for all $i \in N$, $a_i(t) \to M_i(\Sigma_\infty(\Sigma^0))$ as $t \to \infty$; that is, the ω-limit set of the system (13) with initial condition $a(0) = a^0$ contains a single element $a^* \in A$ and for all $i \in N$, $a_i^* = M_i(\Sigma^*)$, where $\Sigma^* = \sum_{i=1}^n a_i^*$. Lemma 1, part (b), then implies $a_i^* = B_i(a_{-i}^*)$ for all $i \in N$; that is, a^* is a Nash equilibrium of g. ■

The intuition behind the global convergence of the adjusted best-reply dynamics defined by equation (13) is simple. By adding up the n differential equations in (13) one obtains a single ordinary differential equation in Σ. Such an equation cannot exhibit any cyclic or chaotic behavior.

4 The Better-Reply Dynamics: Convergence Results

In this section we formally define the better-reply dynamics in discrete time and study its convergence properties in playing-against-the-sum games. We assume that the probability that the randomly selected player i samples a strategy belonging to any subset E of A_i is positive if E has positive Lebesgue measure. Formally, we associate with the strategy space A_i of player i a probability measure P_i defined on the Borel subsets of A_i. For any Borel set $E \subset A_i$ the number $P_i(E)$ expresses the likelihood that player i samples a strategy that belongs to E. The only condition we impose on P_i is that for any open interval $I \subset A_i$ we have $P_i(I) > 0$. Note that this does not exclude singular measures; that is, the measure P_i can have one or more points x where $P_i(\{x\}) > 0$.

Definition 2 THE BETTER-REPLY DYNAMICS. *Consider a continuous game* $g = \langle N, A, U \rangle$*. Let* P_i *be a probability measure on the Borel subsets of* A_i *such that for any open interval* $I \subset A_i$, $P_i(I) > 0$*. At each discrete time period* t *there is a status quo action profile* a^t*. A single player* $i \in N$ *is randomly selected, with all players having positive selection probability. Player* i *randomly samples action* $a_i^E \in A_i$ *according to the probability measure* P_i*. If* $U_i^F(a^t \backslash a_i^E) > U_i^F(a^t)$ *then* $a^t \backslash a_i^E$ *becomes the new status quo,* $a^{t+1} = a^t \backslash a_i^E$*. If* $U_i^F(a^t \backslash a_i^E) \leq U_i^F(a^t)$ *then the status quo does not change,* $a^{t+1} = a^t$*.*

The process described in Definition 2 is essentially the same as the one defined by Friedman and Mezzetti [7], except that they had finite strategy spaces and required all players to have the same probability of being selected to sample a new strategy and all strategies to have the same probability of being sampled. Note that the experimentation of a new strategy on the part of the player sampling at time t has no effect on the other players. In particular, it does not affect the payoff that other players associate with their status quo action. The simplest way to justify this assumption is to think of time as a continuous variable, with players experimenting new actions at (possibly random) discrete points in time. When a player is sampling a new strategy at time t, she has experienced the same payoff for the time interval $(t-1, t)$ and views it as the status quo payoff.

We will derive results that hold for almost all, transversal PAS games. We now make it clear what we mean by a transversal game. Let a^* be a Nash equilibrium of a game $g = \langle N, A, U \rangle$ and $B_i'(a_{-i}^*)$ be the derivative of player i's best reply function at a^*. Player i and j's best reply functions at a^* are said to be *transversal* if they are not tangent; that is, if $B_i'(a_{-i}^*)B_j'(a_{-j}^*) \neq 1$. The game g is called a *transversal game* if at all Nash equilibria a^*, $B_i'(a_{-i}^*)B_j'(a_{-j}^*) \neq 1$ for all i and $j \neq i$ (i.e., if the best-reply functions of all pairs of players are transversal). Transversal games have a finite number of equilibria (equilibria are isolated). Let $S(n, q)$ be the set of n-person, transversal, PAS games with q equilibria, $a^1, ..., a^q$ (we can unambiguously order equilibria by putting $a_1^{h-1} \leq a_1^h$ for all h, $a_2^{h-1} \leq a_2^h$ if $a_1^{h-1} = a_1^h, \ldots$). Consider the map $\xi_q^n : S(n,q) \to \mathbb{R}^{nq}$ that associates to each game $g \in S(n,q)$ the vector s_q^n of the slopes of the best-reply functions at the Nash equilibria of g:

$$s_q^n = \left(B_1'(\Sigma_{-1}^1), \ldots, B_n'(\Sigma_{-n}^1), \ldots, B_1'(\Sigma_{-1}^q), \ldots, B_n'(\Sigma_{-n}^q)\right).$$

Let $I(\xi_q^n)$ be the range of ξ_q^n; clearly $I(\xi_q^n)$ is a subset of $\mathbb{R}^{nq}$ with nonempty interior. We will say that a property $\mathcal{P}$ holds for almost all games in $S(n,q)$ if there exists a subset I^* of $I(\xi_q^n)$ such that (i) $I(\xi_q^n) \cap I^*$ has zero Lebesgue measure in $\mathbb{R}^{nq}$, and (ii) the property $\mathcal{P}$ holds for all games g with $\xi_q^n(g) \in I^*$. A property holds for *almost all transversal, n-person PAS games* if it holds for almost all games in $S(n,q)$, for all q.

Let $a \backslash x_i$ denote the n-tuple $(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) \in A$. The strategy profile $a \backslash x_i \in A$ is a *single-player improvement* over $a \in A$ if and only if the payoff to player i is higher under $a \backslash x_i$ than under a: $U_i(a \backslash x_i) > U_i(a)$. The following lemma shows that for almost all transversal PAS games there is a finite sequence of single-player improvements that ends arbitrarily close to a Nash equilibrium.

Lemma 2 *The following property holds for almost all transversal, n-person, playing-against-the-sum games* $g = \langle N, A, U \rangle$*. Given any* $r > 0$ *and any strategy profile* a^0*, there is a finite sequence of single-player improvements that starts at* a^0 *and ends inside a ball of radius* r *around an isolated Nash equilibrium* a^* *of the game* g*.*

Proof. Since the game g is transversal it follows that the set of Nash equilibria is finite. Take $r > 0$ and any $a^0 \in A$. If a^0 is not already contained in a ball of radius r around a Nash equilibrium, consider the dynamics (13) for the game g with initial condition $a(0) = a^0$. By Theorem 1, there is a time $T > 0$ such that for all $t > T$, the trajectory $a(t)$ lies in some neighborhood V_r contained in a ball of radius r around a Nash equilibrium a^*. The only possible instance in which a^* is not an isolated, asymptotically stable equilibrium of (13) is if (13) has a stable manifold converging to an unstable Nash equilibrium, and a^0 belongs to such a manifold (i.e., it belongs to a trajectory converging to an unstable equilibrium).[4] In such an instance a small deviation (say by player i) from the trajectory $a(t)$ leads to another trajectory that converges that an asymptotically stable, isolated equilibrium of (13). Since,

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[4] A stable manifold of an unstable equilibrium a^*, if one exists, has the property that for all points a in the manifold $\sum_{i=1}^n a_i = \sum_{i=1}^n a_i^* = \Sigma^*$.

as we shall see below, we can always replace the continuous dynamics with a finite sequence of single-player improvements, it is always possible to find a single-player improvement that leads away from a trajectory belonging to a stable manifold of an unstable equilibrium. Hence it is always possible to reach a neighborhood V_r contained in a ball of radius r around an isolated, asymptotically stable equilibrium of (13).

To replace the continuous dynamic with a finite sequence of single-player improvements, we will use a simple Euler scheme to approximate the integral curve $a(t)$. Take an integer $Z \in \mathbb{N}$ and consider the Z-th approximation $a^Z(t)$ of $a(t)$ defined as follows:

$$\begin{array}{lcll} t_z & = & \frac{z}{Z}T, \qquad z = 0, 1, \ldots, Z & \\ a^Z(0) & = & a(0) = a^0 & \\ a^Z(t_{z+1}) & = & a^Z(t_z) + \frac{T}{Z}\left[M\left(\sum_{i=1}^n a_i^Z(t_z)\right) - a^Z(t_z)\right] & z = 0, 1, \ldots, Z-1 \\ a^Z(t) & = & a^Z(t_z), \qquad \text{for} \quad t_z \leq t < t_{z+1} & \end{array} \tag{16}$$

As $Z \to \infty$, $a^Z(t)$ converges to $a(t)$ uniformly on $[0, T]$. In particular there is a Z large enough for which $a^Z(T) \in V_r$. If we show that for each $z = 0, 1, 2, \ldots Z$ there is a finite single-player improvement path from $a^Z(t_z)$ to $a^Z(t_{z+1})$ then we are done, because the existence of a path from $a^Z(0) = a^0$ to $a^Z(t_Z) \in V_r$ follows.

To show that there is an improvement path from $a^Z(t_z)$ to $a^Z(t_{z+1})$, take:

$$\begin{array}{lcll} y^0 & = & a^Z(t_z) & \\ y^h & = & y^{h-1} + e_h\frac{T}{Z}\left[M_h\left(\sum_{i=1}^n a_i^Z(t_z)\right) - a_h^Z(t_z)\right], & h = 1, 2, \ldots n. \end{array}$$

Here e_h is the n-dimensional vector whose h-th element is 1 and all other elements are zero. Since M_h is a continuous function, Z can be chosen large enough to make $M_h\left(\sum_{i=1}^n a_i^Z(t_z)\right)$ as close as desired to $M_h\left(\sum_{i=1}^n y_i^h\right)$ and $a^Z(t_z)$ as close as desired to $a^Z(t_{z+1})$.This implies that

$$\left[M_h\left(\sum_{i=1}^n a_i^Z(t_z)\right) - a_h^Z(t_z)\right]\left[M_h\left(\sum_{i=1}^n y_i^h\right) - y_h^h\right] > 0.$$

By Lemma 1, we have

$$\left[M_h\left(\sum_{i=1}^{n} y_i^h\right) - y_h^h\right]\left[B_h\left(\sum_{i\neq h} y_i^h\right) - y_h^h\right] > 0.$$

This implies that the move from y^{h-1} to y^h by player h is in the direction of his best reply; that is, it is a single-player improvement. This completes the proof. ■

Lemma 2 will be used as the main ingredient in Theorems 2 and 3 to show that with probability one the better-reply dynamics ends up arbitrarily close to a Nash equilibrium of a PAS game. We now present another lemma that will prove useful in studying the convergence properties of the better-reply dynamics. Consider the function γ defined for any $a \in \mathbb{R}^n$ and for $\beta_i \neq 0$, $i = 1, \ldots, n$, as follows:

$$\gamma(a) = \frac{1}{\beta_1}a_1^2 + \frac{1}{\beta_2}a_2^2 + \ldots + \frac{1}{\beta_n}a_n^2 - 2(\sum_{1\leq i<j\leq n} a_i a_j) = \sum_{i=1}^{n} \frac{1+\beta_i}{\beta_i}a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \quad (17)$$

Lemma 3 *(a) The function* $\gamma : \mathbb{R}^n \to \mathbb{R}$ *is bounded from below with greatest lower bound equal to* 0 *if and only if* $\beta_i > 0$ *for* $i = 1, \ldots, n$ *and* $\sum_{i=1}^{n} \beta_i/(1+\beta_i) \leq 1$*. If the second inequality is strict, then* γ *has a unique global minimum at* $(0, 0, \ldots, 0)$*. (b) If* $0 > \beta_i \geq -1$*, for* $i = 1, \ldots, n$*, then the function* γ *is bounded from above with least upper bound equal to* 0*. If the second inequality is strict, then* γ *has a unique global maximum at* $(0, 0, \ldots, 0)$*. (c) If* $n = 2$*, then* γ *is bounded from above with least upper bound equal to* 0 *if and only if* $0 > \beta_i$*,* $i = 1, 2$ *and* $\beta_1\beta_2 \leq 1$*. If the second inequality is strict, then* γ *has a unique global maximum at* $(0, 0)$*.*

Proof. See the Appendix. ■

In the next lemma we provide sufficient conditions for the local convergence to a Nash equilibrium of a playing-against-the-sum game of almost all paths generated by the better-reply dynamics of Definition 2.

Lemma 4 *Let* $g = \langle N, A, U\rangle$ *be an n-person, playing-against-the-sum game. Consider a Nash equilibrium* a^* *of* g*. Let* β_i *be the first derivative of the best-reply function* $B_i(\Sigma_{-i})$ *of player* i *evaluated at* a^**. Assume that either (a)* $\beta_i > 0$ *for all* i *and*

$\sum_{i=1}^{n} \beta_i / (1+\beta_i) < 1$, or (b) $0 > \beta_i$ for all i. Then there exists a neighborhood V of the Nash equilibrium a^* such that almost every path $a^1, a^2, a^3, \ldots$ generated by the stochastic process described in Definition 2 that starts in V stays in V and, moreover, $\lim_{h\to\infty} a^h = a^*$.

Proof. Consider a Nash equilibrium of g. By changing coordinates we can assume, without loss of generality, that this equilibrium is at the point $(0, 0, \ldots, 0) \in \mathbb{R}^n$. For action profiles a sufficiently close to the equilibrium, we can linearize the best-reply function B_i of each player and write $B_i(\Sigma_{-i}) = \beta_i \Sigma_{-i}$. Thus, a move by player i from $a = (a_1, a_2, \ldots, a_n)$ to $\widehat{a} = (a_1, a_2, \ldots, a_{i-1}, \widehat{a}_i, a_{i+1}, \ldots, a_n)$ is payoff improving if and only if

$$|\widehat{a}_i - \beta_i \Sigma_{-i}| < |a_i - \beta_i \Sigma_{-i}|.$$

Geometrically, this means that to improve her payoff player i must move to a point on the line segment parallel to the vector $e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ (with 1 in the i-th position), with one endpoint at $a = (a_i, a_{-i})$, the middle point at $(\beta_i \sum_{j\neq i} a_j, a_{-i})$ and the other endpoint at $b = (b_i, a_{-i}) = (2\beta_i \sum_{j\neq i} a_j - a_i, a_{-i})$. Consider now the function γ defined in equation (17), assuming $\beta_i \neq 0$; we claim that $\gamma(a) = \gamma(b)$. We will check this claim for $i = 1$; we have:

$$\begin{aligned} \gamma(b) - \gamma(a) &= \frac{1}{\beta_1}\left[(2\beta_1\Sigma_{-1} - a_1)^2 - a_1^2\right] - 2\sum_{j=2}^{n}\left[(2\beta_1\Sigma_{-1} - a_1)\, a_j - a_1 a_j\right] \\ &= 4\beta_1\Sigma_{-1}^2 - 4a_1\Sigma_{-1} - 4\beta_1\Sigma_{-1}^2 + 4a_1\Sigma_{-1} = 0. \end{aligned}$$

Now suppose we are in case (a) in the statement of the theorem; that is, $\beta_i > 0$ for all $i = 1, \ldots, n$. and $\sum_{i=1}^{n} \beta_i / (1+\beta_i) < 1$. Then, for any given a_{-i}, the function $a_i \mapsto \gamma(a)$ is quadratic in a_i and goes to $+\infty$ as $|a_i| \to \infty$. Hence, since $\gamma(a) = \gamma(b)$, for all points d on the line segment connecting a and b we have $\gamma(d) \leq \gamma(a)$ with strict inequality if d is inside the segment; a single-player improvement by any player i reduces the value of γ.

We now show that there exists a neighborhood V of the Nash equilibrium $(0, \ldots, 0)$ such that if $a^0 \in V$ then almost all paths $a^0, a^1, a^2, a^3, \ldots$ generated by the stochastic process described in Definition 2 stay all the time in V. Moreover

$$\lim_{k\to\infty} a^k = (0,0,\ldots,0).$$

Let the neighborhood V be given by $\{x : \gamma(x) < c\}$ for some $c > 0$. It follows that any infinite path $a^0, a^1, a^2, a^3, \ldots$ of the stochastic process with $a^0 \in V$ is associated with a nonincreasing sequence of real numbers $\gamma(a^0), \gamma(a^1), \gamma(a^2), \gamma(a^3), \ldots$ and thus $a^k \in V$, for all k. Note that in this sequence we have infinitely many times a strict inequality, since if $a^k \neq (0,0,\ldots,0)$ then there is a positive probability that one of the players samples a strategy that improves his payoff, hence for some $n \geq k$, $\gamma(a^{n+1}) < \gamma(a^n)$. By Lemma 3 the sequence $\gamma(a^0), \gamma(a^1), \ldots$ is bounded from below, since the function γ reaches its strict, global minimum at $(0,0)$. Hence it must converge.

To see that $\lim_{k\to\infty} a^k = (0,0,\ldots,0)$, assume to the contrary that $\lim_{k\to\infty} a^k = a$, with $\gamma(a) = m > 0$. Since $a \neq (0,0,\ldots,0)$, there is $p > 0$ and $\varepsilon > 0$ such that with probability of at least p the stochastic process moves from a to a point b for which $\gamma(b) < \gamma(a) - \varepsilon$. Because of the continuous nature of the game, it must also be true that with probability of at least p the stochastic process moves from a^k to b, where a^k is any point on the path converging to a that is sufficiently close to a. It follows that the probability that the function γ stays above m along a path of the stochastic process is zero. Since this is true for any $m > 0$, for almost any path γ goes to zero and therefore $\lim_{k\to\infty} a^k = (0,0,\ldots,0)$.

In case (b) in the statement of the theorem, first note that (5) and (6) imply that $\beta_i > -1$ for all i. Then we can apply Lemma 3 and the proof is similar to the proof of (a), except that we need to use the function $-\gamma$ in place of γ. $\blacksquare$

Two sets of conditions guarantee the local convergence to a Nash equilibrium a^* of almost all paths generated by the better-reply dynamics. The first condition is that all the derivatives β_i of the best-reply functions evaluated at a^* have the same sign; that is, actions are either locally strategic substitutes or locally strategic complements. The second condition is $\sum_{i=1}^n \beta_i/(1+\beta_i) < 1$. Note that if $\beta_i < 0$ for all $i \in N$ this condition is automatically satisfied, because (5) and (6) guarantee that $\beta_i > -1$ for all $i \in N$. Let C_i be the derivative of the M_i functions defined in equation (11), evaluated at a^*. Equation (11) implies that $C_i = \beta_i/(1+\beta_i)$, hence

$\beta_i > -1$ is equivalent to $C_i < 1$. Furthermore, $\sum_{i=1}^n \beta_i/(1+\beta_i) < 1$ is equivalent to $\sum_{i=1}^n C_i < 1$. This condition is sufficient for a^* to be locally asymptotically stable under the deterministic, continuous-time, adjusted best-reply dynamics defined by equation (13).[5] It is interesting to note that, together with the derivative of the best reply functions having the same sign, $\sum_{i=1}^n \beta_i/(1+\beta_i) < 1$ is also sufficient to guarantee the local convergence of almost all paths of the better-reply dynamic described in Definition 2. Note also that if $0 > \beta_i > -1$ for all i, or $\beta_i > 0$ for all i and $\sum_{i=1}^n \beta_i/(1+\beta_i) < 1$, then at the equilibrium a^* the best reply functions of all players are transversal, since it must be $\beta_i\beta_j < 1$ for all i, j.

The next theorem applies Lemma 4 to 2-person, quasi-concave games. It shows that in such games the better-reply dynamics of Definition 2 globally converges to a Nash equilibrium provided that actions are either locally strategic substitutes or locally strategic complements at all Nash equilibria that are asymptotically stable under the adjusted best-reply dynamics (13). The theorem does not require that condition (6) be satisfied (i.e., the slope of the best-reply functions need not be bounded below by -1), and thus does not require that the game be a PAS game as described in Definition 1. It requires, however, that the game be transversal, so that Nash equilibria are isolated.

Theorem 2 *Let $g = \langle \{1,2\}, A, U \rangle$ be a transversal, quasi-concave, 2-person game. Suppose that $B_1'(a_2^*)B_2'(a_1^*) > 0$ (i.e., the derivatives of the best-reply functions of players 1and 2 have the same sign) at each Nash equilibrium a^* of g such that $B_1'(a_2^*)/\left(1+B_1'(a_2^*)\right) + B_2'(a_1^*)/\left(1+B_2'(a_1^*)\right) < 1$. Then, regardless of the initial position, for almost any path $a^1, a^2, a^3, \ldots$ generated by the stochastic process described in Definition 2 we have: $\lim_{h\to\infty} a^h = a^*$, where a^* is a Nash equilibrium of the game g.*

Proof. Consider the best-reply dynamics (8):

$$\begin{aligned} \dot{a_1} &= B_1(a_2) - a_1 \\ \dot{a_2} &= B_2(a_1) - a_2, \end{aligned} \tag{18}$$

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[5] If $\sum_{i=1}^n C_i > 1$ then a^* is unstable, while if $\sum_{i=1}^n C_i = 1$ then a^* could be either stable or unstable.

with initial condition $(a_1(0), a_2(0)) = a^1$. It follows from Liouville's theorem (see Corchon and Mas-Colell (1996)) that the ω-limit set of this system is a Nash equilibrium a^* of g and that for almost any initial condition $(a_1(0), a_2(0))$ the point in the ω-limit set is a stable equilibrium (a stable manifold of an unstable equilibrium is at most one dimensional).[6] By approximating the system (18) with an Euler scheme as in the proof of Lemma 2, we can then argue that for any neighborhood V of a^* there is a positive probability $p > 0$ that the stochastic path $a^1, a^2, a^3, \ldots$ generated by the better-reply dynamics will eventually end up and stay in V. If by W we denote the union of a set of small neighborhoods of all stable equilibria of the system (18), then there exists an integer k and a number $p > 0$ such that regardless of our initial position the path $a^1, a^2, a^3, \ldots, a^k$ generated by the better-reply dynamics leads to W with probability at least p. Once a path is in W, it stays there indefinitely. On the other hand if $a^k \notin W$ then again with probability at least p the path $a^k, a^{k+1}, a^{k+2}, \ldots, a^{2k}$ leads to W. It follows that eventually almost every path ends in W.

We have shown that the better-reply dynamics leads with probability one to an arbitrarily small neighborhood of a stable equilibrium a^* of the system (18). We now show that a^* is a stable equilibrium of (18) if and only if $B_1'(a_2^*)/(1 + B_1'(a_2^*)) + B_2'(a_1^*)/(1 + B_2'(a_1^*)) < 1$ (which is equivalent to a^* being asymptotically stable under the adjusted best-reply dynamics (13)). To see this, consider the linearization of (18) around a^*:

$$\begin{aligned} \dot{a_1} &= B_1'(a_2^*)a_2 - a_1 \\ \dot{a_2} &= B_2'(a_1^*)a_1 - a_2. \end{aligned} \tag{19}$$

The stability of a^* under (18) implies that the eigenvalues of the linearized system (19) must be non-positive. The characteristic equation of (19) is $(1 + \lambda)^2 = B_1'(a_2^*)B_2'(a_1^*)$ and thus the linearized system has a zero eigenvalue, $\lambda = 0$, if and only if $B_1'(a_2^*)B_2'(a_1^*) = 1$, which is ruled out by the assumption that the game g is transversal ($B_1'(a_2^*)B_2'(a_1^*) \neq 1$ at all Nash equilibria). Thus, the equilibrium a^* of (18) is stable if and only if $B_1'(a_2^*)B_2'(a_1^*) < 1$. Finally note that $B_1'(a_2^*)B_2'(a_1^*) < 1$ is equivalent to $B_1'(a_2^*)/(1 + B_1'(a_2^*)) + B_2'(a_1^*)/(1 + B_2'(a_1^*)) < 1$.

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[6] The equilibrium a^* is stable if for every neighborhood V of a^* there is a neighborhood $V' \subset V$ of a^* such that every trajectory $a(t)$ with $a(0)$ in V' is defines and in V for all $t > 0$.

Now suppose that the inequality $B_1'(a_2^*)B_2'(a_1^*) > 0$ also holds. Then Lemma 4 applies and for almost any path $a^1, a^2, \ldots$ in a small neighborhood of a^* generated by the better-reply dynamics described in Definition 2 we have $\lim_{k\to\infty} a^k = a^*$. This concludes the proof. ■

When there are more than two players, the analog of Theorem 2 holds for playing-against-the-sum games. The stochastic better-reply dynamics described in Definition 2 globally converges if actions are either locally strategic substitutes or locally strategic complements for all players at all Nash equilibria that are asymptotically stable under the deterministic, continuous-time dynamics defined by equation (13). To use Lemma 4 in the proof, we also need to add the technical assumption that $\sum_{i=1}^n B_i'/(1+B_i') \neq 1$ at all asymptotically stable equilibria of the dynamics (13).[7] Note that actions are allowed to be strategic substitutes at a Nash equilibrium and strategic complements at another equilibrium, as we only need to rule out equilibria where actions are strategic complements for some players and strategic substitutes for other players.

Theorem 3 *Let* $g = \langle N, A, U\rangle$ *be a transversal, n-person, playing-against-the-sum game. Suppose that at each Nash equilibrium* a^* *that is asymptotically stable under the adjusted best-reply dynamics defined by equation (13): (a)* $\sum_{i=1}^n B_i'(a_{-i}^*)/[1 + B_i'(a_{-i}^*)] \neq 1$*, and (b) the derivatives of all the best-reply functions have the same sign. Then, regardless of the initial position, for almost any path* $a^1, a^2, a^3, \ldots$ *generated by stochastic process described in Definition 2 (the better-reply dynamics) we have:* $\lim_{h\to\infty} a^h = a^*$*, where* a^* *is a Nash equilibrium of the game* g*.*

Proof. First we must argue as in Lemma 2 that, starting from any nonequilibrium point, the path $a^1, a^2, a^3, \ldots$ will eventually end up in some neighborhood V of a locally asymptotically stable equilibrium a^* of the system (13). Once there we can apply Lemma 4 to conclude the proof. (Recall that at an asymptotically stable equilibrium of (13) it must be $\sum_{i=1}^n B_i'/(1+B_i') \leq 1$; hence $\sum_{i=1}^n B_i'/(1+B_i') \neq 1$ at such an equilibrium implies $\sum_{i=1}^n B_i'/(1+B_i') < 1$.) ■

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[7] Note that this assumption is satisfied by almost all games.

Theorem 3 is the main result of the paper. Why is global convergence obtained and why do we need actions to be either strategic substitutes or strategic complements around a Nash equilibrium? An intuitive explanation consists of two parts. First, in PAS games the stochastic process generated by the better-reply dynamics leads close to a Nash equilibrium of the game with probability one. This is because the process will eventually follow a path close to a trajectory of the adjusted best-reply dynamics. Since the adjusted best-reply dynamics can be reduced to a single ordinary differential equation, it must converge to an equilibrium. Second, if around a stable equilibrium of the adjusted best-reply dynamics actions are strategic substitutes, then an increase in action by player i tends to induce player j to reduce her action, this in turn tends to induce player i to increase her action, etc. Similarly, if actions are strategic complements, then an increase in action by player i tends to induce player j to also increase her action, which in turn induces i to follow with another increase, etc. Thus, if actions are either strategic substitutes or strategic complements near a Nash equilibrium, there is a tendency for each player to always move in the same direction. On the contrary, if actions are strategic substitutes for player i and strategic complements for player j, then after an increase in action by player i, player j will tend to increase her action, which will tend to induce player i to reduce her action, which will induce j to also reduce hers, etc. When the derivatives of the best reply functions have different slopes at a Nash equilibrium, there is no tendency for each player always to move in the same direction. In the next section we will present an example that shows that the better-reply dynamics of Definition 2 need not globally converge when the derivatives of the best-reply functions at a Nash equilibrium have different signs.

Consider again the examples introduced in Section 2.

Example 1. In the case of a homogeneous product, Cournot oligopoly, the slopes of the best-reply functions are:

$$B_i'(a_{-i}) = -\frac{P''(\Sigma)a_i + P'(\Sigma)}{P''(\Sigma)a_i + 2P'(\Sigma) - C_i''(a_i)}.$$

Thus, (7) implies that at a Nash equilibrium a^* all $B_i'(a_{-i}^*)$ have the same sign provided that either $P''(\Sigma^*)a_i^* < -P'(\Sigma^*)$ or $P''(\Sigma^*)a_i^* > -P'(\Sigma^*)$ for all i. A sufficient condition for this to be satisfied is $P''(\Sigma^*) \leq 0$ (i.e., at the total output level corresponding to a Nash equilibrium the slope of the inverse demand function is decreasing) in which case all $B_i'(a_{-i}^*)$ have a negative sign.

Example 2. In a differentiated product, price competition oligopoly, the slopes of the best-reply functions are:

$$B_i'(a_{-i}) = -\frac{\frac{\partial Q_i}{\partial \Sigma} - C_i'' \frac{dQ_i}{da_i} + (a_i - C_i')\left(\frac{\partial^2 Q_i}{\partial a_i \partial \Sigma} + \frac{\partial^2 Q_i}{\partial \Sigma^2}\right)}{\left(2 - C_i'' \frac{dQ_i}{da_i}\right)\frac{dQ_i}{da_i} + (a_i - C_i')\left(\frac{\partial^2 Q_i}{\partial a_i^2} + 2\frac{\partial^2 Q_i}{\partial \Sigma \partial a_i} + \frac{\partial^2 Q_i}{\partial \Sigma^2}\right)}.$$

A sufficient condition for them to have the same (positive) sign at a Nash equilibrium a^* is

$$\left(\frac{\partial^2 Q_i(a_i^*, \Sigma^*)}{\partial a_i \partial \Sigma} + \frac{\partial^2 Q_i(a_i^*, \Sigma^*)}{\partial \Sigma^2}\right) \geq 0.$$

Note that if, for example, the demand functions are linear, then this condition holds globally.

Example 3. In a collective action problem, the slopes of the best-reply functions are

$$B_i'(a_{-i}) = -\frac{V_i''(\Sigma)}{V_i''(\Sigma) - C_i''(a_i)}.$$

Thus, they have the same (negative) sign at a Nash equilibrium a^* provided each player's marginal benefit function is decreasing in Σ^*, $V_i''(\Sigma^*) < 0$.

Example 4. In an economy with catching-up-with-the-Joneses, the slopes of the best reply functions are

$$B_i'(a_{-i}) = \frac{\frac{2(n-1)}{2n^2} V_i'((a_i - \Sigma/n)^2) + \frac{4(n-1)}{n^2}(a_i - \Sigma/n)^2 V_i''((a_i - \Sigma/n)^2)}{\frac{2(n-1)^2}{n^2} V_i'((a_i - \Sigma/n)^2) + \frac{4(n-1)^2}{n^2}(a_i - \Sigma/n)^2 V_i''((a_i - \Sigma/n)^2) - C_i''(a_i)}.$$

They have positive slope at a Nash equilibrium a^* provided $V_i''((a_i^* - \Sigma^*/n)^2) < 0$ and $C_i''(a_i^*) > 0$.

In the next example, the slopes of the best-reply functions of the two players have different signs.

Example 5. There are two players; player 1 is an anti-conformist, player 2 is a conformist. Each player chooses an action $a_i \in [-1, 1]$. The payoff functions are

$$U_1(a) \ = \ -\left(a_1 + \beta_1 a_2\right)^2,$$

$$U_2(a) \;=\; -\left(\beta_2 a_1 - a_2\right)^2 .$$

where $1 > \beta_1 > 0$ and $\beta_2 > 0$. The functions D_i are

$$\begin{aligned}\frac{dU_1(a)}{da_1} &= D_1(a_1, \Sigma) = -2(a_1 + \beta_1 a_2) = -2(1-\beta_1)a_1 - 2\beta_1\Sigma, \\ \frac{dU_2(a)}{da_2} &= D_2(a_2, \Sigma) = 2(\beta_2 a_1 - a_2) = -2(1+\beta_2)a_2 + 2\beta_2\Sigma.\end{aligned}$$

Then, for $i = 1, 2$

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} < 0,$$

and (6) is satisfied. Note also that the functions U_i are strictly concave (hence also strictly quasi-concave) in a_i. Finally, since $B_1(a_2) = -\beta_1 a_2$ and $B_2(a_1) = \beta_2 a_1$, we have $B_1'(a_2) = -\beta_1 < 0$ and $B_2'(a_1) = \beta_2 > 0$.

5 An Example of Non-Convergence of The Better-Reply Dynamics in Discrete Time

In this section we construct an example of a 2-person, quasi-concave game g^E with a unique Nash equilibrium $a^* = (0, 0)$ at which the derivatives of the best-reply functions have different signs. We know that in this game the continuous-time, deterministic, best-reply and adjusted best-reply dynamics defined by (8) and (13) globally converge to a^* (see Corchon and Mas-Colell [3] and Theorem 1). We now show that, on the contrary, for almost all paths $a^1, a^2, \ldots$ the stochastic process described in Definition 2 does not converge to the equilibrium. First we introduce a needed lemma.

Lemma 5 *Let $\rho_1, \rho_2, \ldots$ be an infinite path of a discrete time Markov process with $r > \rho_1 > 0$.*[8] *Suppose the probability law governing the stochastic process in the interval $(0, r)$ satisfies the following inequalities:*

$$P\left(\frac{\rho_{t+1}}{\rho_t} \geq 2^{2^4}\right) \;>\; \frac{1}{4}, \tag{20}$$

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[8] The stochastic process considered in this lemma may depend on some hidden, time-varying variables, provided that their values do not influence the validity of inequalities (20) and (21).

$$P\left(\frac{\rho_{t+1}}{\rho_t} \leq \varepsilon\right) \quad < \quad \varepsilon^2 \quad \textit{for any} \quad \varepsilon > 0. \tag{21}$$

Then for almost all path $\rho_1, \rho_2, \ldots$ *there exists* T *such that* $\rho_T > r$.

Proof. See the Appendix. ■

Now we are ready to construct our example, a variation of Example 5 at the end of Section 4. The strategy set of each player $i = 1, 2$ is $A_i = [-2, 2]$. Let $\beta_1, \beta_2 > 0$ and define the best-reply functions of the players as follows:

$$B_1(a_2) = \begin{cases} -\beta_1 a_2 & \text{for } a_2 \in \left[-\frac{1}{\beta_1}, \frac{1}{\beta_1}\right], \\ -\frac{2\beta_1+\beta_1 a_2-2}{2\beta_1-1} & \text{for } a_2 > \frac{1}{\beta_1}, \\ -\frac{2+\beta_1 a_2-2\beta_1}{2\beta_1-1} & \text{for } a_2 < -\frac{1}{\beta_1}, \end{cases} \qquad B_2(a_1) = \begin{cases} \beta_2 a_1 & \text{for } a_1 \in \left[-\frac{1}{\beta_2}, \frac{1}{\beta_2}\right], \\ \frac{2\beta_2+\beta_2 a_1 x-2}{2\beta-1} & \text{for } a_1 > \frac{1}{\beta_2}, \\ \frac{2+\beta_2 a_2-2\beta_2}{2\beta_2-1} & \text{for } a_1 < -\frac{1}{\beta_2}. \end{cases}$$

Let the utility function of each player be the square of the Euclidean distance from the best reply. This defines a two-player, continuous game g^E having a unique Nash equilibrium at the point $a^* = (0, 0)$. If $\beta_1 < 1$, then condition (6) is satisfied and the game is a PAS game. In this game the continuous best-reply and adjusted best-reply dynamics defined by (8) and (13) always converge to a^*. Also, around the equilibrium the game has linear best-reply functions with $B_1'(0)B_2'(0) = -\beta_1\beta_2 < 0$.

We will show that in the game g^E, if the evolution of the action profile a follows the stochastic process described in Definition 2 (the better-reply dynamics), then play will not converge to the Nash equilibrium a^*. For simplicity we will suppose that each player's sampling probability is uniform on $A_i = [-2, 2]$ and that the probability that each player is selected to sample a new strategy is 1/2. We will denote with $a^{d(t)}$, with $d(t) > t$, the first action profile different from a^t in a path of the process, and with $P(a^{d(t)} \in S \subset A | a^t = a)$ the probability that $a^{d(t)}$ belongs to the set S given that $a = a^t$ (i.e., a is the status quo at time t).

Theorem 4 *In the game* g^E, *consider the stochastic process generated by the better-reply dynamics described in Definition 2. Assume that each player's sampling probability is uniform on* $A_i = [-2, 2]$ *and that the probability that each player is selected to sample a new strategy is* 1/2. *For all* $\beta_1 > 0$ *there exists* $\beta^0 > 0$ *such that if* $\beta_2 > \beta^0$,

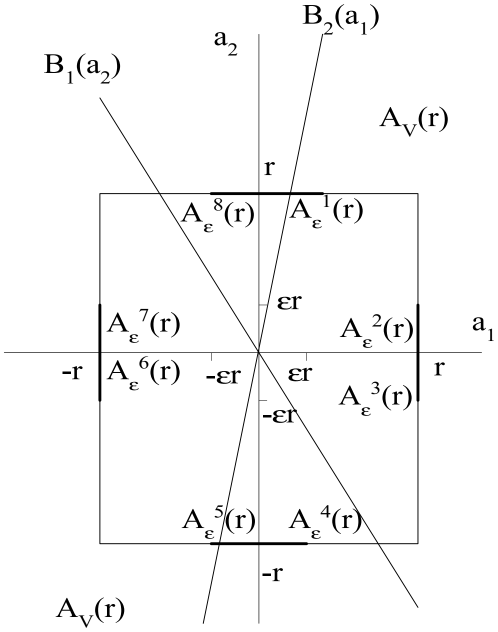

Figure 1: The set $A(r) = \{a \in A : \rho(a) = r\}$.

then for almost all paths $a^1, a^2, \ldots,$ *with* $a^1 \neq (0,0)$*, the stochastic process does not converge to the Nash equilibrium* a^* *of* g^E*.*

Proof. Define the distance $\rho(a)$ of a point $a = (a_1, a_2) \in [-2,2] \times [-2,2]$ from the origin $(0,0)$ by

$$\rho(a) = \max\{|a_1|, |a_2|\}. \tag{22}$$

By letting $\rho_t = \rho(a^t)$ and $\rho_{t+1} = \rho(a^{d(t)})$, where $a^{d(t)}$ is the first action profile different from a^t in a path of the stochastic process described in Definition 2, we can define a new process that keeps track of the evolution of the distance ρ. Let $A(r) = \{a \in A : \rho(a) = r\}$, $A^-(r) = \{a \in A : \rho(a) \leq r\}$ and $A^+(r) = \{a \in A : \rho(a) \geq r\}$. Define $\lambda^{A(r)}$ as the Lebesque (or uniform) probability measure over $A^{(r)}$; then we have

$$\begin{aligned} P(\rho_{t+1} &\leq s|\rho_t = r) = \int_{A(r)} P(a^{d(t)} \in A^-(s)|a^t = a)d\lambda^{A(r)}, \\ P(\rho_{t+1} &\geq s|\rho_t = r) = \int_{A(r)} P(a^{d(t)} \in A^+(s)|a^t = a)d\lambda^{A(r)}. \end{aligned}$$

Let $V = \{a : \rho(a) < \varepsilon^V\}$ be a small neighborhood of the point $(0,0)$. To prove our claim about nonconvergence we need to show that almost all paths starting in V leave V; that is, there is T for which $a^T \notin V$. To establish this we will show that the probability law of the process governing the evolution of the distance ρ satisfies the inequalities in Lemma 5.

First note that we can take $\varepsilon^V > 0$ small enough so that in the neighborhood V around $(0,0)$ the game has linear best-reply functions. Second, recall that if a^t is the prevailing strategy profile, then the strategies that improve player 1's payoff are the strategies belonging to the interval $I_1(a^t)$ with endpoints a_1^t and $-2\beta_1 a_2^t - a_1^t$, while the strategies improving player 2's payoff are the ones in the interval $I_2(a^t)$ with endpoints a_2^t and $2\beta_2 a_1^t - a_2^t$.

We will begin by showing that in V we have $P(\rho_{t+1} \geq 2^{2^4} r | \rho_t = r) > 1/4$ for all r with $\varepsilon^V \geq r > 0$ and hence equation (20) holds. Suppose $\rho_t = r$, or equivalently $a^t \in A(r)$. Let $A_V(r) = \{a \in A(r) : |a_1| = r\}$ be the vertical sides of the square $A(r)$ in $\mathbb{R}^2$ (see Figure 1) and $\lambda^{A_V(r)}$ be the Lebesque (or uniform) probability measure over $A_V^{(r)}$. Observe that

$$P(\rho_{t+1} \geq 2^{2^4} r | \rho_t = r) \geq \frac{1}{2} \int_{A_V(r)} P(a^{d(t)} \in A^+(2^{2^4} r) | a^t = a) d\lambda^{A_V(r)}.$$

We will show that for $\beta_2 > 2^{2^4} + 2 + \beta_1$ it is $P(a^{d(t)} \in A^+(2^{2^4} r) | a^t \in A_V(r)) > 1/2$ and hence $P(\rho_{t+1} \geq 2^{2^4} r | \rho_t = r) > 1/4$. For $\beta_2 > 2^{2^4} + 2 + \beta_1$ we have

$$P(a^{d(t)} \in A^+(2^{2^4} r) | a^t \in A_V(r)) > \frac{2\beta_2 r - a_2^t - 2^{2^4} r}{2\beta_2 r - 2a_2^t} \cdot \frac{2\beta_2 r - 2a_2^t}{2\beta_2 r - 2a_2^t + |2\beta_1 a_2^t + 2r|},$$

where the first term is the probability that $a_2^{d(t)} > 2^{2^4} r$ given that player 2 is the first to move, and the second term is the probability that player 2 is the first to move. It follows that $P(a^{d(t)} \in A^+(2^{2^4} r) | a^t \in A_V(r)) > 1/2$ provided that

$$\begin{aligned} 2\left(2\beta_2 r - a_2^t - 2^{2^4} r\right) &> 2\beta_2 r - 2a_2^t + \left|2\beta_1 a_2^t + 2r\right|, \quad \text{or} \\ \beta_2 r &> 2^{2^4} r + \left|\beta_1 a_2^t + 2r\right| \end{aligned}$$

which holds if $\beta_2 > 2^{2^4} + 2 + \beta_1$. This completes the proof that in V equation (20) holds.

It remains to show that equation (21) also holds in V. Take $a^t \in A(r)$, so that $\rho_t = r$. We need to show that $P(\rho_{t+1} \leq \varepsilon r | \rho_t = r) < \varepsilon^2$ for all $\varepsilon^V \geq r > 0$ and all $1 \geq \varepsilon > 0$. The only way the distance from the origin can decrease rapidly, that is $\rho_{t+1} \leq \varepsilon r$, is if $|a_i^t| \leq \varepsilon r$ for some $i = 1, 2$, and player $j \neq i$ moves to $a_j^{d(t)}$ with $\left| a_j^{d(t)} \right| \leq |a_i^t|$. Let

$$\begin{aligned}
A_\varepsilon^1(r) &= \{a \in A(r) : 0 \leq a_1^t \leq \varepsilon r \text{ and } a_2^t = r\}, \\
A_\varepsilon^2(r) &= \{a \in A(r) : a_1^t = r \text{ and } 0 \leq a_2^t \leq \varepsilon r\}, \\
A_\varepsilon^3(r) &= \{a \in A(r) : a_1^t = r \text{ and } -\varepsilon r \leq a_2^t \leq 0\}, \\
A_\varepsilon^4(r) &= \{a \in A(r) : 0 \leq a_1^t \leq \varepsilon r \text{ and } a_2^t = -r\}, \\
A_\varepsilon^5(r) &= \{a \in A(r) : -\varepsilon r \leq a_1^t \leq 0 \text{ and } a_2^t = -r\}, \\
A_\varepsilon^6(r) &= \{a \in A(r) : a_1^t = -r \text{ and } -\varepsilon r \leq a_2^t \leq 0\}, \\
A_\varepsilon^7(r) &= \{a \in A(r) : a_1^t = -r \text{ and } 0 \leq a_2^t \leq \varepsilon r\}, \\
A_\varepsilon^8(r) &= \{a \in A(r) : -\varepsilon r \leq a_1^t \leq 0 \text{ and } a_2^t = r\},
\end{aligned}$$

(see Figure 1). Let $\lambda^{A_\varepsilon^i(r)}$ be the uniform probability measure over $A_\varepsilon^i(r)$; and observe that

$$P(\rho_{t+1} \leq \varepsilon r | \rho_t = r) = \frac{\varepsilon}{8} \sum_{i=1}^{8} \int_{A_\varepsilon^i(r)} P(a^{d(t)} \in A^-(\varepsilon r) | a^t = a) d\lambda^{A_\varepsilon^i(r)}. \tag{23}$$

There are four different cases. (1) If either $a \in A_\varepsilon^1(r)$ or $a \in A_\varepsilon^5(r)$, then we have

$$P(a^{d(t)} \in A^-(\varepsilon r) | a^t = a) \leq \frac{2\varepsilon r}{(r + \varepsilon r) + (2a_1^t + 2\beta_1 r)} < \frac{2\varepsilon}{1 + 2\beta_1}. \tag{24}$$

(2) If either $a \in A_\varepsilon^2(r)$ or $a \in A_\varepsilon^6(r)$, then we have

$$P(a^{d(t)} \in A^-(\varepsilon r) | a^t = a) \leq \frac{2\varepsilon r}{(2\beta_2 r - 2a_2^t) + (2r + 2\beta_1 a_2^t)} = \frac{\varepsilon r}{(1 + \beta_2) r + (\beta_1 - 1) a_2^t}, \tag{25}$$

which is less than ε provided that $\beta_2 > |\beta_1 - 1| \varepsilon$.

(3) If either $a \in A_\varepsilon^3(r)$ or $a \in A_\varepsilon^7(r)$, then we have

$$P(a^{d(t)} \in A^-(\varepsilon r) | a^t = a) \leq \frac{2\varepsilon r}{(2\beta_2 r - 2a_2^t) + r + \varepsilon r} < \frac{2\varepsilon}{1 + 2\beta_2}. \tag{26}$$

Finally, (4) if either $a \in A_\varepsilon^4(r)$ or $a \in A_\varepsilon^8(r)$, then we have

$$P(a^{d(t)} \in A^-(\varepsilon r) | a^t = a) \leq \frac{2\varepsilon r}{(2\beta_2 a_1^t + 2r) + 2\left|\beta_1 r - a_1^t\right|} < \varepsilon. \tag{27}$$

Adding up the left hand sides of equations (24) and (26) we obtain

$$\frac{2\varepsilon}{1 + 2\beta_1} + \frac{2\varepsilon}{1 + 2\beta_2} < 2\varepsilon \quad \text{for} \quad \beta_2 > \frac{1}{4\beta_1}.$$

Thus, for a sufficiently large β_2 equation (23) implies that $P(\rho_{t+1} \leq \varepsilon r | \rho_t = r) < \varepsilon^2$. This completes the proof that equation (20) holds in V.

Applying Lemma 5 to the stochastic process governing the evolution of the distance of the state of the system from the origin, we see that for any $(0,0) \neq a^t \in V$ and for a sufficiently large k, $a^{n+k} \notin V$. Hence we cannot have $(0,0) = a^* = \lim_{t \to \infty} a^t$. This concludes the proof. ■

Lemma 2 shows that there is a positive probability that in the game g^E discussed in this section the stochastic process described in Definition 2 enters any small neighborhood U of the equilibrium $(0,0)$; Theorem 4 shows that it is also the case that almost any path of the process will leave the neighborhood U. This explains why in Theorem 2 we must impose the condition $B_1'(a_2^*)B_2'(a_1^*) > 0$ at a Nash equilibrium a^* to guarantee that the system globally converges.

Theorem 4 is related to a result by Gale and Rosenthal [10]. They studied a model with an experimenter and an imitator. At each point in time, the experimenter samples new actions and moves to a better response, while the imitator adjusts her action towards the current action of the experimenter. The experimenter's (player 1) best-reply function is $B_1 = \gamma a_2$, where γ could be positive or negative. Gale and Rosenthal [10] showed that if γ is negative (and sufficiently small) then the system leaves any sufficiently small neighborhood of the unique Nash equilibrium $(0,0)$ with probability one, while if γ is positive then the system globally converges to the equilibrium. To relate this result to our model, note that we can think that the imitator acts as if her best-reply function were $B_2 = a_1$. Then, applying our better-reply dynamics, we would also obtain global convergence when $\gamma > 0$ and no convergence for a sufficiently small, negative γ.

6 The Better-Reply Dynamics for Finite Games

So far we have considered games with continuous strategy sets. In reality, we often model discrete games as being continuous to simplify the analysis. For example, even though in many instances a firm is only able to produce goods and price in discrete units, we express the relationship between price and demand of a good as a continuous function and we let the quantity demanded and the price be real numbers. In this section we show that there are important differences in the convergence properties of the better-reply dynamics with a discrete and with a continuous state space.

First, in Section 6.1 we demonstrate that almost all playing-against-the-sum games have the weak FIP; that is, starting from any action profile there is a finite sequence of single-player improvements that leads to a Nash equilibrium. In Section 6.2 we show that this implies that in any sufficiently fine discretization of almost all PAS games the better reply dynamics converges in finite time to a point arbitrarily close to a Nash equilibrium. Thus, weaker conditions are needed to guarantee global convergence to Nash equilibrium with discrete than with continuous strategy sets (with discrete strategy sets actions need not be strategic complements or substitutes at the Nash equilibria that are stable under (13)).

6.1 The Weak Finite Improvement Property

Recall that the strategy profile $a \backslash x_i \in A$ is a single-player improvement over $a \in A$ if and only if $U_i(a \backslash x_i) > U_i(a)$. Friedman and Mezzetti [7] introduced the following definition.

Definition 3 *The game* $g = \langle N, A, U \rangle$ *has the* weak finite improvement property (weak FIP) *if from all action profiles* $a \in A$ *there exists a finite sequence of single-player improvements that ends in a pure strategy Nash equilibrium.*

The weak FIP should be contrasted with the *finite improvement property* (FIP) as defined by Monderer and Shapley [17] (they proved that any finite ordinal potential game has the FIP). A game has the FIP if any sequence of single-player improvements ends after a finite number of steps; that is, if there are no single-player improvement cycles. In contrast, a game with the weak FIP can have single-player improvement cycles, provided that there is a single-player improvement leading out of any such cycle. See Friedman and Mezzetti [7] for an example of a game that has the weak FIP, but not the FIP.

Friedman and Mezzetti [7] proved the following theorem.

Theorem 5 *Any transversal,* 2*-person, quasi-concave game* $g = \langle \{1,2\}, A, U \rangle$ *has the weak FIP.*

Recall that, by definition, all quasi-concave 2-person games that satisfy condition (6) are playing-against-the-sum-games. We will extend Theorem 5 by showing that almost all transversal, n-person, playing-against-the-sum-games also have the weak FIP. We begin with a lemma that deals with 2-person games with linear best-reply and M_i functions:

$$\begin{aligned} B_i(a_j) &= \beta_i a_j \\ M_i(a_1 + a_2) &= C_i(a_1 + a_2) \end{aligned} \tag{28}$$

where β_i and C_i are constants and $C_i = \beta_i/(1+\beta_i)$, $i = 1, 2$, $j \neq i$.

Lemma 6 *Let* $\mathcal{C}$ *be the set of all pairs* C_1, C_2 *with* $C_i < 1$ *and* $C_1 + C_2 < 1$ *and let* $g = \langle \{1,2\}, A, U \rangle$ *be a transversal,* 2*-person game with best-reply and* M_i *functions given by equation (28) with* $(C_1, C_2) \in \mathcal{C}$.[9] *For almost all* $(C_1, C_2) \in \mathcal{C}$ *(i.e., with the possible exception of a subset of* $\mathcal{C}$ *having zero Lebesgue measure) the following claims hold: (a) Given any action profile* $a^0 = (a_1^0, a_2^0)$ *and any number* $\theta \in (-1, 1)$*, there exists a finite sequence of single-player improvements* $\{a^0, a^1, \ldots, a^T\}$ *such that* $a_1^T + a_2^T = \theta(a_1^0 + a_2^0)$*. (b) If* $\theta = 0$*, the sequence* $\{a^0, a^1, \ldots, a^T\}$ *can be chosen so that* $a^T = (0, 0)$*.*

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[9] Note that the game g is transversal. For 2-person games the best reply functions at a Nash equilibrium are transversal if and only if $\beta_1 \beta_2 \neq 1$, or equivalently $C_1 + C_2 \neq 1$.

Proof. See the Appendix. ■

Let $\ell(\widehat{a})$ be the line in $\mathbb{R}^2$ with slope -1 which intercepts the segment with endpoints a^0 and $-a^0$ at the interior point $\widehat{a}$. Part (a) of Lemma 6 says that it is possible to find a single-player improvement path that starts at a^0 and reaches a point a^T on the line $\ell(\widehat{a})$ after T steps. Part (b) says that if the line goes through the origin we can choose $a^T = (0,0)$; that is, there is a single-player improvement path from a^0 to the Nash equilibrium $(0,0)$. The lemma is needed in the proof of the following theorem, which is the main result of this section.

Theorem 6 *Almost all transversal, n-person, playing-against-the-sum games $g = \langle N, A, U \rangle$ have the weak finite improvement property.*

Proof. First note that by Lemma 2 a finite number of single-player improvements are sufficient to move the n players from an arbitrary starting point a^0 to a ball of any given radius $r > 0$ around an isolated Nash equilibrium a^*. We now proceed by induction. We know from Theorem 5 that every transversal, 2-person, playing-against-the-sum game has the weak FIP. More precisely, the proof in Friedman and Mezzetti [7] shows that from any small neighborhood of an asymptotically stable Nash equilibrium there is a finite, single-player, improvement path leading to the equilibrium. Suppose that almost all transversal, $(n-1)$-person, playing-against-the-sum games have this property. We will show then that the property must also hold for n-person games.

Consider a transversal, n-person, playing-against-the-sum game g. By Lemma 2, from any starting point a^0, we will reach a point a^r that lies in a neighborhood V_r around an asymptotically stable Nash equilibrium a^*. By changing coordinates we can assume, without loss of generality, that the Nash equilibrium a^* is at the origin: $a^* = (0, \ldots, 0)$. Since r is arbitrary, we can choose it small enough so that the players' payoff functions are closely approximated by quadratic functions. This implies that the best-reply functions are of the form

$$B_i(a_{-i}) \simeq \beta_i \Sigma_{-i} \tag{29}$$

which in turn implies that

$$\frac{dU_i(a)}{da_i} = D_i(a_i, \Sigma) \simeq \gamma_i \left(\beta_i \Sigma - (1+\beta_i) a_i\right)$$

where γ_i is a constant. Then, by equation (11), the M_i functions are

$$M_i(\Sigma) = C_i \Sigma \qquad \text{where } C_i = \frac{\beta_i}{1+\beta_i}. \tag{30}$$

Quasi-concavity of U_i (Condition (5)) implies $\gamma_i > 0$, while condition (6) requires $\gamma_i(1+\beta_i) > 0$. Hence we have $\beta_i > -1$ and $1 - C_i = 1/(1+\beta_i) > 0$, or, equivalently, $C_i < 1$. Furthermore, since a^* is asymptotically stable under the dynamics defined by equation (13), it must be $\sum_{i=1}^n C_i \leq 1$.

We need to show that from a^r there is a finite sequence of single-player improvements leading to $(0, \ldots, 0)$. Note that $\sum_{i=1}^n C_i \leq 1$ implies that there are at least two players i and j such that $C_i + C_j \leq 1$; without loss of generality, we will assume that $C_{n-1} + C_n \leq 1$. Furthermore, since transversality of the game g implies $\beta_{n-1}\beta_n \neq 1$ or, equivalently, $C_{n-1} + C_n \leq 1$, it must be $C_{n-1} + C_n < 1$. Define a new game $\widetilde{g} = \langle \{1, 2, \ldots, n-1\}, \widetilde{A}_i, \widetilde{U}_i \rangle$ with $(n-1)$ players as follows. The first $n-2$ players are as in game g; that is, the strategy sets are $\widetilde{A}_i = A_i$ and the payoff functions are $\widetilde{U}_i(\widetilde{a}_i, \widetilde{a}_{-i}) = U_i(\widetilde{a}_i, \widetilde{a}_{-i})$ for $i = 1, 2, \ldots, n-2$. Player $(n-1)$ in $\widetilde{g}$ has the strategy set $\widetilde{A}_{n-1} = A_{n-1} + A_n$, and his payoff function is:

$$\widetilde{U}_{n-1}(\widetilde{a}) \simeq \widetilde{\mu}_{n-1} - \left(\frac{C_{n-1} + C_n}{1 - C_{n-1} - C_n} \sum_{j=1}^{n-2} \widetilde{a}_j - \widetilde{a}_{n-1} \right)^2$$

where $\widetilde{\mu}_{n-1}$ is a constant. Letting $\widetilde{\Sigma}_{-(n-1)} = \sum_{j=1}^{n-2} \widetilde{a}_j$, the best-reply function of player $(n-1)$ in $\widetilde{g}$ is given by:

$$\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)}) = \frac{C_{n-1} + C_n}{1 - C_{n-1} - C_n} \widetilde{\Sigma}_{-(n-1)}.$$

Let $\widetilde{a}^r$ be the strategy profile in $\widetilde{g}$ corresponding to a^r in g: $\widetilde{a}^r = \left(\widetilde{a}_1^r, \ldots, \widetilde{a}_{n-2}^r, \widetilde{a}_{n-1}^r\right)$ $= \left(a_1^r, \ldots, a_{n-2}^r, a_{n-1}^r + a_n^r\right)$. Note that $\widetilde{a}^r$ is in a small neighborhood of the $(n-1)$-dimensional zero vector, which is a Nash equilibrium of the game $\widetilde{g}$. Hence, by the induction hypothesis, there is a finite sequence $\widetilde{S}$ of single-player improvements in $\widetilde{g}$ starting at $\widetilde{a}^r$ and leading to $\left(\widetilde{a}_1^*, \ldots, \widetilde{a}_{n-1}^*\right) = (0, \ldots, 0)$. Observe that each step in this sequence in which the improving player is $i < n-1$ also corresponds to an improvement for player i in game g. Next, consider a step, say from $\widetilde{a}$ to $\widetilde{b} = \widetilde{a} \backslash \widetilde{b}_{n-1}$, in the sequence $\widetilde{S}$ in which the improving player in $\widetilde{g}$ is $(n-1)$. Let $\widetilde{a}_{n-1} = a_{n-1}^0 + a_n^0$. We will show that we can find a finite sequence S in g going from $(\widetilde{a}_1, \ldots, \widetilde{a}_{n-2}, a_{n-1}^0, a_n^0)$ to $(\widetilde{a}_1, \ldots, \widetilde{a}_{n-2}, a_{n-1}^T, a_n^T)$, where $a_{n-1}^T + a_n^T = \widetilde{b}_{n-1}$, in which at each step either player $(n-1)$ or player n improves her payoff.

First, note that the payoff of player $(n-1)$ in game $\widetilde{g}$ must have improved in moving from $\widetilde{a}$ to $\widetilde{b}$; that is, $\widetilde{b}_{n-1}$ must be closer to player $(n-1)$'s best reply $\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)})$ than $\widetilde{a}_{n-1}$. This implies that

$$\begin{aligned}
\widetilde{b}_{n-1} &= \lambda \widetilde{a}_{n-1} + (1-\lambda)\left(2\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)}) - \widetilde{a}_{n-1}\right) \\
&= \lambda\left(a_{n-1}^0 + a_n^0\right) + (1-\lambda)\left[2\frac{C_{n-1}+C_n}{1-C_{n-1}-C_n}\widetilde{\Sigma}_{-(n-1)} - \left(a_{n-1}^0 + a_n^0\right)\right] \qquad (31) \\
&= (2\lambda - 1)\left(a_{n-1}^0 + a_n^0\right) + 2(1-\lambda)\frac{C_{n-1}+C_n}{1-C_{n-1}-C_n}\widetilde{\Sigma}_{-(n-1)}
\end{aligned}$$

for some $\lambda \in (0,1)$, where $2\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)}) - \widetilde{a}_{n-1}$ is the point on the line going through $\widetilde{a}_{n-1}$ and $\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)})$ whose distance from $\widetilde{B}_{n-1}(\widetilde{a}_{-(n-1)})$ is the same as $\widetilde{a}_{n-1}$. Next, consider the 2-person game $\widehat{\widehat{g}} = \langle \{n-1, n\}, X_{n-1} \times X_n, \{U_{n-1}, U_n\} \rangle$ derived from g by forcing players $i = 1, \ldots, n-2$ to play actions $\widetilde{a}_i$ and by changing the $n-1$ and n coordinate as follows:

$$x_{n-1} = a_{n-1} - \frac{C_{n-1}\widetilde{\Sigma}_{-(n-1)}}{1-C_{n-1}-C_n} \qquad\qquad x_n = a_n - \frac{C_n\widetilde{\Sigma}_{-(n-1)}}{1-C_{n-1}-C_n}. \qquad (32)$$

The strategy spaces in $\widehat{\widehat{g}}$ are $X_i = A_i - C_i\widetilde{\Sigma}_{-(n-1)}/(1-C_{n-1}-C_n)$. Using (29), (30) and (32), simple algebra shows that the best reply and the M_i functions in $\widehat{\widehat{g}}$ are:

$$B_i(x_j) = \frac{C_i}{1-C_i}x_j \qquad i,j = n-1, n, \qquad i \neq j$$

$$M_i(x_{n-1} + x_n) = C_i(x_{n-1} + x_n) \qquad i,j = n-1, n, \qquad i \neq j$$

Let x_{n-1}^0 and x_n^0 be the actions corresponding to a_{n-1}^0 and a_n^0 under the new coordinates, and let $\widetilde{y}_{n-1}$ correspond to $\widetilde{b}_{n-1}$. By equations (31) and (32)

$$\widetilde{y}_{n-1} = \widetilde{b}_{n-1} - \frac{C_{n-1} + C_n}{1 - C_{n-1} - C_n} \widetilde{\Sigma}_{-(n-1)} = (2\lambda - 1) \left(x_{n-1}^0 + x_n^0 \right)$$

Since $(2\lambda - 1) \in (-1, 1)$, Lemma 6 then implies that for almost all games $\widehat{\widehat{g}}$ there exists a finite sequence $\widehat{\widehat{S}}$ of single-player improvements from (x_{n-1}^0, x_n^0) to (x_{n-1}^T, x_n^T), where $x_{n-1}^T + x_n^T = (2\lambda - 1) \left(x_{n-1}^0 + x_n^0 \right)$. This sequence corresponds to a sequence S in game g going from $(\widetilde{a}_1, \ldots, \widetilde{a}_{n-2}, a_{n-1}^0, a_n^0)$ to $(\widetilde{a}_1, \ldots, \widetilde{a}_{n-2}, a_{n-1}^T, a_n^T)$, where $a_{n-1}^T + a_n^T = \widetilde{b}_{n-1}$.

The profile $\left(\widetilde{a}_1^*, \ldots, \widetilde{a}_{n-1}^* \right)$ in $\widetilde{g}$ corresponds to the profile $(0, \ldots, 0, a_{n-1}, a_n)$, with $a_{n-1} + a_n = 0$, in g. Thus, combining the sequences $\widetilde{S}$ and S we obtain a finite sequence of single-player improvements in the game g going from a^r to some profile $(0, \ldots, 0, a_{n-1}, a_n)$, where the projection (a_{n-1}, a_n) of this profile on the last two coordinates is in a small neighborhood of $(0, 0)$. Since $(0, 0)$ is a Nash equilibrium of the 2-person game $\widehat{g}$ derived from g by forcing players $i = 1, \ldots, n-2$ to play action $a_i = 0$, we know from Theorem 5 that there is a finite sequence $\widehat{S}$ of single-player improvements in $\widehat{g}$ leading to $(0, 0)$. Each step of the sequence $\widehat{S}$ corresponds to an improvement by either player $(n-1)$ or player n in game g and thus there is a finite sequence of single-player improvements in g going from $(0, \ldots, 0, a_{n-1}, a_n)$ to the Nash equilibrium $(0, \ldots, 0)$. This concludes the proof. ■

We will use Theorem 6 to show that in any sufficiently fine, finite discretization of almost all continuous, playing-against-the-sum games g, the better-reply dynamics described after Definition 3 converges in finite time to a point arbitrarily close to a Nash equilibrium of g.

6.2 The Better-Reply Dynamics

We begin by modifying the better reply dynamics described in Definition 2 to fit the case in which the strategy space of each player is finite.

Definition 4 THE BETTER-REPLY DYNAMICS FOR FINITE GAMES. *Consider a finite game $g^F = \langle N, A^F, U^F \rangle$. At each discrete time period t there is a status quo action profile a^t. A single player $i \in N$ is randomly selected, with all players having* *positive selection probability. Player i randomly samples an action $a_i^E \in A_i^F$, with all the elements of A_i^F having positive probability of being sampled. If $U_i^F(a^t \backslash a_i^E) > U_i^F(a^t)$ then $a^t \backslash a_i^E$ becomes the new status quo, $a^{t+1} = a^t \backslash a_i^E$. If $U_i^F(a^t \backslash a_i^E) \leq U_i^F(a^t)$ then the status quo does not change, $a^{t+1} = a^t$.*

The weak FIP is an important property in the study of adaptive dynamics in finite games. If the game g is finite and has the weak FIP, then the better-reply dynamics of Definition 4 will converge to an equilibrium of g.

We now discretize the strategy sets of a continuous game g. In order not to introduce artificial instability, we assume that all the actions corresponding to a Nash equilibrium in the original continuous game are available to players in the discretized version.

Definition 5 *Let $A_i = [\underline{a}_i, \overline{a}_i]$ be the strategy set of player i in the continuous game $g = \langle N, A, U \rangle$. We say that a partition $\underline{a}_i = a_i^0 < a_i^1 < a_i^2 < \ldots < a_i^H = \overline{a}_i$ of A_i is ε-fine if for all $h = 1, \ldots, H$: $\left| a_i^h - a_i^{h-1} \right| < \varepsilon$. We call a finite game $g^F = \left\langle N, A^F, U^F \right\rangle$, where $A^F = A_1^F \times \ldots \times A_n^F$ an ε-fine discretization of the continuous game g if the following properties hold. (a) Each set A_i^F is an ε-fine partition of A_i. (b) If $a^* = (a_1^*, \ldots, a_n^*)$ is a Nash equilibrium of g, then $a_i^* \in A_i^F$ for all $i \in N$. (c) The payoff functions U_i^F of the game g^F are the restrictions of the payoff functions U_i of the game g to the set A^F.*

All transversal, continuous games g have a finite number of Nash equilibria and thus admit at least one ε-fine discretization g^F, for any $\varepsilon > 0$. By choosing ε sufficiently small, the finite game g^F can be made arbitrarily close to the continuous game g.

We now show that for a sufficiently fine discretization g^F of a playing-against-the-sum game g, the Markov process generated by the better-reply dynamics of Definition 4 converges in finite time to a Nash equilibrium of g^F which lies within a small distance from a Nash equilibrium of g.

Theorem 7 *For almost all transversal, n-person, playing-against-the-sum games $g = \langle N, A, U \rangle$ and for all $r > 0$ there exists $\varepsilon_0 > 0$ such that if $g^F = \left\langle N, A^F, U^F \right\rangle$ is an ε-fine discretization of g and $0 < \varepsilon < \varepsilon_0$, then g^F has the weak finite improvement*

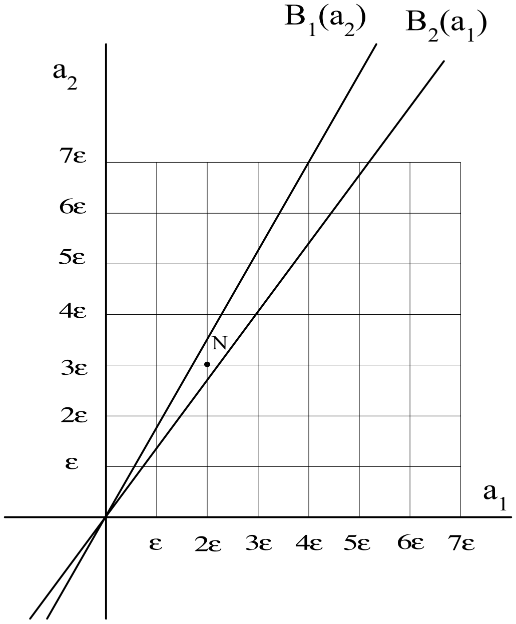

Figure 2: Nash Equilibria of a Discretized Game.

property and the Markov process described in Definition 4 converges in finite time to a Nash equilibrium a^F *of* g^F *which is contained in a ball of radius* r *around a Nash equilibrium* a^* *of* g.

Proof. Theorem 6 shows that, given any $r > 0$, each trajectory $a(t)$ of the dynamical system (13) can be replaced by a finite sequence of single-player improvements leading first inside a ball of radius r around a Nash equilibrium a^* of g and then into a^*. This implies that, given r, if ε is sufficiently small (i.e., $\varepsilon < \varepsilon_0$), then any ε-fine discretization g^F of g has the property that starting from all $a^1 \in A^F$ there is a finite sequence of single-player improvements leading inside a ball of radius r around a Nash equilibrium of g. Observe, however, that inside the ball there may be Nash equilibria of the discretized game g^F that are not Nash equilibria of g. For example, in Figure 2 the profile N is not a Nash equilibrium of the original game, but it is an equilibrium of an ε-fine discretization. We can nevertheless conclude that g^F has the weak finite improvement property. It follows that, starting from any state $a^1 \in A^F$, almost all paths of the stochastic process described in Definition 4 will reach a Nash equilibrium of g^F in finite time. ■

Comparing Theorem 7 with Theorem 3 reveals that global convergence to Nash equilibrium requires less stringent conditions when the state space is discrete than when it is continuous (actions need not be strategic complements or substitutes at any Nash equilibrium). This may seem puzzling. In particular, the convergence result of Theorem 7 for a discrete state space, and the non-convergence of the example in Section 5 with a continuous state space (see Theorem 4) may seem in conflict. However, observe that we can use the proof of Theorem 4 to show that in an ε-fine discretization of the game g^E described in Section 5, the average time that the better-reply dynamics of Definition 5 takes to converge to the Nash equilibrium goes to infinity as ε goes to zero. Thus, there is really no conflict between Theorem 7 and Theorem 4.

7 Conclusions

We have studied the global convergence properties of a stochastic adjustment process, the better-reply dynamics, in which at each discrete point in time a player is randomly selected to sample one of her available actions. The player only changes her current action if the sampled action improves her payoff.

Our convergence results are considerably stronger than existing results on the convergence of the deterministic best-reply dynamics. Gabay and Moulin [9] and Moulin [18] (see also Moulin [19]) showed that the deterministic, discrete-time, best-reply dynamics globally converges to the unique Nash equilibrium if players' payoff functions are strictly concave and an additional condition on the second derivatives of the payoff function is satisfied. This condition requires that the sum of the absolute values of the cross partial derivatives of a player's payoff function with respect to her own action and the other players' actions is less than the absolute value of the second derivative of the player's payoff function with respect to her own action. Al-Nowaihi and Levine [1] proved global convergence to the unique Nash equilibrium for the continuous-time version of the best-reply dynamics of the homogeneous-product, Cournot model when the difference between price and marginal cost is a decreasing function of the firm's output, the best-reply functions have negative slope everywhere and there are at most 5 firms (Al-Nowaihi and Levine show that the claim made by Hahn [12] and Okuguchi [20] that this result holds for any number of firms is incorrect). Dastidar [5] showed that if there is a unique Cournot equilibrium, then the equilibrium is locally stable under fairly general conditions. Vives [23] observed that a result of Hirsch [13] implies that the continuous-time, best-reply dynamics globally converges to a Nash equilibrium if the signs of the partial derivatives of the best-reply functions of all players are positive everywhere. Thorlund-Petersen [22] studied a variant of the deterministic, discrete-time, best-reply dynamics of the Cournot model, in which players best reply to the time average of the total output of their opponents, rather than to their current total output. This dynamics is analogous to the process known as fictitious play in finite games (see Fudenberg and Levine [8] for a survey of results on fictitious play in finite games). Thorlund-Petersen [22] showed that if the difference between price and marginal cost is a decreasing function of the firm's output and the best-reply functions have negative slope everywhere, then his dynamics globally converges to the unique Nash equilibrium, independently of the number of firms.

We have shown that when the action space is continuous, global convergence to a Nash equilibrium in a playing-against-the-sum game occurs provided that the actions of all players are either locally strategic complements or locally strategic substitutes at all Nash equilibria that are stable under the deterministic, continuous-time, adjusted best-reply dynamics defined by equation (13). We used an example to show that if the slopes of the best-reply functions at a Nash equilibrium have different signs, then the better-reply dynamics may not converge to a Nash equilibrium. We also showed that in any discretization of a continuous PAS game the better-reply dynamics converges to an action profile that is close to a Nash equilibrium of the original game.

Appendix

Proof of Lemma 3.

(a) If $\beta_i < 0$ for some i then γ is unbounded from below ($\gamma \to -\infty$ if $a_i \to \infty$ and $a_j = 0$ for $j \neq i$). Hence, if γ is bounded from below it must be $\beta_i > 0$ for all i. We proceed by induction. When $n = 2$, $\beta_i > 0$ and $\beta_1\beta_2 \leq 1$, then γ is a convex function with a minimum at $(0,0)$; if $\beta_1\beta_2 < 1$ then γ is strictly convex and the minimum at $(0,0)$ is strict and unique. When $\beta_i > 0$, $\beta_1\beta_2 \leq 1$ is equivalent to $\sum_{i=1}^{2} \beta_i/(1+\beta_i) \leq 1$. To see that $\beta_1\beta_2 \leq 1$ is also necessary for γ to be unbounded from below, suppose that $\beta_1\beta_2 > 1$ and take $a_2 = \beta_2 a_1$. Then we have

$$\gamma(a_1, \beta_2 a_1) = \frac{1}{\beta_1} a_1^2 + \beta_2 a_1^2 - 2\beta_2 a_1^2 = a_1^2 \left(\frac{1 - \beta_1\beta_2}{\beta_1} \right) \to -\infty \qquad \text{as} \qquad a_1 \to \infty.$$

Suppose now that for $n-1$ it is also true that $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is bounded from below with an infimum of 0 if and only if $\beta_i > 0$ and $\sum_{i=1}^{n-1} \beta_i/(1+\beta_i) \leq 1$. We will show that the same property holds for $\gamma : \mathbb{R}^n \to \mathbb{R}$. Fix a_{-n} and view $a_n \mapsto \gamma(a)$ as a function of one variable. Since $\beta_n > 0$, this function attains its minimum at the point $a_n = \beta_n(a_1 + \ldots + a_{n-1})$. By replacing this value in equation (17) we can define the function $\widetilde{\gamma} : \mathbb{R}^{n-1} \to \mathbb{R}$ as follows

$$\begin{aligned}
\widetilde{\gamma}(\cdot) &= \frac{1}{1+\beta_n} \gamma(a_1, \ldots, a_{n-1}, \beta_n(a_1 + \ldots + a_{n-1})) \qquad (33) \\
&= \frac{1}{1+\beta_n} \left[\sum_{i=1}^{n-1} \frac{1}{\beta_i} a_i^2 + \beta_n \left(\sum_{i=1}^{n-1} a_i \right)^2 - 2 \left(\sum_{1 \leq i < j \leq n-1} a_i a_j \right) - 2\beta_n \left(\sum_{i=1}^{n-1} a_i \right)^2 \right] \\
&= \frac{1}{1+\beta_n} \sum_{i=1}^{n-1} \left(\frac{1}{\beta_i} - \beta_n \right) a_i^2 - 2 \left(\sum_{1 \leq i < j \leq n-1} a_i a_j \right) \\
&= \sum_{i=1}^{n-1} \frac{1}{\widetilde{\beta}_i} a_i^2 - 2 \left(\sum_{1 \leq i < j \leq n-1} a_i a_j \right) \qquad \text{where} \quad \widetilde{\beta}_i = \frac{(1+\beta_n)\beta_i}{1 - \beta_i\beta_n}
\end{aligned}$$

Note that $\widetilde{\beta}_i > 0$ provided $\beta_i\beta_n < 1$ and that $\sum_{i=1}^{n} \beta_i/(1+\beta_i) \leq 1$ implies $\beta_i\beta_j < 1$ for all i, j, $i \neq j$. Thus, if we apply the induction hypothesis to the function $\widetilde{\gamma} : \mathbb{R}^{n-1} \to \mathbb{R}$

we have that $\widetilde{\gamma}$ is bounded from below with an infimum of 0 if and only if:

$$\sum_{i=1}^{n-1} \frac{\widetilde{\beta}_i}{1+\widetilde{\beta}_i} = (1+\beta_n) \sum_{i=1}^{n-1} \frac{\beta_i}{1+\beta_i} \leq 1.$$

By dividing both side by $1 + \beta_n$ and adding $\beta_n/(1 + \beta_n)$ the inequality above is equivalent to

$$\sum_{i=1}^{n} \frac{\beta_i}{1+\beta_i} \leq \frac{1}{1+\beta_n} + \frac{\beta_n}{1+\beta_n} = 1. \tag{34}$$

Since $\gamma : \mathbb{R}^n \to \mathbb{R}$ is bounded from below if and only if $\widetilde{\gamma}$ is bounded from below, equation (34) yields the desired result. One can check that the minimum is attained at $(0, \dots, 0)$. If the inequality in equation (34) is strict, then we can find $\widehat{\beta}_i > \beta_i$, such that

$$\frac{1}{\beta_i} = \frac{1}{\widehat{\beta}_i} + \varepsilon_i, \quad \varepsilon_i > 0, \quad \text{and} \quad \sum_{i=1}^{n} \frac{\widehat{\beta}_i}{1+\widehat{\beta}_i} \leq 1.$$

Hence it follows that

$$\gamma(a) = \sum_{i=1}^{n} \left(\frac{1}{\widehat{\beta}_i} + \varepsilon_i \right) a_i^2 - 2 \left(\sum_{1 \leq i < j \leq n} a_i a_j \right) \geq \sum_{i=1}^{n} \varepsilon_i a_i^2$$

and thus the minimum at $(0, \dots, 0)$ is strict and unique.

(c) If $\beta_i > 0$ for some i then γ is unbounded from above ($\gamma \to \infty$ if $a_i \to \infty$ and $a_j = 0$ for $j \neq i$). Thus, it must be $\beta_i < 0$ for all i. If $n = 2$, $\beta_i < 0$ and $\beta_1\beta_2 \leq 1$, then γ is a concave function with a maximum at $(0, 0)$; if $\beta_1\beta_2 < 1$ then γ is strictly concave and the maximum at $(0, 0)$ is strict and unique. To see that $\beta_1\beta_2 \leq 1$ is also necessary, suppose that $\beta_1\beta_2 > 1$ and take $a_2 = \beta_2 a_1$. Then we have

$$\gamma(a_1, \beta_2 a_1) = \frac{1}{\beta_1} a_1^2 + \beta_2 a_1^2 - 2\beta_2 a_1^2 = a_1^2 \left(\frac{1 - \beta_1\beta_2}{\beta_1} \right) \to \infty \qquad \text{as} \qquad a_1 \to \infty.$$

(b) We already know from (c) that $\beta_i < 0$ is a necessary condition for γ to be bounded above and that when $n = 2$, if $0 > \beta_i \geq -1$ then γ is bounded from above with a supremum of 0; note that $\beta_i \geq -1$, $i = 1, 2$, implies $\beta_1\beta_2 \leq 1$. As in the proof of (a), we proceed by induction: suppose that if $0 > \beta_i \geq -1$ for all i, then $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is bounded from above with a supremum of 0. We will show that the same property holds for n. Note that if $\beta_i = -1$ for all i, then $\gamma(a) = -\left(\sum_{i=1}^n a_i\right)^2 \leq 0$. Suppose now that $\beta_i > -1$ for some i; without loss of generality, say $\beta_n > -1$. Fixing a_{-n}, the function of one variable $a_n \mapsto \gamma(a_1, \ldots, a_n)$ has a global maximum at $a_n = \beta_n(a_1 + \ldots + a_{n-1})$. We can then proceed as in (a) to obtain equation (33). If we apply the induction hypothesis to the function $\widetilde{\gamma} : \mathbb{R}^{n-1} \to \mathbb{R}$ we have that $\widetilde{\gamma}$ is bounded from above with a supremum of 0 if:

$$0 > \widetilde{\beta}_i = \frac{(1+\beta_n)\beta_i}{1-\beta_i\beta_n} \geq -1 \qquad \text{for all } i \tag{35}$$

It is easy to see that if $0 > \beta_i \geq -1$, for $i = 1, \ldots, n < 0$ then equation (35) holds and thus $\widetilde{\gamma}$ is bounded from above. Since $\beta_i\beta_n < 1$, the first inequality in equation (35) holds; the second inequality also holds, because it is equivalent to $\beta_i \geq -1$. It then follows from (33) that $\gamma : \mathbb{R}^n \to \mathbb{R}$ is also bounded from above. To show that if $\beta_i > -1$ for all i, then the maximum at $(0, \ldots, 0)$ is strict and unique, we can proceed as in the last part of (a). ■

Proof of Lemma 5.

To prove this lemma it is sufficient to prove that if the inequalities (20) and (21) are satisfied on the interval $(0, \infty)$, then

$$\lim_{k\to\infty} P\left(\frac{\rho_{t^0+k+1}}{\rho_{t^0}} > 2^{\sqrt{k}}\right) = 1 \quad \text{for all } t^0 \in \mathbb{N}, \tag{36}$$

and thus $\lim_{t\to\infty} \rho_t = \infty$.

By the central limit theorem for the binomial distribution (e.g., see Billingsley [2]), if p is the probability of a random event, k is the number of independent draws, and X is the random variable that counts the occurrence of the event, then

$$\frac{X-kp}{\sqrt{kp(1-p)}} \sim N(0,1) \quad \text{as} \quad k \to \infty, \tag{37}$$

where $N(0,1)$ is the standard normal distribution with distribution function

$$\Phi(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Y} \exp\left\{-\frac{s^2}{2}\right\} ds. \tag{38}$$

It is convenient to take $k = 2^{4K}$ for some integer $K > 0$. Suppose that the random event is $\rho_{t+1}/\rho_t \geq 2^{2^4}$. By (20) the probability p of this event is greater than 1/4; then we have

$$P\left(\left\{\#t : \frac{\rho_{t+1}}{\rho_t} \geq 2^{2^4}\right\} < 2^{4K-2}\right) = \Phi\left(\frac{2^{4K-2} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) \to 0 \quad \text{as} \quad K \to \infty \tag{39}$$

By (21), the probability p of the random event $\rho_{t+1}/\rho_t \in [1/2^{m+1}, 1/2^m]$ is less than $1/2^{2m}$, for $m = 1, \ldots, 3K/2 - 1$, and hence

$$P\left(\left\{\#t : \frac{\rho_{t+1}}{\rho_t} \in \left[\frac{1}{2^{m+1}}, \frac{1}{2^m}\right]\right\} \leq \frac{2^{4K+1}}{2^{2m}}\right) = \Phi\left(\frac{2^{4K+1-2m} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) \to 1 \quad \text{as } K \to \infty. \tag{40}$$

Similarly, the probability p of the random event $\rho_{t+1}/\rho_t \in \left[1/2^{5K/2}, 1/2^{3K/2}\right]$ is

$$p < P\left(\frac{\rho_{t+1}}{\rho_t} \leq \frac{1}{2^{3K/2}}\right) < \frac{1}{2^{3K}},$$

and by (37) and (38) we have

$$\begin{aligned} P\left(\left\{\#t : \frac{\rho_{t+1}}{\rho_t} \in \left[\frac{1}{2^{5K/2}}, \frac{1}{2^{3K/2}}\right]\right\} > 2k\frac{1}{2^{3K}} = 2^{K+1}\right) &= \\ 1 - \Phi\left(\frac{2^{K+1} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) &\to 0 \quad \text{as } K \to \infty. \end{aligned} \tag{41}$$

Finally, the probability p of the random event $\rho_{t+1}/\rho_t \geq 1/2^{5K/2}$ is

$$p = P\left(\frac{\rho_{t+1}}{\rho_t} \geq \frac{1}{2^{5K/2}}\right) > \frac{2^{5K} - 1}{2^{5K}}$$

and thus

$$\begin{aligned} P\left(\left\{\#t : \frac{\rho_{t+1}}{\rho_t} < \frac{1}{2^{5K/2}}\right\} = 0\right) &= P\left(\left\{\#t : \frac{\rho_{t+1}}{\rho_t} \geq \frac{1}{2^{5K/2}}\right\} > 2^{4K} - 1\right) \\ &= 1 - \Phi\left(\frac{2^{4K}(1-p) - 1}{\sqrt{2^{4K}p(1-p)}}\right) \to 1 \quad \text{as } K \to \infty. \end{aligned} \tag{42}$$

Equation (39) says that as $K \to \infty$, for almost all paths $\rho_1, \rho_2, \ldots \rho_{2^{4K}}$ the number of indices $t \in \{1, 2, \ldots, 2^{4K}\}$ for which $\rho_{t+1}/\rho_t \geq 2^{2^4}$ is at least 2^{4K-2}. Clearly, since $k = 2^{4K}$, the number of indices t for which $\rho_{t+1}/\rho_t \in [1/2, 1]$ is at most 2^{4K}. As $K \to \infty$, by (40) the number of indices t for which $\rho_{t+1}/\rho_t \in [1/2^{m+1}, 1/2^m]$, $m = 1, 2, \ldots, 3K/2 - 1$, is at most $2^{4K+1-2m}$, and by equation (41) the number of indices t for which $\rho_{t+1}/\rho_t \in [1/2^{5K/2}, 1/2^{3K/2}]$ is at most 2^{K+1}. Finally, by equation (42), as $K \to \infty$ the number of indices t for which $\rho_{t+1}/\rho_t < 1/2^{5K/2}$ is zero. Then we can estimate that for a sufficiently large K:

$$\frac{\rho_{t^0+2^{4K}+1}}{\rho_{t^0}} \simeq \prod_{t=t^0}^{t^0+2^{4K}} \frac{\rho_{t+1}}{\rho_t} \geq \left(2^{2^4}\right)^{2^{4K-2}} 2^{-2^{4K}} \prod_{m=1}^{3K/2-1} \left(\frac{1}{2^{m+1}}\right)^{2^{4K+1-2m}} \left(\frac{1}{2^{5K/2}}\right)^{2^{K+1}} \tag{43}$$

To evaluate the product on the right hand side of (43) we first use logarithms to change the product into a sum:

$$\begin{aligned} \log\left(\prod_{m=1}^{3K/2-1} \left(\frac{1}{2^{m+1}}\right)^{2^{4K+1-2m}}\right) &= 2^{4K+1}\left(\sum_{m=1}^{3K/2-1} 2^{-2m}(-m-1)\right)\log 2 \qquad (44) \\ &> -2^{4K+1}\left(\sum_{m=1}^{\infty} \frac{m+1}{2^{2m}}\right)\log 2 \geq -2^{4K+1}\log 2 \end{aligned}$$

where the inequality follows from

$$\sum_{m=1}^{\infty} \frac{m+1}{2^{2m}} < \sum_{m=1}^{\infty} \frac{2^m}{2^{2m}} = \sum_{m=0}^{\infty} \frac{1}{2^m} - 1 = 1.$$

Using inequality (44) in equation (43) gives:

$$\begin{aligned} \frac{\rho_{t^0+2^{4K}+1}}{\rho_{t^0}} &\geq \left(2^{2^4}\right)^{2^{4K-2}} 2^{-2^{4K}} 2^{-2^{4K+1}} \left(2^{-5K/2}\right)^{2^{K+1}} = 2^{2^{4K+2}} 2^{-(3)2^{4K}} 2^{-(5K)2^K} \\ &= 2^{-(5K)2^K} 2^{2^{4K}} = 2^{2^{2K}(2^{2K}-5K2^{-K})} > 2^{2^{2K}} = 2^{\sqrt{k}} \end{aligned}$$

This is exactly what we claimed in (36). Hence the lemma follows. ■

Proof of Lemma 6.

Without loss of generality we can assume that $C_1 \leq C_2$. Also, since we only need to prove that (a) and (b) in the lemma hold for almost all $(C_1, C_2) \in \mathcal{C}$, we can assume that $C_1 \neq C_2$ and $C_1, C_2 \neq 0$. Therefore, there are three distinct cases we have to consider. All cases must satisfy $C_1 + C_2 < 1$.

1. $0 < C_1 < C_2 < 1$, or, equivalently $0 < \beta_1 < \beta_2 < \infty$.
2. $C_1 < 0 < C_2 < 1$, or, equivalently $-\infty < \beta_1 < 0 < \beta_2 < \infty$.
3. $C_1 < C_2 < 0$, or, equivalently $-\infty < \beta_1 < \beta_2 < 0$.

Recall that $1/\beta_1$ is the slope of the best-reply function of the first player and β_2 is the slope of the best-reply function of the second player in the $a_1 - a_2$ plane. The lemma says that, starting from any point a^0, and for any given $c \in (-1, 1)$, it is possible to construct a sequence of single-player improvements that reaches some point on the straight line $a_1^0 + a_2^0 = c$.

Case 1. When $0 < C_1 < C_2$, condition $C_1 + C_2 < 1$ is equivalent to $1/\beta_1 > \beta_2$; that is, the slope in the $a_1 - a_2$ plane of player 1's best-reply function is greater than the slope of the best-reply function of player 2. Consider the function γ defined by:

$$\gamma(a) = \frac{1}{\beta_1} a_1^2 + \frac{1}{\beta_2} a_2^2 - 2a_1 a_2.$$

For any $c > 0$, the set $E = \{a \in \mathbb{R}^2 : \gamma(a) = c\}$ is an ellipse centered at the origin. Denote by E_+ and E_- the intersections of E with the region that lies between the best-reply functions in the first and third quadrant, respectively. Clearly, E_+ and E_- are symmetric with respect to the origin. Moreover, the slope of the ellipse in the $a_1 - a_2$ plane is zero for the two points on the best-reply function of player 1, and it is infinity for the two points on the best-reply function of player 2 (see Figure 3). The symmetry of the ellipse, the slope of player 1's best-reply function being greater than the slope of player 2's best-reply function, and the slope of the ellipse being zero or infinity at the intersections with the best-reply functions imply that starting from any $a \in E_+$ we can define two finite sequences $\left\{a^0, a^1, a^2, \ldots, a^{T_2}\right\}$ and $\left\{b^0, b^1, b^2, \ldots, b^{T_1}\right\}$ of points on the ellipse E with the following properties:

$a^0 = a \in E_+$, $a^{T_2} \in E_-$ and $b^0 = a \in E_+$, $b^{T_1} \in E_-$;

$\{a^0, a^1, a^2, \ldots, a^{T_2}\} = \{(a_1, a_2), (a_1, 2\beta_2 a_1 - a_2), (2\beta_1 (2\beta_2 a_1 - a_2) - a_1, 2\beta_2 a_1 - a_2), \ldots, a^{T_2}\}$ and $\{b^0, b^1, b^2, \ldots, b^{T_1}\} = \{(a_1, a_2), (2\beta_1 a_2 - a_1, a_2), (2\beta_1 a_2 - a_1, 2\beta_2 (2\beta_1 a_2 - a_1) - a_2), \ldots, b^{T_1}\}$;

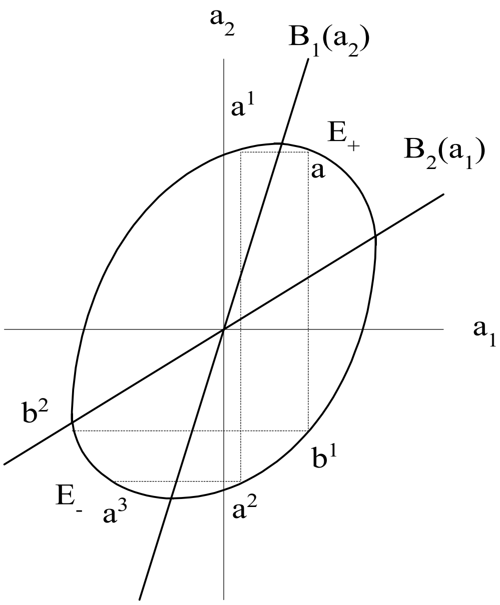

Figure 3: The Ellipse $E = \{a \in \mathbb{R}^2 : \gamma(a) = c\}$.

$a^h \notin E_+ \cup E_-$ for $h = 1, \ldots, T_2 - 1$ and $b^h \notin E_+ \cup E_-$ for $h = 1, \ldots, T_1 - 1$.

Each step in the sequences consists of a payoff neutral change by one of the two players. Players take turns changing action; in the sequence $\left\{a^0, a^1, a^2, \ldots, a^{T_2}\right\}$ player 2 is the first to change action, in the other sequence the first to change action is player 1.

By letting $\varphi_1(a) = a^{T_1}$ and $\varphi_2(a) = a^{T_2}$ we can define two continuous maps from E_+ into E_-. Consider a smooth parameterization $f_+ : [0,1] \to E_+$ of the arc E_+ of the ellipse E and note that by letting $f_- = -f_+$ we obtain a smooth parametrization of the arc E_-, $f_- : [0,1] \to E_-$. We are now ready to define two continuous map $\Phi_h : [0,1] \times \{1,2\} \to [0,1] \times \{1,2\}$, $h = 1, 2$

$$\Phi_h(t,i) = \left(f_-^{-1}\left(\varphi_i\left(f_+(t)\right)\right), j(i,t,h)\right) \qquad \text{for any } t \in [0,1],\ i = 1,2 \text{ and } h = 1,2$$

where, for all $t \in [0,1]$, $j(i,t,h)$ is defined as follows:

$$j(i,t,h) \quad = \quad h \qquad \text{if} \qquad \varphi_1\left(-\varphi_i\left(f_+(t)\right)\right) = -f_+(t) \tag{45}$$

$$j(i,t,h) \quad = \quad 3-h \quad \text{if} \quad \varphi_1\left(-\varphi_i\left(f_+(t)\right)\right) \neq -f_+(t) \quad \text{and} \quad \varphi_2\left(-\varphi_i\left(f_+(t)\right)\right) = -f_+(t)$$

Note that the symmetry of the ellipse (see Figure 3), implies that one of the two conditions in equation (45) must be true. In fact, it is only when $-\varphi_i\left(f_+(t)\right)$ coincides with an endpoint of the arc E_+ that the functions φ_1 and φ_2 take on the same values (in this case we set $j=1$).

Continuity of the maps f_+, φ_1 and φ_2 implies that Φ_h is also continuous. We now argue that Φ_h is a homeomorphism, that is, a continuous bijection. First, we show that the map is onto; that is, given any point $(\tau, j) \in [0,1] \times \{1,2\}$ we can find $(t,i) \in [0,1] \times \{1,2\}$ such that $\Phi_h(t,i) = (\tau, j)$. To see this, let $t = f_+^{-1}(\varphi_j(f_-(\tau))$ and note that either $\Phi_h(t,1) = (\tau, j)$, or $\Phi_h(t,2) = (\tau, j)$. That Φ_h is 1-to-1 follows from f_-, f_+ and φ_i being 1-to-1. In fact, the map Φ_h is of class C^2.

Consider the set $[0,1] \times \{1,2\}$; by identifying, or gluing together, the point $(0,1)$ with $(0,2)$ and the point $(1,1)$ with $(1,2)$ we can view the set $[0,1] \times \{1,2\}$ as a circle S^1 and the map Φ_h as a homeomorphism from S^1 to S^1. Let s_1, s_2, s_3 be three points on the circle S_1 and suppose that as we move clockwise on the circle starting from s_1 we encounter first s_2 and then s_3. We say that the map Φ_h is *orientation preserving* if as we move clockwise on the circle starting from $\Phi_h(s_1)$ we encounter first $\Phi_h(s_2)$ and then $\Phi_h(s_3)$. The map Φ_h is *orientation reversing* if as we move clockwise on the circle starting from $\Phi_h(s_1)$ we encounter first $\Phi_h(s_3)$ and then $\Phi_h(s_2)$. Since it is a homeomorphism, Φ_h must be either orientation preserving, or orientation reversing. In fact, if Φ_1 is orientation preserving, then Φ_2 is orientation reversing and vice versa. In the reminder of the proof we will use the orientation preserving map and denote it simply as Φ.

Recall that the covering space of a circle S^1 is the real line; that is, we can find a homeomorphism $h : [0,1) \to S^1$ with $\lim_{x \to 1} h(x) = h(0)$ and then define the map $H : \mathbb{R} \to S^1$ by letting $H(x+z) = h(x)$ for all $x \in [0,1)$ and all integers $z \in Z$. The *lift* of the orientation preserving map Φ is the function $\widetilde{\Phi} : \mathbb{R} \to \mathbb{R}$ defined by $\widetilde{\Phi}(x+z) = h^{-1}(\Phi(H(x+z)))$ for all $x \in [0,1)$ and all integers $z \in Z$. (We could add any integer q to $\widetilde{\Phi}$; the lift of Φ is uniquely defined up to the addition of an integer). Let $\widetilde{\Phi}^n = \widetilde{\Phi} \circ \widetilde{\Phi}^{n-1}$ and define the following limit:

$$r(\Phi, x) = \lim_{n \to \infty} \frac{1}{n} \left(\widetilde{\Phi}^n(x) - x \right) \quad \text{for } x \in \mathbb{R}.$$

This definition was proposed by Poincare, who also showed that this limit exists and is independent of x (e.g., see Milnor [16]) (if we added an integer to the lift than the limit would only be unique up to addition of an integer); that is, $r(\Phi, x) = r(\Phi)$ for all $x \in \mathbb{R}$. We call $r(\Phi)$ the rotational number of the map Φ. Except for a zero measure set of cases, the rotational number of our map Φ is irrational. In fact for fixed β_1 there are only countably many choices of β_2 that yield a rational rotational number.

A *rotation by* α is a map $r_\alpha : S^1 \to S^1$ whose lift $\widetilde{r}_\alpha : \mathbb{R} \to \mathbb{R}$ is $\widetilde{r_\alpha}(x) = x + \alpha$. Let $r_\alpha^n = r_\alpha \circ r_\alpha^{n-1}$. If α is an irrational number, then for all $t \in S^1$ the set of points in the infinite sequence $r_\alpha(t), r_\alpha^2(t), r_\alpha^3(t), \ldots$ is dense in S^1. A theorem by Denjoy [6] implies that a C^2 homeomorphism Φ with an irrational rotation number $r(\Phi) = \alpha$ is conjugate to a rotation by α; that is, there exists a homeomorphism $g : S^1 \to S^1$ such that $\Phi = g^{-1} \circ r_\alpha \circ g$. This implies that for all $t \in S^1$ the set of points in the infinite sequence $\Phi(t), \Phi^2(t), \Phi^3(t), \ldots$ is dense in S^1, where $\Phi^n = \Phi \circ \Phi^{n-1}$. As a consequence, given any $a^0 \in \mathbb{R}^2$ and any $\varepsilon > 0$, we can find a finite sequence $a^0, a^1, \ldots, a^m$ of payoff neutral single-player moves from a^0 to a^m where $|a^m + a^0| < \varepsilon$; that is, the points a^0 and a^m are almost symmetric with respect to the origin $(0, 0)$. By continuity and quasi-concavity of the players' payoffs, given any $\delta > 0$, we can then find a finite sequence of single-player improvements $a^0, \widehat{a}^1, \ldots, \widehat{a}^m$ such that $\left|\widehat{a}^h - a^h\right| < \delta$ for all $h = 1, \ldots, m$; that is, the finite sequence of single-player improvements can be chosen to be as close as desired to the sequence of payoff neutral single-player moves. By choosing $\delta = \varepsilon - |a^m + a^0|$ we obtain that $|\widehat{a}^m + a^0| < \varepsilon$; that is, we can construct a finite sequence of single-player improvements from any point a^0 to a point arbitrarily close to $-a^0$. Think of this sequence as a sequence of horizontal and vertical steps, for $\theta \in (-1, 1)$ at least one of this steps must cross the line $x_1 + x_2 = \theta\,(a_1^0 + a_2^0)$, say it crosses at a^* in the step from a^h to a^{h+1}. Quasi-concavity of the payoff functions then implies that the sequence $a^0, a^1, \ldots, a^h, a^*$ is a finite sequence of single-player improvements. To conclude the proof, we only need to show that if $\theta = 0$ we can reach the origin. This simply follows from the fact that there will be a first step, let say from a^k to a^{k+1}, when the sequence from a^0 to a point arbitrarily close to $-a^0$ must cross one of the axis. Let b^{k+1} be the point on the intersection of line segment with endpoints a^k, a^{k+1} and one of the axis. Again, quasi-concavity of the payoff functions implies that $a^0, a^1, \ldots, a^k, b^{k+1}, (0,0)$ is a finite sequence of single-player improvements. This concludes the proof of this case.

Case 2. If we graph the best-reply functions in the $a_1 - a_2$ plane, the line corresponding to the best-reply function of player 1 passes through the second and fourth quadrant, whereas that of player 2 goes through the first and third quadrant. It is sufficient to show that from any starting point a^0 it is possible to construct a sequence of single-player improvements that spirals away from the equilibrium $(0,0)$, since this implies that such a sequence crosses the region between the lines $a_1^0 + a_2^0 = 1$ and $a_1^0 + a_2^0 = -1$. Then, there is another sequence that reaches a point on any line $a_1^0 + a_2^0 = c$, with $-1 < c < 1$. There is no loss of generality in choosing a starting point $a^0 = (a_1^0, \beta_2 a_1^0)$ on the best reply of player 2 (it is always possible to reach such a point with a finite sequence of single-player improvements). Let $0 < \mu < 1$ and consider the following sequence of single-player improvements (each time a players moves it changes action in the direction of his best reply by an amount less than twice the distance between its current action and his best reply, see Figure 4) $a^0, a^1, a^2, a^3, a^4, \ldots$ where

$$a_1^1 = \left[2\mu\beta_1\beta_2 - (2\mu - 1)\right] a_1^0, \quad \text{and} \quad a_2^1 = a_2^0 = a_1^0\beta_2,$$

$$a_1^2 = a_1^1 \quad \text{and} \quad a_2^2 = \left[4\mu^2\beta_1\beta_2^2 - (4\mu^2 - 1)\beta_2\right] a_1^0,$$

$$a_1^3 = \left[8\mu^3\beta_1^2\beta_2^2 - (8\mu^3 + 4\mu^2 - 4\mu)\beta_1\beta_2 + 4\mu^2 - 4\mu + 1\right] a_1^0 \quad \text{and} \quad a_2^3 = a_2^2$$

$$a_1^4 = a_1^3 \quad \text{and} \quad a_2^4 = \beta_2 a_1^3.$$

For μ sufficiently close to 1, a^4, which is on player 2's best-reply function, is further away from $(0,0)$ than a^0. To see this note that $\lim_{\mu \to 1} |a_1^3| = \left[1 - 8\beta_1\beta_2\left(1 - \beta_1\beta_2\right)\right] |a_1^0|$ $> |a_1^0|$. By iterating this construction we can obtain a sequence with any finite number of steps, spiraling away from the origin.

Finally, from any starting point a^0 the equilibrium $(0,0)$ can be reached by two single-player improvement; first a player moves to one of the axis and then the other moves from the axis to $(0,0)$.

Case 3. Change the choice variable of player 2 from a_2 to $-a_2$. More precisely, let $x_1 = a_1$, $x_2 = -a_2$, and $\alpha_i = -\beta_i$ for $i = 1, 2$, and view the game g as one in which player i chooses x_i. The best-reply functions in this game are $B_1(x_2) = \alpha_1 x_2$ and $B_2(x_1) = \alpha_2 x_1$. Since we now have $0 < \alpha_2 < \alpha_1$, we are in the same situation as in

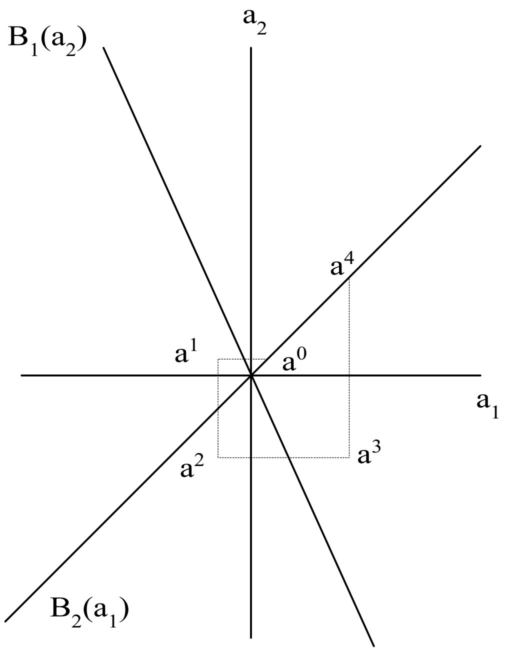

Figure 4: A Sequence of Single-Player Improvements.

Case 1, modulo a permutation of player 1 with player 2, and we can use its proof. (Note that after the change of variable: (i) the numbers corresponding to C_1 and C_2 are $\alpha_1/(1+\alpha_1)$ and $\alpha_2/(1+\alpha_2)$; (ii) $0 < \alpha_2/(1+\alpha_2) < \alpha_1/(1+\alpha_1) < 1$; (iii) $\alpha_1/(1+\alpha_1) + \alpha_2/(1+\alpha_2) < 1$). This concludes the proof. ■

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