

Equilibrium Reserve Prices in Sequential Ascending Auctions*

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Running title: *Sequential ascending auctions*

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Abstract

We study a model in which the same set of bidders, with perfectly correlated valuations across units, compete for two units of a good in two sequential ascending-price auctions. The seller sets a reserve price before the beginning of each auction. Surprisingly, the equilibrium has a simple structure; strategic non-disclosure of information (i.e., pooling) only takes the form of non-participation in the early auction by bidders with valuations below a threshold, while bidders with valuations above the threshold participate and bid truthfully; that is, they stay in the auction until the price reaches their true valuations. The participation threshold is strictly higher than the reserve price in the first auction, so some buyers who would find it profitable to buy at the reserve price select not to participate in order to attempt to decrease the reserve price in the second auction. Participation in the first auction is lower than under full commitment, but the probability of at least one bidder participating in the second auction is higher.

Keywords: repeated auctions, ratchet effect, participation, reserve price.

JEL: D44 & D82.

1 Introduction

In recent years there has been a surge of theoretical, empirical and practical interest in auctions (see, for example, Klemperer's survey (1999) and Milgrom's book (2001)). A great deal of attention has been devoted to the simultaneous ascending auction of multiple units to bidders with multi-unit demands due, at least in part, to the successful use of such auctions by the FCC to allocate spectrum rights. A consensus has emerged that this auction format, with or without combinatorial bids, is particularly appropriate for designing procedures to allocate the rights to operate in newly created markets.

One of the important arguments, besides those related to the Linkage Principle,¹ in favor of an ascending auction as opposed to a first-price or a Vickrey auction (Vickrey (1961)), is that bidders are reluctant to reveal their true valuations. Rothkopf et al. (1990, pp. 108) first note: "Vickrey auctions are rare because bidders are reluctant to follow the truth-revealing strategies that the 'proper' operation of such auctions would require. Bidders have good reasons to be reluctant when they may lose a fraction of the economic rent revealed by the sealed second-price format in subsequent negotiations."² If revealing one's valuation leads to potential future losses, then equilibrium in a first-price and a Vickrey auction involves substantial pooling of bids and complex equilibrium strategies. At work here is the well-known ratchet effect from the planning and principal-agent literature. Laffont and Tirole (1988), for example, show that when information takes a continuum of values, local dynamic incentive compatibility implies that separation of agent types is not feasible. A lot of pooling occurs in equilibrium; that is, agents undertake identical actions for different values of their information.

In this paper we analyze a setting in which bidders are justified in their reluctance to reveal their

¹See in particular the seminal paper by Milgrom and Weber (1982).

²Ausubel and Milgrom (2001, pp. 6-7) elaborate on this: "the revelation of bidders' maximum willingness to pay during the auction can be problematic ... Winning bidders may fear that information revealed by their bids will be used by auctioneers to cheat them or by third parties to disadvantage them in some negotiation ... A bidder's motive to conceal its information can destroy the dominant strategy property that accounts for much of the appeal of the Vickrey auction ... a similar case can be made against ordinary first-price auctions, since the theoretical bid functions are invertible to reveal bidders' values. In this respect, ascending auctions are theoretically superior to both kinds of sealed bid auctions because they better conceal the winning bidder's valuation."

valuations. We study sequential auctions in which bidders have correlated valuations for multiple units, while the seller can freely set the reserve price at the beginning of each auction. Contrary to sequential first-price and Vickrey auctions, and in spite of the fact that a bidder's decision to participate and stay in the first auction does reveal some information to the seller, we prove that the ratchet effect takes a simple form in sequential ascending auctions.

Our setting is of clear practical importance. Many goods, services and contracts are allocated in sequential auctions, sometimes with quite long time periods between two consecutive auctions, sometimes with several auctions almost in a row. As documented in the literature, estate (art, stamps, books, etc.), cattle, fish, vegetables, timber and wine are often allocated in comparable lots at sequential auctions, to a quite well-established and limited group of potential buyers.³

There are several reasons that explain the use of sequential auction procedures in practice. First, the units may not be available at the same time and yet, when available, they may be perishable and would then have to be sold separately. This is a particularly relevant explanation for fish, flowers, timber, vegetables, etc.⁴ Second, when the auctioning party is a government or a public authority, committing to a future reserve price, or pricing policy, is often impossible or illegal. This is relevant for the auction of many kinds of concession contracts, operating licenses and leases in which the duration of the contract is of several years, and it is anticipated that another auction will take place at the renewal stage.

We assume that bidders' valuations for the units, in two sequential auctions, are perfectly correlated across periods, since strategic retention of information is particularly severe under these conditions. Nevertheless, we are able to characterize the equilibrium path and to show that in our model strategic non-disclosure of information (i.e., pooling) only takes the form of non-participation in the early auction.

Bidders with valuations below a threshold do not participate in the first auction, while bidders with

³See, e.g., Cassady (1967), Ashenfelter (1989), Donald, Paarsch and Robert (1997, 2001), Laffont, Ossard and Vuong (1995) and Milgrom (2001).

⁴See, e.g., the discussion in Weber (1983) and, for details concerning timber auctions, Donald, Paarsch and Robert (2001).

valuations above the threshold participate and bid truthfully; that is, they stay in the auction up to the point where the price reaches their true valuation. Intuitively, this is because in an ascending auction the winner only reveals a lower bound on his valuation. The participation threshold is strictly higher than the reserve price in the first auction, so some buyers who would find it profitable to buy at the reserve price select not to participate in order to attempt to decrease the reserve price in the second auction.

If it were possible, the seller would want to commit to the same reserve price in both auctions. This reserve corresponds to the reserve price in the optimal static auction for a single item. We show that in our model participation in the first auction is lower than under full commitment, but that the probability of at least one bidder participating in the second auction is higher than in the optimal auction. Finally, we show that our results are specifically due to the ascending auction format since, in a sequence of two second-price auctions, other things being identical, the ratchet effect prevents information disclosure in the early auction and induces pooling.

Our analysis of the seller's opportunism in sequential auctions with multi-unit demands is related to a few papers that study the choice of a reserve price in *single-unit* auctions with some dynamic considerations. McAfee and Vincent (1997) study the seller's choice of a reserve price when she cannot commit not to re-auction the object if a sale fails. They prove the revenue equivalence of a first and a second-price auction. They also show that if the time between auctions goes to zero, then the seller's expected revenue converges to the revenue of a static auction without a reserve price. Burguet and Sakovics (1996) study a first-price auction, with a large number of potential bidders and endogenous entry. Bidders who decide to participate must pay a cost to learn their valuations; the seller may set a reserve price, but if the object goes unsold, she must offer it for sale at another auction without a reserve price. They show that a reserve price that restricts participation may be optimal. Haile (2000) studies a second-price auction with a reserve price between two bidders with imperfect information about their

valuations. If the good is sold, then all bidders' valuations become known and efficient bargaining takes place if there are gains to be realized (e.g., the item is not in the hands of the highest valuation bidder).

Haile shows that in equilibrium there is partial pooling of bids at the reserve price.

In the literature on sequential auctions with multi-unit demands and correlation, attention has rather been devoted to the study of the strategic use of information *among bidders*. Ortega Reichert's (1968) pioneering article first analyzes a two-bidder, two-period, sequential first-price sealed bid auction with positive correlation of bidders' valuations across periods and across bidders. He shows that a bidder in the first auction is less aggressive than in a one-shot auction, so as to induce more pessimistic beliefs by his rivals about his likely future valuation and thus reduce competition in the second auction. In a similar model, Hausch (1986) shows that, despite this underbidding effect, the seller's revenue may be higher in sequential rather than in simultaneous auctions precisely because more information may be revealed.⁵ Other contributions on sequential auctions with multi-unit demands either assume perfect commitment by the auction designer,⁶ or they do not allow for an active role for the seller,⁷ or else they focus on models where bidders' valuations for the different units are independent (but ranked in decreasing order) so that a bidder who wins a given auction is not concerned about revealing his valuation.⁸

This paper proceeds as follows. In the next section we introduce the model. In Section 3 we study the continuation equilibria for a fixed, first-auction reserve price. In Section 4 we complete the characterization of the equilibrium of our sequential ascending auction. In Section 5 we show that equilibrium in a sequential second-price auction exhibits a lot of pooling; the first-auction bids never reveal the bidders' valuations. Section 6 concludes.

⁵See also Weber (1983).

⁶E.g. Lutan and McAfee (1986).

⁷Such is the case for the papers about the so-called "declining price anomaly": see e.g. Ashenfelter (1989), Black and deMeza (1992), Gandal (1997), McAfee and Vincent (1993), Gale and Hausch (1994), Gale and Stegeman (2001), von-der-Fehr and Morch (1994), Jeitschko (1999) and Gale, and Hausch and Stegeman (2000).

⁸E.g. Donald, Paarsch and Robert (1997, 2001), Montmarquette and Robert (1999), and Katzman (1999).

2 The Model

A seller wants to sell two units of a good to n potential bidders, indexed by i , in a sequence of two ascending-price auctions. While being committed to the use of ascending price auctions, at the start of each auction the seller is free to choose a reserve price, below which she will not sell the good. For simplicity we will assume that each player has a discount factor equal to 1.⁹

Bidder i 's valuation for each unit of the good is v_i ; that is, a bidder's valuations for the units are perfectly correlated. Because the focus of this paper is the study of ratchet effects in sequential auctions when the seller cannot commit to future reserve prices, it is important that there be positive correlation between a bidder's value for the two objects. If, on the other hand, valuations were independent, or stochastically equivalent as in Engelbrecht-Wiggans (1994), then there would be no ratchet effect; information revealed by the bidding behavior in the first auction would have no effect on the seller's choice of the reserve price in the second auction.

Bidders' valuations are private information and are identically and independently distributed across bidders according to the c.d.f. $F(\cdot)$, with positive density $f(\cdot)$, over $[\underline{v}, \bar{v}]$. Let $G(\cdot)$ (respectively, $g(\cdot)$) denote the c.d.f. (respectively, the density) of the highest order statistics $v_{-i}^{(1)} = \max\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ for a given i : $G(v) = [F(v)]^{n-1}$. Let a bidder's virtual valuation function be given by

$$J(v) \equiv v - \frac{1 - F(v)}{f(v)}.$$

We will make the standard regularity assumption that $J(v)$ is increasing in v . A sufficient condition for this assumption to be satisfied is that the hazard rate $f(v)/[1 - F(v)]$ is increasing. The seller's valuation for each unit of the good is a constant smaller than \underline{v} which we normalize to 0.

In this symmetric and regular setting, the optimal (revenue-maximizing) auction procedure to sell a *single* object can be implemented by an ascending-price auctions (in fact by any of several standard

⁹Assuming that the discount factor is less than 1 would not change the analysis in any significant way.

auctions) with a reserve price R_0 given by (Myerson ,1981):¹⁰

$$R_0 = \begin{cases} J^{-1}(0) & \text{if } J(\underline{v}) < 0 \\ \underline{v} & \text{otherwise.} \end{cases}$$

More generally, if the seller gets ex ante the additional information that all bidders have valuations in a sub-interval $[v_L, v_H] \subset [\underline{v}, \bar{v}]$, the seller's updates his beliefs so that bidders' valuations are viewed as i.i.d. according to the c.d.f. $\frac{F(\cdot) - F(v_L)}{F(v_H) - F(v_L)}$ on $[v_L, v_H]$ with virtual valuation function given by:

$$J_{v_L, v_H}(v) \equiv v - \frac{F(v_H) - F(v)}{f(v)}.$$

It is simple to check that $J_{v_L, v_H}(v)$ is a decreasing function of v_H and, given our assumption that $J(\cdot)$ is increasing, an increasing function of v . Hence, given the ex ante information that $v \in [v_L, v_H]$, the optimal reserve price R_{v_L, v_H} for the sale of a single unit is determined by:

$$R_{v_L, v_H} = \begin{cases} J_{v_L, v_H}^{-1}(0) & \text{if } J_{v_L, v_H}(v_L) < 0 \\ v_L & \text{otherwise.} \end{cases}$$

R_{v_L, v_H} is non-decreasing in v_H and in v_L (in fact, if $J_{v_L, v_H}(v_L) \leq 0$, then R_{v_L, v_H} does not depend on v_L). Note also that this optimal reserve price does not depend on the number of bidders.

Consider now the case of repeated ascending-price auctions; the setting where the seller can commit to the sequence of reserve prices R_1 and R_2 in the successive auctions (with R_2 possibly depending on the outcome of the first auction) constitutes a useful benchmark. By the revelation principle, the seller cannot raise a higher revenue than the one he would obtain in an optimal static mechanism in which buyers are asked to report their valuations. Since buyers have the same valuation for the item in each period, the optimal mechanism for the seller is Myerson's optimal auction of the bundle composed by the two units of the good, with a reserve price of $2R_0$. If the seller must sell the goods in sequence but can commit to a sequence of reserve prices, then Myerson's optimal auction can be implemented by a repeated standard auction with a reserve price of R_0 in both periods.

¹⁰We consider the specific form of ascending auctions called Japanese auctions. The seller sets the reserve price R . If all bidders are "out" at this price, there is no sale. If only one bidder is "in", he gets the good at the reserve price. If at least two bidders are "in", the price continuously increases until all bidders, except one, leave the process. The winner pays the highest price at which his most aggressive rival got out. All ties are resolved by random draws.

In the following, however, we suppose that the seller cannot commit to the sequence of reserve prices. She can only commit to a reserve price in each auction and not to re-auction an object that goes unsold. (See McAfee and Vincent (1997) for an analysis of the case in which the seller cannot commit not to re-auction the object if a sale fails.) So, she sets R_1 for the first ascending-price auction and then, after having observed the first-auction outcome, R_2 for the second auction.

From the study of the ratchet effect (e.g., see Laffont and Tirole (1988)), we know that agents do not want to fully reveal their information in the initial stages of their relationship with a principal that cannot fully commit to a future course of action. In a similar fashion, we should expect pooling, that is strategic non-disclosure of information, to take place in the first of our sequential auctions. The questions we ask are: How does the ratchet effect affect bidding behavior in the early auction? Can information about the buyers' valuation be revealed through the first auction? And are there equilibria with a simple structure in which pooling in the first auction only takes the form of strategic non-participation? More precisely, we study whether there exist symmetric, pure-strategy, perfect Bayesian equilibria in which bidders use a strictly increasing bid function whenever they participate, which we call equilibria with full separation under participation. Note that we do not look for a full characterization of equilibria: we do not consider mixed strategy equilibria and, among pure strategy equilibria, we are only interested in the possibility of local separation.

3 Bidding Behavior for a Given First Auction Reserve Price

In this section, we take the first auction reserve price R_1 as given and investigate the existence of a continuation equilibrium with “full separation under participation”, that is, where bidders follow a symmetric bidding strategy $\beta(\cdot)$ such that $\beta(\cdot)$ is strictly increasing above some participation threshold v_* .

The second auction is a standard ascending auction with private values. So, irrespective of their

beliefs, it is a weakly dominant strategy for bidders to participate as long as their valuation exceeds the reserve price R_2 , and to exit when the price reaches their true valuation v_i . The second auction optimal reserve price, however, depends on the seller's beliefs about the bidders' valuations; each bidder has then an incentive to deviate from the one-shot truthful bidding strategy in the first ascending auction in order to try to induce the seller to set a lower reserve price in the second auction. It is therefore critical to analyze the amount of information that filters out of the first auction. The seller knows at what price each loser dropped out, who won the auction, and what was the final price reached in the process. How these data are interpreted depends upon equilibrium bidding behavior in the first auction.

As pointed out, this setting exhibits strong similarities with dynamic contracting models without commitment, such as Laffont and Tirole (1988). In their model no characterization of the continuation equilibrium is available. It is only known that local separation of types is impossible and a lot of pooling occurs in equilibrium. As will appear below, a remarkable result in our setting is that there exists an equilibrium with a simple structure, namely with full separation under participation.

Lemma 1 : *Suppose the continuation game following a given reserve price R_1 in the first auction exhibits full separation under participation, characterized by a strictly increasing bid function $\beta(\cdot)$ and a participation threshold v_* . Then, $\beta(v) = v$ for all $v \in [v_*, \bar{v}]$.*

Proof. When there is active bidding in the first auction, it becomes known that the highest valuation is at least equal to v_* . Therefore $R_2 \geq v_*$ and type v_* cannot make any profit in the second auction. Type v_* then bids in the first auction as in a static auction: $\beta(v_*) = v_*$ (which imposes $v_* \geq R_1$). Suppose that a type $v' > v_*$ bids $\beta(v') \neq v'$ and consider a deviation to bidding v' . Let v_A be defined by $v_A = \bar{v}$ if $\beta(v) < v'$ for all v and by $\beta(v_A) = v'$ otherwise. There are two cases.

Case 1: $\beta(v') > v'$. In this case $v_A < v'$. If $v_{-i}^{(1)} \in (v_A, v')$ then the deviation increases the first auction payoff of bidder i (he does not get the object at a price higher than v') and reduces the second auction

reserve price R_2 . If $v_{-i}^{(1)} < v_A$ then the deviation has no impact on either the first auction payoff of bidder i or R_2 . If $v_{-i}^{(1)} > v'$ then bidder i gets zero payoff in each auctions either if he follows the equilibrium bid $\beta(v')$, or the deviation. This shows that the deviation is profitable. So it cannot be that $\beta(v') > v'$.

Case 2: $\beta(v') < v'$. In this case $v_A > v'$. If $v_{-i}^{(1)} \in (v', v_A)$ then the deviation increases the first auction payoff of bidder i (he gets the object at a price lower than v'); it also increases the second auction reserve price R_2 , but this does not affect the second auction payoff of bidder i , because $v' < v_{-i}^{(1)}$ implies that i will lose the second auction. If $v_{-i}^{(1)} < v'$ then the deviation has no impact. If $v_{-i}^{(1)} > v_A$ then bidder i never wins an object. Thus the deviation is profitable and it cannot be that $\beta(v') < v'$. This concludes the proof. ■

Bayesian updating of the seller's beliefs provides, as a corollary, a characterization of the optimal reserve price in the second auction.

- If no bidder participates at R_1 , then the seller updates his beliefs conditional on $v_i \in [\underline{v}, v_*)$ for all i and optimally sets a reserve price $R_*(v_*)$ given by:

$$R_*(v_*) \equiv R_{\underline{v}, v_*} = \begin{cases} J_{\underline{v}, v_*}^{-1}(0) & \text{if } J_{\underline{v}, v_*}(\underline{v}) < 0 \\ \underline{v} & \text{otherwise.} \end{cases} \quad (1)$$

Note that, given our regularity assumption, if $R_*(v_*) > \underline{v}$, then:

$$R'_*(v_*) = \frac{f(v_*)}{2f(R_*) + R_*f'(R_*)} > 0. \quad (2)$$

- If there is participation and the ending price of the first auction is $p \geq R_1$, then the seller gets the information that one bidder has valuation above p while all other bidders have valuations smaller than p (even smaller than v_* , for those who do not participate). Consequently, the optimal reserve price in the second auction $R_2(p)$ corresponds to:

$$R_2(p) \equiv R_{p, \bar{v}} = \max \{R_0, p\}.$$

- If (as will be true) $v_* > R_1$, a winning price $p \in (R_1, v_*)$ is an event off the equilibrium path, for which the seller's beliefs and hence the second auction reserve price are not pinned down by the equilibrium characterization.

Given this continuation strategy by the seller, we are in a position to characterize the set of reserve prices R_1 for which there exists a continuation equilibrium with full separation under participation. Such an equilibrium necessarily exhibits “truthful participation”; that is, participating bidders exit exactly when the price reaches their true valuation.

Proposition 2 : *Fix R_1 . There exists a continuation equilibrium with truthful participation by a positive measure of types if and only if*

$$R_1 < r_0 \equiv \int_{\underline{v}}^{\bar{v}} \max \{R_0, y\} dG(y) = E \left(\max \left\{ R_0, v_{-i}^{(1)} \right\} \right).$$

If this condition holds, bidders participate whenever their valuation is higher than v_ defined by:*

$$\int_{\underline{v}}^{v_*} (v_* - \max \{y, R_*(v_*)\}) dG(y) = (v_* - R_1) G(v_*). \quad (3)$$

If $R_1 \geq r_0$, there exists no equilibria with full separation under participation by a positive measure of types; there exists a continuation equilibrium where no bidder participates in the first auction.

Proof. From Lemma 1, it remains to show that truthful bidding above some v_* is an equilibrium and to characterize v_* . Suppose all bidders but bidder i bid accordingly and consider bidder i of type v_i .

What is bidder i 's expected profit from not participating? If $v_{-i}^{(1)} \geq v_*$, the second auction reserve price will be set at $\max \{v_*, v_{-i}^{(2)}, R_0\}$, where $v_{-i}^{(2)}$ is the second rank-order statistics among $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

The second-auction sale price conditional on bidder i 's winning the second auction will then equal:

$$\max \left\{ v_*, v_{-i}^{(2)}, R_0, v_{-i}^{(1)} \right\} = \max \left\{ R_0, v_{-i}^{(1)} \right\} = R_2(v_{-i}^{(1)}).$$

If $v_{-i}^{(1)} \leq v_*$, however, the reserve price in the second auction will be set equal to $R_*(v_*)$. Overall, bidder i 's expected profit from not participating in the first auction is:¹¹

$$\int_{\underline{v}}^{v_*} (v_i - \max\{y, R_*(v_*)\})^+ dG(y) + \int_{v_*}^{v_i} (v_i - R_2(y))^+ dG(y) \quad (4)$$

where the last integral appears only when $v_i > v_*$.

What if bidder i participates and decides to drop out at price b ? Suppose first that $b \geq v_*$; that is, b is on the candidate equilibrium path. Either bidder i wins, i.e. $b > v_{-i}^{(1)}$, and the second auction reserve price will be set equal to $\max\{v_*, v_{-i}^{(1)}, R_0\} = \max\{v_*, R_2(v_{-i}^{(1)})\}$, hence independently of b . Or he loses, i.e., $b < v_{-i}^{(1)}$, in which case the second auction reserve price is set equal to $\max\{b, v_{-i}^{(2)}, R_0\}$. So, in the event where he wins the second auction, that is when $v_i > v_{-i}^{(1)}$, bidder i will pay a price equal to $\max\{v_*, R_2(v_{-i}^{(1)})\}$. So, by bidding up to $b \geq v_*$, bidder i 's expected profit is:

$$(v_i - R_1) G(v_*) + \int_{v_*}^b (v_i - y) dG(y) + \int_{\underline{v}}^{v_i} (v_i - \max\{v_*, R_2(y)\})^+ dG(y). \quad (5)$$

Suppose then that b is off the equilibrium path: $b \in [R_1, v_*)$. Either bidder i wins, i.e. $v_{-i}^{(1)} < v_*$, and the second-auction reserve price will be set equal to $R_2(v_*)$, independently of b . Or, he loses, i.e., $b < v_* \leq v_{-i}^{(1)}$, and then we assume the seller holds beliefs about the bidders' types that leads her to fix $R_2 = r \geq R_0$ (e.g., the seller believes that i 's valuation is at least v_*). The price bidder i will pay if he wins the second auction is then $\max\{v_{-i}^{(1)}, r\} \geq R_2(v_{-i}^{(1)})$ and his expected profit from bidding $b \in [R_1, v_*)$ is:

$$(v_i - R_1) G(v_*) + \int_{\underline{v}}^{v_*} (v_i - \max\{R_2(v_*), y\})^+ dG(y) + \int_{v_*}^{v_i} (v_i - \max\{y, r\})^+ dG(y), \quad (6)$$

where the last integral appears only when $v_i > v_*$.

Note that $\max\{R_2(v_*), y\} = \max\{v_*, R_2(y)\}$. Hence, for $v_i \geq v_*$, it is immediate from (5) and (6) that bidding $b \geq v_*$ is preferred to bidding $b \in [R_1, v_*)$. Moreover, optimizing w.r.t. to b in (5) yields

¹¹We let $(x)^+$ denote $\max\{x, 0\}$.

$b = v_i$. For $v_i < v_*$, it is also immediate from above that bidding $b = v_*$ is at least as good as bidding any $b \in [R_1, v_*)$, given that $\max\{v_{-i}^{(1)}, r\} \geq R_2(v_{-i}^{(1)})$ and preferable to any $b > v_*$. It follows that, if he participates, then the best reply of player i of type v_i is to bid $\max\{v_i, v_*\}$.

Given the slopes in v_i of the expressions in (4) and (5), and the fact that $[v - R_2(v)]^+ = 0$ for all v , there is a unique participation cutoff valuation v_* determined by:

$$\int_{\underline{v}}^{v_*} (v_* - \max\{y, R_*(v_*)\}) dG(y) = (v_* - R_1) G(v_*), \quad (7)$$

and truthful bidding above this v_* is indeed part of a continuation equilibrium, given R_1 .

Note that for $v_* = R_1$, the LHS of (7) is positive while the RHS is null. The function given by the difference between the RHS and the LHS in (7) has a total derivative w.r.t. to v_* equal to

$$R'_*(v_*) [G(v_*) - G(R_*(v_*))] + (v_* - R_1)g(v_*)$$

which is positive for $v_* \geq R_1$. Hence this function is monotone, negative for $v_* = R_1$. Since $R_*(\bar{v}) = R_0$, it follows that a solution exists if and only if:

$$R_1 < \int_{\underline{v}}^{\bar{v}} \max\{R_0, y\} dG(y) = r_0.$$

It remains to show that non-participation is an equilibrium strategy when $R_1 \geq r_0$. If nobody else participates, bidder i with valuation v_i then simply expects the following equilibrium payoff:

$$\int_{\underline{v}}^{v_i} (v_i - R_2(y))^+ dG(y)$$

from non-participating. By deviating, i.e. by participating, he could win the first auction for a price of R_1 . Since participation is off the candidate equilibrium path, assume the seller's beliefs after observing participation are concentrated on the highest type \bar{v} , which results in a take-it-or-leave-it offer at price \bar{v} in the second auction. (Note that these beliefs provide the least incentives for participation and thus are the most favorable for the existence of a pooling equilibrium with no bidders participating.) The bidder

then earns no profits from the second auction. The equilibrium condition is then: for all v_i ,

$$\int_{\underline{v}}^{v_i} (v_i - R_2(y))^+ dG(y) \geq v_i - R_1.$$

Given that the LHS increases less steeply with v_i than the RHS, a necessary and sufficient condition is therefore that the above condition holds for \bar{v} :

$$R_1 \geq \int_{\underline{v}}^{\bar{v}} R(y) dG(y) = r_0.$$

This completes the proof. ■

Note that $R_0 < r_0$, so that for $R_1 \leq R_0$, an equilibrium with a positive measure of participating types always exists. Note also that since $R_*(v_*) < v_*$, if $R_1 > \underline{v}$ then we have $v_* > R_1$; that is, some bidders with valuations higher than the reserve price do not participate in the first auction.

In the first auction, a bidder is concerned that if he turns out to be the highest bidder and to win the first unit, then he might disclose so much information about his valuation that the seller would be in a position to capture all his rent in the second auction, by setting a reserve price equal to his valuation. An ascending-price auction format, however, is ample enough protection against the seller's opportunism, as it only provides a lower bound on the winner's valuation. Bidding truthfully in an ascending auction amounts to perfectly revealing one's own valuation only when one does not win; but if a participating bidder does not win the first auction, it is because there is another bidder with a higher valuation. Thus, when losing in the first auction the bidder knows that he will also lose in the second auction and does not care about revealing his valuation. In other words, there is no point for a bidder to get out of the first auction when another bidder is still in and the price has not yet reached his true valuation.

However, if no other bidder participates in the first auction, it may make sense for a bidder with a value above the first-auction reserve to not participate either, in order to induce the seller to decrease the second-auction reserve price. By not participating when nobody else does, a bidder induces the seller

to hold more pessimistic beliefs; namely, to believe that all bidders have valuation below some threshold v_* instead of being identified as the only bidder with valuation higher than v_* . When v is close to R_1 , the bidder has not much to expect from the first auction; by not participating he can influence the second-auction reserve price and raise his expected profits. This explain why the participation threshold v_* is strictly higher than the first-auction reserve R_1 . The threshold value v_* is precisely the type who is indifferent between $v_* - R_1$ with probability $G(v_*)$ today (the RHS of equation (3)) and the expected profit that he would make tomorrow by not participating today (the LHS of equation (3)).

Note that although truthful bidding is an equilibrium behavior for the participating bidders in the first auction, it is by no means a dominant strategy. As we shall see in Section 5, in sealed-bid auctions, as in the standard contracting literature without commitment, truthful bidding is impossible, since monotone bidding functions would reveal the winning bidder's true valuation through his first-auction bid, and hence would wipe out his informational rent in the second auction.

When no bidders participate in the first auction, it is clear that in equilibrium the seller does not update his beliefs and will optimally fix a reserve price equal to R_0 in the second auction. Unsurprisingly, if the reserve price in the first auction is sufficiently large, namely above r_0 , then bidders do not participate. Their prospects from first-auction participation are too low compared to the potential benefits of winning the second unit in a second auction with a low reserve price. If, on the other hand, the reserve in the first auction is sufficiently low, below r_0 , then in equilibrium bidders having high valuations (i.e., values $v_i \geq v_*$) participate in the first auction and bid up to their valuations.

Lemma 1 and Proposition 2 would not hold in general in a model with imperfect correlation of the buyers' valuations for the two units. Even a first-auction loser could then expect non-zero gains in the second auction and could benefit from manipulating R_2 . So, under imperfect correlation, although the valuation for the first unit provides less information on the bidder's valuation for the second unit, the ratchet effect

would induce a more complicated bidding equilibrium with some pooling among participating types, and hence less revelation of information.

4 Equilibrium and its Properties

Restricting attention to symmetric continuation equilibria with full separation under participation or no participation at all, it is now a straightforward task to characterize a global equilibrium, namely the equilibrium reserve price R_1 endogenously chosen by the seller and the ensuing continuation equilibrium.

We will show that the seller always induces participation by a positive measure of bidders' types; formally, $v_* < \bar{v}$. We also show that, in the first auction, there is less equilibrium participation than in the optimal one-shot auction; formally, $v_* > R_0$. On the contrary, the probability that at least one bidder participates in the second auction is higher than in the optimal auction; formally, $R_*(v_*) < R_0$.

Recall first that (3) provides a one-to-one mapping from R_1 to the ensuing equilibrium participation threshold v_* . It is indeed more convenient to think that the seller sets directly the participation threshold v_* . Rearranging (3) we then obtain that the reserve price in the first auction is equal to:

$$R_1 = \int_{\underline{v}}^{v_*} \frac{\max\{y, R_*(v_*)\}}{G(v_*)} dG(y) = E\left(\max\{v_{-i}^{(1)}, R_*(v_*)\} | v_{-i}^{(1)} \leq v_*\right). \quad (8)$$

Recall also that (1) determines $R_*(v_*)$, that is the second-auction reserve price whenever no participation took place in a first auction characterized by the participation threshold v_* , while $R_2(p) = \max\{p, R_0\}$ is the equilibrium reserve price in the second auction after the first auction concluded at a final price $p \geq v_*$. Characterizing the global equilibrium amounts then to determining the equilibrium participation threshold v_* . To simplify the notation, in the rest of this section we will omit the argument of $R_*(v_*)$ and simply write R_* .

Proposition 3 : *The seller's optimal participation threshold v_* solves:*

$$G(R_*) R_*' [1 - F(R_*) - R_* f(R_*)] + G(v_*) [1 - F(v_*) - v_* f(v_*)] = 0. \quad (9)$$

Furthermore, if $f(\bar{v}) > 0$ and $R_0 > \underline{v}$ the following strict inequalities hold:

$$R_* < R_0 < v_* < \bar{v},$$

while if $R_0 = \underline{v}$, then $v_* = R_* = R_0 = \underline{v}$.

Proof. Using the fact that the joint density of $(v^{(1)}, v^{(2)})$ is given by $nf(v^{(1)})g(v^{(2)})\mathbf{1}_{\{v^{(2)} \leq v^{(1)}\}}$, we can write the seller's expected profit as follows (the first integral relates to $v^{(2)}$, the second to $v^{(1)}$):

$$\begin{aligned} \pi(R_*, v_*) &= n \int_{\underline{v}}^{v_*} \int_{\max\{R_*, y\}}^{v_*} \max\{R_*, y\} f(x)g(y) dx dy \\ &\quad + nR_1 \int_{\underline{v}}^{v_*} \int_{v_*}^{\bar{v}} f(x)g(y) dx dy + nR_2(v_*) \int_{\underline{v}}^{v_*} \int_{R_2(v_*)}^{\bar{v}} f(x)g(y) dx dy \\ &\quad + n \int_{v_*}^{\bar{v}} \int_y^{\bar{v}} y f(x)g(y) dx dy + n \int_{v_*}^{\bar{v}} \int_{R_2(y)}^{\bar{v}} R_2(y) f(x)g(y) dx dy. \end{aligned}$$

Integrating over x and replacing R_1 using equation (8), one obtains:

$$\begin{aligned} \pi(R_*, v_*) &= n \int_{\underline{v}}^{v_*} \max\{R_*, y\} [F(v_*) - F(\max\{R_*, y\})] g(y) dy \\ &\quad + n [1 - F(v_*)] \left[v_* G(v_*) - \int_{R_*}^{v_*} G(y) dy \right] + nR_2(v_*) [1 - F(R_2(v_*))] G(v_*) \\ &\quad + n \int_{v_*}^{\bar{v}} y [1 - F(y)] g(y) dy + n \int_{v_*}^{\bar{v}} R_2(y) [1 - F(R_2(y))] g(y) dy. \end{aligned}$$

Recalling that $J(v) = v - \frac{1-F(v)}{f(v)}$, it follows that:

$$\frac{1}{n} \frac{\partial \pi}{\partial v_*} = -f(v_*) J(v_*) G(v_*) \mathbf{1}_{\{v_* \geq R_0\}},$$

$$\frac{1}{n} \frac{\partial \pi}{\partial R_*} = -G(R_*) f(R_*) J(R_*).$$

Suppose first that $v_* < R_0$, then the monotonicity of $J(\cdot)$ implies: $J(R_*) < J(v_*) < J(R_0) = 0$. Since $R_*' > 0$, it follows that $\frac{d\pi}{dv_*} > 0$. It is not optimal for the seller to choose R_1 so as to induce $v_* < R_0$.

Suppose then that $v_* \geq R_0 > \underline{v}$. For $v_* = R_0$, $J(R_*) < J(v_*) = J(R_0) = 0$ and so,

$$\frac{1}{n} \frac{d\pi}{dv_*} \Big|_{v_*=R_0} = -f(R_*) J(R_*) G(R_*) R_*'(R_0) > 0.$$

This implies that in equilibrium $v_* > R_0$. On the other hand, for $v_* = \bar{v}$, $R_* = R_0$ and $J(R_*) = 0$ while $J(v_*) > 0$. It follows that

$$\frac{1}{n} \frac{d\pi}{dv_*} \Big|_{v_*=\bar{v}} = -f(\bar{v})\bar{v} < 0.$$

Hence, in equilibrium, $v_* < \bar{v}$ and there is participation in the first auction. The equilibrium participation threshold v_* is given by the maximum condition $v_* \in \arg \max_x \pi(R_*(x), x)$. It follows that the FOC:

$$-f(R_*)J(R_*)G(R_*)R'_* - f(v_*)J(v_*)G(v_*) = 0$$

holds, where $R_*(v_*)$ depends on v_* . This implies that if $R_0 > \underline{v}$, then $R_*(v_*) < R_0 < v_* < \bar{v}$. If $R_0 = \underline{v}$, then, clearly, $v_* = R_* = R_0 = \underline{v}$. This completes the proof. ■

Proposition 3 asserts that in equilibrium the seller strategically chooses the first-auction reserve price so as to induce a positive measure of bidders' types to participate, but less than in a one-shot auction; that is, less than if she were able to commit not to use in the second-auction the information revealed by the different bids in the first auction. Non-commitment therefore induces a reduction in participation in the first auction. Although truthful bidding is still an equilibrium strategy for participating bidders, strategic retention of information takes the form of intermediate-valuation bidders refraining from competing in the first auction to avoid revealing information on their types, compared to the situation where this information would not be used.

Although participation is the relevant measure of the first-auction performance, note that our results do not imply that the actual reserve price chosen in the first auction is larger than when the seller can perfectly commit to the sequence of reserve prices. Indeed, it is possible that $R_1 < R_0$ and still $R_0 < v_*$. A simple examination of (8) allows us to assert that the difference $(v_* - R_1)$ decreases when n increases, but no result with respect to the comparison with R_0 is available.¹²

¹²It is also immediate and no surprise that: $\lim_{n \rightarrow \infty} v_* = \lim_{n \rightarrow \infty} R_1 = R_0$.

It is useful to present an example of our equilibrium. If F is uniform on $[0, 1]$ then $R_0 = \frac{1}{2}$, $R_* = \frac{v_*}{2}$,

$$R_1 = \frac{(1 + 2^n)(n2^n - 2^n + 1)}{n2^n(1 + 2^{n+1})}, \text{ and}$$

$$v_* = \frac{1 + 2^n}{1 + 2^{n+1}}.$$

It is clear that the seller cannot make a higher expected revenue in the repeated ascending auction without commitment than in the optimal dynamic auction, with commitment on reserve prices. We show, indeed, that the seller makes strictly less; that is, there is a positive cost for the seller to her lack of commitment power over reserve prices.

Proposition 4 : *The seller's expected revenue in the repeated ascending auction is strictly less than in the optimal sequential auction under full commitment.*

Proof. From the proof of Proposition 3, $\frac{\partial \pi}{\partial R_*}(R, v) > 0 > \frac{\partial \pi}{\partial v_*}(R, v)$ for all (R, v) such that $R_* \leq R < R_0 < v \leq v_*$. Therefore, $\pi(R_*, v_*) < \pi(R_0, R_0)$. Since $\pi(R_0, R_0)$ is precisely the seller's expected revenue in the optimal sequential auction under full-commitment, the proposition follows. ■

5 Sequential Sealed-Bid Auctions

We now focus on a situation with two sequential, second-price auctions. The critical difference with the case of sequential ascending auctions is that the seller observes all n bids, as opposed to just observing the $n - 1$ losing bids. This has important implications for the reserve price that the seller sets in the second auction and, consequently, on the equilibrium in the first auction. We prove that full separation of any set of types with positive measure is impossible. In contrast with the previous sections, this result is perfectly in line with the literature on the ratchet effect, in particular with Laffont-Tirole [1988].

Proposition 5 : *There exists no symmetric, (weakly) monotone, pure-strategy equilibrium in which a positive measure of types is revealed at the end of the first auction.*

Proof. Suppose, to the contrary, that there exists a symmetric, monotone, pure-strategy equilibrium, with a first-auction bidding function $\beta(\cdot)$ which is strictly increasing in an interval $[a, b]$, with $a \geq R_1$, the first auction reserve price. Consider type $v \in (a, b)$; if he bids $\beta(v)$ in the first auction then his type becomes known and he makes no profit in the second auction. This implies that $\beta(v) = v$ for all $v \in (a, b)$, otherwise type v would have an incentive to deviate and bid his valuation. Now consider a deviation by type $v \in (a, b)$ to a bid $v - \varepsilon > a$. The loss in the first auction is equal to

$$\int_{v-\varepsilon}^v (v - y)dG(y),$$

while the gain in the second auction, due to a lower reserve price, is

$$\varepsilon G(v - \varepsilon) + \int_{v-\varepsilon}^v (v - y)dG(y).$$

Thus, the deviation is profitable. This concludes the proof. ■

6 Conclusions

While we have not analyzed sequential, first-price auctions, it seems clear that the ratchet effect would be at least as serious as in second-price auctions. In addition to having an incentive to conceal information from the seller, bidders would also want to conceal information from each other, in order to affect bidding in their favor in the second auction.

On the contrary, the sequential ascending auction has a simple, symmetric, pure-strategy equilibrium in which the only pooling consists of strategic non-participation in the first auction by low valuation bidders. Thus, we have provided additional content to the view expressed by Ausubel and Milgrom (2001) that, from the point of view of protecting bidders' against future exploitation, the ascending auction is theoretically superior.¹³

¹³However, as argued first by Robinson (1985), the oral ascending auction is more susceptible than sealed-bid auctions to bidders' collusion (see Brusco and Lopomo (2002) for a recent application).

Two important limitations of our model are that it only considers a sequence of two auctions and that bidders' valuations are perfectly correlated across time. Once perfect correlation is assumed, it is natural to look at a sequence of two auctions, for the following reason. With perfect correlation of valuations, any sequence of auctions in which the highest bidder becomes known degenerates into a sequence of fixed-price offers by a single seller to a single buyer. Thus, before analyzing a model with an arbitrary, finite sequence of auctions we would need to solve the single-seller, single-buyer model. Unfortunately, as pointed out by Laffont and Tirole (1993, p. 407), "little is known about the equilibrium path for arbitrary horizons" of the single-seller, single-buyer, fixed-price contracting model without commitment. In our view, a more promising research direction is to relax perfect correlation of a bidder's valuation across time. With imperfect correlation, a plausible generalization of our results would then be that the ratchet effect induces less pooling in ascending-price auctions than in sealed-bid auctions. We plan to pursue this line of research in the future.

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