The Moving Mirror: Creation of Particles by an Accelerating Reflecting Point in (1+1) Flat Spacetime

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The motion of a single reflecting boundary can create particles. This is similiar to an accelerating observer in the familiar Minkowski vacuum state who observes particles (i.e. the Unruh effect). This provides a useful model for understanding the appearance of a thermal spectrum of particles in curved spacetime.

The moving mirror example does not generalize to massive fields or to four dimensional spacetime. In 4D only special trajectories will give exact solutions. Also, for general trajectories the solutions are not exact but only approximate. For more on this look at Ford and Vilenkin 1982. This review follows that of Davies QFTCS book, and the '77 moving mirror papers by Fulling and Davies.

A Planck spectrum with temperature

$$T = \frac{\kappa}{2\pi k_B}$$

occurs for this thermal radiation. We set $\hbar = c = 1$. k_B is the Boltzmann constant and κ is a constant with a relationship to the acceleration of the mirror. This temperature reveals the innate thermal nature of the vacuum in quantum field theory.

Consider a moving reflecting point(which we will sometimes call a mirror) in two-dimensional Minkowski spacetime moving along the trajectory:

$$x = z(t)$$

with speed less than light, $|\dot{z}(t)| < 1$ and things get interesting after time zero, that is, the mirror is at rest before time zero, $z(t) = 0$ for $t < 0$.

Let us work only with the theory for a massless scalar field, so that the Klein-Gordon wave equation becomes the equation of motion for the field. The massless scalar field ϕ using null coordinates $u = t - x$ and $v = t + x$ has the field equation

$$\Box\phi = \frac{\partial^2\phi}{\partial u \partial v} = 0$$

that is constrained to vanish at the mirror. That is, we impose from the outset, (somewhat unsatisfyingly as we have a resulting theory that is not via the minimization of any action) a boundary condition:

$$\phi(t, z(t)) = 0$$

that will physically represent consistency with reflection.

Allowing the mirror to be static in some inertial frame, (i.e. z(t) = 0), positive frequency 'in' modes have the form:

$$u_k^{in} = \frac{i}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega u}) = \frac{\sin \omega x}{\sqrt{\pi\omega}} e^{-i\omega t}$$

This is a positive frequency mode(i.e. we have a negative sign on the i in $e^{-i\omega t}$ which by convention is called 'positive frequency') with respect to Minkowski time t. We have the left moving, $e^{-i\omega v}$ and the right moving, $e^{-i\omega u}$ pieces of the mode. The label 'in' is useful because the state $|0, in\rangle$ has no particles in it. This is only true when the mirror is not accelerating, i.e. $t \leq 0$. But of course, if we are interested in the field once the mirror accelerates, we need a more general form for the modes. That general form is

$$u_k^{in} = \frac{i}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega(2\tau_u - u)})$$

The field, ϕ may be expanded in terms of these modes and we define a vacuum state:

$$\phi = \sum_{k>0}[a_k u_k^{in} + a_k^\dagger (u_k^{in})^*]$$

$$a_k|0, in\rangle = 0$$

This vacuum state is the initial 'no particle state'. It is the familiar one of standard quantum field theory.

Brief note on 'no particles': If we had some constant velocity particle detector, it would detect nothing. The two point function forms the basis of the algebraic approach to quantum field theory in curved spacetime. We can show that the mirror with constant velocity will detect no particles by using the two point function (i.e. the Wightman function) for the 'in' region:

$$\langle in, 0|\phi(x)\phi(x')|0, in\rangle = -\frac{1}{4\pi} ln[\frac{(u - u' - i\epsilon)(v - v' - i\epsilon)}{(v - u' - i\epsilon)(u - v' - i\epsilon)}]$$

and substituting into the detector response function,

$$F(E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-eE(\tau - \tau')} D^+(x(\tau), x(\tau')).$$

The quickest way to see this is to assume some inertial path of the detector for ($x = x_0 + v_0 t$) you can choose the privilaged path $v_0 = 0$ and $x_0 = 0$ so that $v = t + x$ and $u = t - x$ becomes $v = t$ and $u = t$ and the two point function is seen to be equal to a $\ln(1) = 0$. There is no flux so we have no creation of particles.

What we are most interested in, however, is when the mirror starts moving. Once the mirror accelerates we have something akin to gravity. When the mirror accelerates as $t > 0$ the field modes will suffer distortion for $u = t - x > 0$ from

$$u_k^{in} = \frac{i}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega u})$$

to our more general form

$$u_k^{in} = \frac{i}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega(2\tau_u - u)})$$

Here τ_u is determined implicitly by the trajectory $x = z(t)$ through $\tau_u - z(\tau_u) = u$. There is an asymmetry between u and v chosen to correspond to the usual retarded boundary condition. Notice that the right moving waves, the ones that are reflected from the mirror are complicated because of the Doppler shift suffered. Note again, that for the static mirror $z = 0$ and thus $\tau_u = u$ we have reduced the general answer back to the resting mirror case.

This moving mirror plays the same role as a time-dependent background geometry, that is, a gravitational field. What makes it different is that the $2\tau_u - u$ term is unchanged all along the null ray as the light ray goes out forever, and the distortion of the modes occurs suddenly after reflection rather than gradually over a long time as is the case in a gravitational geometrical disruption.

So lets look at this again for a moment. Look at the right moving piece:

$$e^{-i\omega(2\tau_u - u)}$$

This reduces in the region $u < 0$ to the standard form for a right-moving wave, $e^{-i\omega u}$. For $t < 0$ we associate the empty particle vacuum state, defined by

$$a_k|0, in\rangle = 0$$

with it, as long as $u < 0$. But in the region $u > 0$, it does not reduce to the standard right moving waves. The vacuum state becomes a state with particles! The Doppler complication ends up exciting field modes and particles come into existence. Moving mirrors therefore create particles, which travel away to the right along null rays, $u = constant$.

A thermal spectrum results from the calcuation of Bogolubov coefficients. The outline of the calculation is as follows:

Allow the mirror to have specific trajectory of asymptotic form as $t \to \infty$

$$z(t) \to -t - Ae^{-2\kappa t} + B$$

Null rays with $v < B$ reflect while rays with $v > B$ arn't in the lightcone of the mirror. The rays with $v = B$ is the horizon. Define $p(u) \equiv 2\tau_u - u$. Use $\tau_u - z(\tau_u) = u$ and the above trajectory to get:

$$p(u) = 2\tau_u - u = \tau_u + z(\tau_u) = B - Ae^{-2\kappa\tau_u} = B - Ae^{-\kappa(u+p(u))}$$

As $u \to \infty$

$$p(u) \to B - Ae^{-\kappa(u+B)}$$

Define the inverse $p(u)^{-1} \equiv f(v)$ and for $v < B$

$$f(v) \to -\kappa^{-1}\ln[(B - v)/A] - B$$

The original standing waves (in-going) for $t \leq 0$:

$$u_\omega^{in} = (\pi\omega)^{-1/2}\sin\omega x e^{-i\omega t}$$

The outgoing modes consist of the outgoing plane wave, $(4\pi\omega')^{-1/2}e^{-i\omega' u}$ valid for $v \geq B$ and the reflection wave, $(4\pi\omega')^{-1/2}e^{-i\omega' f(v)}$ valid for $v < B$. We don't care about the light that is outside the lightcone of the mirror thats accelerating away. We are only concerned with the reflected light, so we will set $u_k^{out} = 0$ for $v \geq B$ and the out-going modes will be

$$u^{out}_{\omega'} = i(4\pi\omega')^{-1/2}[e^{-i\omega' f(v)}]$$

The spacelike surface chosen for the Bogolubov tranformation is up to us. For our trajectory we will choose $t = 0$. The reason $f(v)$ is used for the outmodes is because the portions of the standard $e^{-i\omega u}$ right moving waves traced back in time will correspond to left-moving waves that crowd up along $v = B$. The function $f(v)$ is rapidly varying in this region. By symmetry the function $f(v)$ will be the inverse of the function $p(u)$.

By evaluating the Bogolubov transformation between the in and out modes the thermal spectrum results. Using only the region $v < B$ for the out modes, the Bogolubov coefficients are solved using

$$\alpha_{\omega\omega'} = (u^{in}_{\omega}, u^{out}_{\omega'}) \qquad \beta_{\omega\omega'} = -(u^{in}_{\omega}, u^{out*}_{\omega'})$$

Evaluating the coefficients at $t = 0$ is the convenient Cauchy (hypersurface) and for $\beta_{\omega\omega'}$:

$$\beta_{\omega\omega'} = -((\pi\omega)^{-1/2}\sin(\omega x)e^{-i\omega t}, [i(4\pi\omega')^{-1/2}e^{-i\omega' f(v)}]^*)$$

Pull out constants, and complex conjugate the u^{out}

$$\beta_{\omega\omega'} = -(2\pi)^{-1}(\omega\omega')^{-1/2}(\sin(\omega x)e^{-i\omega t}, -ie^{i\omega' f(v)})$$

The scalar field inner product for (1+1) is defined by $(\phi_1, \phi_2) = i\int_t(\phi_2^*\partial_t\phi_1 - \phi_1\partial_t\phi_2^*)dx$ where t signifies a spacelike hyperplane of simultaneity at instant t. This gives

$$\beta_{\omega\omega'} = -(2\pi)^{-1}(\omega\omega')^{-1/2} i\int[ie^{-i\omega' f(v)}\partial_t(\sin(\omega x)e^{-i\omega t}) - \sin(\omega x)e^{-i\omega t}\partial_t(ie^{-i\omega' f})]dx$$

Take care of the derivatives, after pulling out an i:

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega\omega')^{-1/2}\int[e^{-i\omega' f(v)}\sin\omega x(-i\omega)e^{-i\omega t} - \sin\omega x e^{-i\omega t}(-i\omega' f'(v))e^{-i\omega' f(v)}]dx$$

Bring out i in front and at the instant $t = 0$ integrating as far as $v = B$ only, we have

$$\beta_{\omega\omega'} = i(2\pi)^{-1}(\omega\omega')^{-1/2}\int_0^B dx[\omega' f'(x) - \omega]e^{-i\omega' f(x)}\sin\omega x$$

This bears fruit by integrating by parts on the first term (1^{st}) and performing some algebra:

$$1^{st} = i(2\pi)^{-1}(\omega\omega')^{-1/2}\int_0^B [\omega' f'(x)e^{-i\omega' f(x)}\sin\omega x]dx$$

Where $\int udv = uv - \int vdu$, here $dv = (-i\omega' f')e^{-i\omega' f}dx$ thus $v = e^{-i\omega' f}$, and $u = \sin\omega x$ thus $du = \omega\cos\omega x dx$:

$$1^{st} = -(2\pi)^{-1}(\omega\omega')^{-1/2}\int[(-i\omega' f'(x))e^{-i\omega' f(x)}\sin\omega x]dx$$

This gives, including the boundary term:

$$1^{st} = -(2\pi)^{-1}(\omega\omega')^{-1/2}\left[e^{-i\omega' f(x)}\sin\omega x\Big|_0^B - \int_0^B e^{-i\omega' f(x)}\omega\cos(\omega x)dx\right]$$

The boundary term vanishes as most of the contribution comes from the region near $x = B$, as $f(x)$ is rapidly varying there. Including the second term, while ignoring the boundary term gives:

$$\beta_{\omega\omega'} = \left[i(2\pi)^{-1}(\omega\omega')^{-1/2} \int_0^B -\omega e^{-i\omega' f(x)} \sin(\omega x) dx \right] - \left[-(2\pi)^{-1}(\omega\omega')^{-1/2} \int_0^B e^{-i\omega' f(x)} \omega \cos(\omega x) dx \right]$$

Simplify,

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega/\omega')^{1/2} \int_0^B e^{-i\omega' f(x)}(-i \sin \omega x + \cos \omega x) dx$$

Euler's formula:

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega/\omega')^{1/2} \int_0^B e^{-i\omega' f(x) - i\omega x} dx$$

Obscure approximations now follow: assume our trajectory $f(x)$ is used for all x, assume it is valid to replace lower limit 0 of the integral by $-\infty$. Let $\omega \to \infty$ and recover an ordinary Γ-function instead of evaluating the integral explicitly in terms of incomplete Γ-functions.

Evaluate integral with these approximations:

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega/\omega')^{1/2} \int_0^B e^{-i\omega'[-\kappa^{-1} \ln{(B-x)/A}] + Bi\omega' - i\omega x} dx$$

Using $D = \kappa^{-1} \ln A - B$

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega/\omega')^{1/2} \int_0^B e^{-i\omega'[-\kappa^{-1} \ln{(B-x)}] + D] - i\omega x} dx$$

Continue via

$$\beta_{\omega\omega'} = (2\pi)^{-1}(\omega/\omega')^{1/2} \int_0^B (B-x)^{i\omega'\kappa^{-1}} e^{-i\omega' D - i\omega x} dx$$

Pulling out the D term

$$\beta_{\omega\omega'} = (4\pi^2)^{-1/2}(\omega/\omega')^{1/2} e^{-i\omega' D} \int_0^B (B-x)^{i\omega'\kappa^{-1}} e^{-i\omega x} dx$$

We look toward evaluating the integral so use a substitution $t = B - x$, $dt = -dx$, and the integral is

$$\int_B^0 -dt\, t^{i\omega'/\kappa} e^{-i\omega(B-t)} = -e^{-i\omega B} \int_B^0 dt\, t^{i\omega'/\kappa} e^{i\omega t}$$

Apply another substitution, say $u = -i\omega t$, $du = -i\omega dt$, the integral becomes

$$-e^{-i\omega B} \int_{-i\omega B}^0 du(-i\omega)^{-1}(-u/i\omega)^{i\omega'/\kappa} e^{-u} = e^{i\omega B}(-i\omega)^{-1}(-1)^{i\omega'/\kappa}(i\omega)^{-i\omega'/\kappa} \int_{-i\omega B}^0 du\, u^{i\omega'/\kappa} e^{-u}$$

All together, our coeffiecient is

$$\beta_{\omega\omega'} = (4\pi^2)^{-1/2}(\omega/\omega')^{1/2} e^{-i\omega' D} \left[e^{i\omega B}(-i\omega)^{-1}(-1)^{i\omega'/\kappa}(i\omega)^{-i\omega'/\kappa}(-) \int_0^\infty du\, u^{i\omega'/\kappa} e^{-u} \right]$$

The integral is now a complete gamma function

$$\beta_{\omega\omega'} = -(4\pi^2)^{-1/2}(\omega/\omega')^{1/2} e^{-i\omega' D + i\omega B}(-i\omega)^{-1}(-1)^{i\omega'/\kappa}(i\omega)^{-i\omega'/\kappa} \Gamma(1 + i\omega'/\kappa)$$

Grouping together terms

$$\beta_{\omega\omega'} = -i(4\pi^2\omega\omega')^{-1/2}e^{-i\omega' D+i\omega B}(-1)^{i\omega'/\kappa}(i)^{-i\omega'/\kappa}(\omega)^{-i\omega'/\kappa}\,\Gamma(1+i\omega'/\kappa)$$

Using $e^{i\pi} = 1$

$$\beta_{\omega\omega'} = -i(4\pi^2\omega\omega')^{-1/2}e^{-i\omega' D+i\omega B}e^{-\pi\omega'/2\kappa}(\omega)^{-i\omega'/\kappa}\,\Gamma(1+i\omega'/\kappa)$$

From here we may calculate $|\beta_{\omega,\omega'}|^2$,

$$|\beta_{\omega,\omega'}|^2 = (4\pi^2\omega\omega')^{-1}e^{-\pi\omega'/\kappa}|\Gamma(i\omega'/\kappa)i\omega'/\kappa|^2$$

Using $|\Gamma(ix)|^2 = \pi/x\sinh(\pi x)$ we have

$$|\beta_{\omega,\omega'}|^2 = (4\pi^2\omega\omega')^{-1}e^{-\pi\omega'/\kappa}\left(\frac{\omega'}{\kappa}\right)^2\frac{\kappa}{\omega'}\frac{\pi}{\sinh(\pi\omega'/\kappa)}$$

Using $2\sinh = e^x - e^{-x}$

$$|\beta_{\omega,\omega'}|^2 = (4\pi^2\kappa\omega)^{-1}e^{-\pi\omega'/\kappa}\frac{2\pi}{e^{\pi\omega'/\kappa}-e^{-\pi\omega'/\kappa}}$$

So

$$|\beta_{\omega,\omega'}|^2 = \frac{1}{2\pi\kappa\omega}\left(\frac{1}{e^{2\pi\omega'/\kappa}-1}\right)$$

Thus

$$T = \frac{\kappa}{2\pi k_B}$$