# Casimir Effect

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The Casimir effect is the result of a force between two objects not due to the electromagnetic force, the gravitational force, or the exchange of any particles. It is a 'vacuum' force, or 'zero-point energy' force due to the resonance of the ever-present energy fields in the space between two objects. It is a force due to quantization, similiar to the 'force' that keeps negative electrons from pummeling into the positive nucleus.

Fields contribute to the energy of the vacuum, whether they be scalar fields, the electromagnetic field or whatever. If we 'disturb the vacuum' by introducing boundary conditions we can observe the Casimir effect by adding up the energies of the standing waves between the two objects, renormalize and measure the finite energy shift.

## 1 (1+1) massless scalar field Casimir effect

The standing waves are:

$$\psi_n(x,t) = e^{-i\omega_n t}\sin(k_n x)$$

There is no need to worry about $y$ or $z$ as we only have one spatial dimension. The calculation will not involve the electromagnetic field so there is no need to worry about polarization, etc. The wave vector $k_n$ is

$$k_n = \frac{n\pi}{a}$$

where $a$ is the distance between the two objects. ( i.e. delta point functions in 1+1). We have via $\omega = kc$

$$\omega_n = \frac{n\pi c}{a}$$

The vacuum energy of all the standing waves is the sum of the excitation modes:

$$E = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n$$

This is divergent as there is no cut off value for the highest energy waves. We now have an infinity to deal with. In real life though, nothing can stop ultra high frequency waves from leaking out. So we need to account for this vital piece of physics. We do this by introducing a factor to destroy the high energy waves that will leak out. This is called regularizing. We make this sum finite by introducing a regulator, manipulate the sum, then follow it up by taking a limit that will remove the regulator.

$$E = \frac{\hbar}{2}\sum_{n}^{\infty}\frac{n\pi c}{a} = \frac{\hbar\pi c}{2a}\sum_{n}^{\infty}\frac{1}{n^{-1}}$$

Invoke the Reimann zeta function regulator of $-1$. This is called renormalization by zeta function regularization. Here

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Reimann zeta function and in our case, $\xi(-1) = -\frac{1}{12}$. So our regularization is considered complete. Our sum is now

$$E = \frac{\hbar \pi c}{2a}(-\frac{1}{12}) = -\frac{\hbar \pi c}{24a}$$

The force between the two objects is found by $F = -\partial_a E$, that is

$$F = -\partial_a(-\frac{\hbar \pi c}{24a})$$

$$\boxed{F = -\frac{\hbar \pi c}{24a^2}}$$

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This is the Casimir effect. Note that the effect is attractive as indicated by the negative sign and that $\hbar$ reveals the quantum nature of this force.

## 2 Canonical (1+1) massless scalar field Casimir effect

Scalar field obeys boundary conditions

$$\psi(0,t) = \psi(a,t) = 0$$

Scalar field wave equation

$$\frac{1}{c^2}\partial_t^2 \psi(x,t) - \partial_x^2 \psi(x,t) + \frac{m^2c^2}{\hbar^2}\psi(x,t) = 0$$

where the field will be massless, $m = 0$. The solutions are

$$\psi_n^{\pm}(x,t) = \left(\frac{c}{a\omega_n}\right)^{1/2} e^{\pm i\omega_n t} \sin k_n x$$

$$\omega_n = \left(\frac{m^2c^4}{\hbar^2} + c^2k_n^2\right)^{1/2} = ck_n, \quad k_n = \frac{\pi n}{a}, \quad n = 1, 2, ...$$

Quantization occurs

$$\psi(x,t) = \sum_n [\psi_n^{(-)}(x,t)a_n + \psi_n^{(+)}(x,t)a_n^+]$$

$$[a_n, a_{n'}^+] = \delta_n^{n'}, \quad [a_n, a_{n'}] = [a_n^+, a_{n'}^+] = 0$$

$$a_n|0\rangle = 0$$

The operator of the energy density is

$$T_{00} = \frac{\hbar c}{2}\left(\frac{1}{c^2}[\partial_t \psi]^2 + [\partial_x \psi]^2\right)$$

Using the above, you obtain

$$\langle 0|T_{00}|0\rangle = \frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n$$

Total vacuum energy

$$E_0(a) = \int_0^a \langle 0|T_{00}|0\rangle dx = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n$$

Now we are the standard starting point. Allow us to regularize this infinite result by the damping term $e^{-\delta\omega_n}$. We will apply the limit $\delta \to 0$ to remove the regularization.

$$E_0(a,\delta) = \frac{\hbar c\pi}{2a}\sum_{n=1}^{\infty} n e^{-\delta c\pi n/a}$$

You can solve this by

$$E_0(a,\delta) = -\frac{\hbar}{2}\frac{\partial}{\partial\delta}\sum_{n=1}^{\infty} e^{-\delta\pi nc/a}$$

$$E_0(a,\delta) = -\frac{\hbar}{2}\frac{\partial}{\partial\delta}\frac{e^{-\delta\pi c/a}}{1-e^{-\delta\pi c/a}} = \frac{\hbar\pi c}{2a}\frac{e^{\delta\pi c/a}}{(e^{\delta\pi c/a}-1)^2}$$

$$E_0(a,\delta) = \frac{\hbar\pi c}{2a}\frac{1}{(e^{\delta\pi c/2a}-e^{-\delta\pi c/2a})^2} = \frac{\hbar\pi c}{8a}\sinh^{-2}\frac{\delta\pi c}{2a}$$

In limit of small $\delta$,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

$$\sinh^2 x = x^2 + \frac{x^4}{3} + ...$$

$$\sinh^{-2} x \approx \frac{x^{-2}}{1+x^2/3} \approx x^{-2}(1-x^2/3)$$

$$E_0(a,\delta) = \frac{\hbar\pi c}{8a}\sinh^{-2}\frac{\delta\pi c}{2a} = \frac{\hbar a}{2\pi c\delta^2} - \frac{\hbar\pi c}{24a} + O(\delta^2)$$

Compare with unbounded interval,

$$\psi_k^{\pm}(x,t) = \left(\frac{c}{4\pi\omega}\right)^{1/2} e^{\pm i(\omega t - kx)} \quad \omega = ck \quad -\infty < k < \infty$$

$$\psi(x,t) = \int \frac{dk}{2\pi}[\psi_k^{(-)}(x,t)a_k + \psi_k^{(+)}(x,t)a_k^{+}]$$

$$[a_k, a_{k'}^{+}] = \delta(k-k'), \quad [a_k, a_{k'}] = [a_k^{+}, a_{k'}^{+}] = 0$$

$$a_k|0_M\rangle = 0 \qquad \langle 0_M|T_{00}|0_M\rangle = \frac{\hbar}{2\pi}\int_0^{\infty}\omega dk$$

Total vacuum energy

$$E_{0M}(-\infty,\infty) = \frac{\hbar}{2\pi}\int_0^\infty \omega dk L$$

Interval of interest

$$E_{0M}(a) = E_{0M}(-\infty,\infty)\frac{a}{L} = \frac{\hbar a}{2\pi}\int_0^\infty \omega dk$$

Regularize

$$E_{0M}(a) = \frac{\hbar a}{2\pi}c\int_0^\infty k e^{-\delta c k} dk = \frac{\hbar a}{2\pi c \delta^2}$$

Renormalized energy is therefore

$$E_0^{ren}(a) = \lim_{\delta\to 0}[E_0(a,\delta) - E_{0M}(a,\delta)] = -\frac{\hbar\pi c}{24a}$$

$$F = -\frac{\partial E(a)}{\partial a}$$

$$\boxed{F = -\frac{\hbar\pi c}{24a^2}}$$

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Same result as the renormalization by zeta function regularization.

## 3 (3+1) electromagnetic field Casimir effect

Here the standing waves are

$$\psi_n(x,y,z,t) = e^{-i\omega_n t} e^{ik_x x + ik_y y}\sin(k_n z)$$

The plates are in the x-y plane. Polarization and magnetic components are ignored.

$$k_n = \frac{n\pi}{a}$$

and

$$\omega_n = c\sqrt{k_x^2 + k_y^2 + \frac{n^2\pi^2}{a^2}}$$

Summing over all possible modes to get the vacuum energy per unit area yields

$$\frac{E}{A} = 2\cdot\int \frac{dk_x dk_y}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{\hbar\omega_n}{2}$$

where the factor of 2 is for the two possible polarizations of the wave. It can be seen already that this integral will diverge. Introducing a regulator, particularly, the zeta function regulator $|\omega|^{-s}$ will make our result physically meaningful.[1] So therefore:

$$\frac{E}{A} = \frac{\hbar}{4\pi^2}\int dk_x dk_y \sum_{n=1}^{\infty}\omega_n$$

[1] Relevant to our group, the zeta function regulator is not very useful in numerical calculations, but great for theoretical calculations. The Gaussian regulator $e^{-t^2|\omega_n|^2}$ is better suited for numerical calculations because of its superior convergence properties but harder to use in theoretical calculations.

becomes

$$\frac{E}{A} = \frac{\hbar}{4\pi^2}\int dk_x dk_y \sum_{n=1}^{\infty} \omega_n |\omega_n|^{-s}$$

and we will take the limit and set the complex number $s = 0$ after we manipulate this infinite sum. Convert to polar coordinates to get rid of the double integral:

$$\frac{E}{A} = \frac{\hbar c^{1-s}}{4\pi^2}\sum_{n=1}^{\infty}\int_0^{\infty} 2\pi q dq |q^2 + \frac{\pi^2 n^2}{a^2}|^{(1-s)/2}$$

Here $q^2 = k_x^2 + k_y^2$, the $2\pi$ comes from integration over $\phi$, and the $q$ in front is the Jacobian. Simplifying,

$$\frac{E}{A} = \frac{\hbar c^{1-s}}{2\pi}\sum_{n=1}^{\infty}\int_0^{\infty} q dq |q^2 + \frac{\pi^2 n^2}{a^2}|^{(1-s)/2}$$

and making $u^2 = q^2 + \pi^2 n^2/a^2$, where $udu = qdq$ the integral is evaluated as

$$\frac{E}{A} = \frac{\hbar c^{1-s}}{2\pi}\sum_{n=1}^{\infty}\int_{\pi n/a}^{\infty} u^{2-s} du$$

This is

$$\frac{E}{A} = -\frac{\hbar c^{1-s}}{2\pi}\sum_{n=1}^{\infty}\frac{\pi^{3-s} n^{3-s}}{(3-s)a^{3-s}}$$

The infinite vacuum energy of the quantized electromagnetic field in free Minkowski space was subtracted (i.e. we are looking for the shift of the disturbed vacuum).

$$\frac{E}{A} = -\frac{\hbar c^{1-s}\pi^{2-s}}{2(3-s)a^{3-s}}\sum_{n=1}^{\infty}\frac{1}{n^{s-3}}$$

The sum may be taken to be the Riemann zeta function. It can be shown that the use of the $\xi(-3)$ is equivalent to the renormalization of the vacuum energy of our situation under consideration. For different and more complicated geometric configurations additional renormalization is generally needed. Taking the limit as $s \to 0$ and using $\xi(-3) = 1/120$,

$$\lim_{s\to 0}\frac{E}{A} = -\frac{\hbar c \pi^2}{6a^3}\xi(-3)$$

$$\frac{E}{A} = -\frac{\hbar c \pi^2}{720 a^3}$$

The force per area between the plates is then $F/A = -\partial_a(E/A)$:

$$\boxed{\frac{F}{A} = -\frac{\hbar c \pi^2}{240 a^4}}$$

This is the mysterious and celebrated Casimir effect. This force, though small, has been measured in the lab. It is one of few macroscopic quantum effects, soon to rival the attention that has been given to the other macroscopic quantum effects like superfluidity, superconductivitiy and quantum Hall effect.