Curved Space Quantization

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This short review will outline a few of the basic applications of quantum field theory in curved spacetime. It will summarize the treatment of the creation of particles from gravitational fields, moving mirror radiation and the Hawking effect. I will follow the QFTCS lectures of Ford.

1 Gravitational Field Particle Creation

Let's stick to a simple, standard model. The real massive scalar field with Lagrangian:

$$L = \frac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2 - \xi R \phi^2)$$

This model uses the metric $(+---)$ and $G = c = \hbar = 1$. The equation of motion that results is:

$$\Box \phi + m^2 \phi + \xi R \phi = 0$$

This is a wave equation where R denotes scalar curvature, ξ is a coupling constant (for minimal coupling we set $\xi = 0$, and for conformal coupling we set $\xi = 1/6$) and $\Box$ is the d'Alembertian operator, $\Box = \nabla_\mu \nabla^\mu$.

Now this Klein-Gordon wave equation has an inner product for a pair of solutions, f_1 and f_2. It is:

$$(f_1, f_2) \equiv i \int (f_2^* \overleftrightarrow{\partial_\mu} f_1) d\Sigma^\mu$$

Here $d\Sigma^\mu = d\Sigma n^\mu$ and $d\Sigma$ is the volume element in a given spacelike hypersurface, n^μ is the timelike unit vector normal to this hypersurface. Remember this important point: this inner product is independent of the choice of hypersurface. That is

$$(f_1, f_2)_{\Sigma_1} = (f_1, f_2)_{\Sigma_2}$$

Where Σ_1 and Σ_2 are different, non-intersecting hypersurfaces. Canonical quanitzation include defining the canonical momentum by $\pi = \frac{\partial L}{\partial \dot{\phi}}$, imposing the canonical commutation relation $[\phi(x), \pi(x')] = i\delta(x, x')$. The field operator is now writen like

$$\phi = \sum_j (a_j f_j + a_j^\dagger f_j^*)$$

with $[a_j, a_{j'}^\dagger] = \delta_{j,j'}$. This prescription comes with a defined vacuum state, $|0\rangle$, where $a_j|0\rangle = 0$. In flat space the positive normalized solutions are positive frequency solutions, $f_j \propto e^{-i\omega t}$. There is no unique choice for the modes or vacuum state in curved spacetime.

Imagine a spacetime which in the far past and far future is completely flat, but has some curvature at the present time, (i.e. asymptotically flat but a curved intermediate region). Now we can describe the real massless scalar field by the solutions to the wave equation everywhere in spacetime. The solutions are choosen to be positive and orthonormal. There are 'in-region' solutions f_j that exist in the past, and 'out-region' solutions F_j that exist in the future. It's interesting to note that we defined these functions by their asymptotic properties in different regions, but that they are still solutions to the wave equation everywhere. Their orthonormal properties demand:

$$(f_j, f_{j'}) = (F_j, F_{j'}) = \delta_{jj'}$$
$$(f_j^*, f_{j'}^*) = (F_j^*, F_{j'}^*) = -\delta_{jj'}$$
$$(f_j, f_{j'}^*) = (F_j, F_{j'}^*) = 0$$

You may expand the in-modes in terms of the out-modes:

$$f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*)$$

And when you insert this into the orthorgonality conditions you will find:

$$\sum_k (\alpha_{jk}\alpha^*_{j'k} - \beta_{jk}\beta^*_{j'k}) = \delta_{jj'}$$
$$\sum_k (\alpha_{j'k}\beta_{jk} - \beta_{j'k}\alpha_{jk}) = 0$$

The out-modes in terms of the in-modes looks like:

$$F_k = \sum_j (\alpha^*_{jk} f_j - \beta_{jk} f_j^*)$$

We can expand our field in either the past (f_j) or future modes (F_j):

$$\phi = \sum_j (a_j f_j + a_j^\dagger f_j^*) = \sum_j (b_j F_j + b_j^\dagger F_j^*)$$

Note the labels for the annihilation and creation operators are a's for the in-region and b's for the out-region. If we have no partilces initially we have

$$a_j |0\rangle_{in} = 0 \quad \forall j$$

for the in-region vacuum state. For no particles in the far future, we have likewise,

$$b_j |0\rangle_{out} = 0 \quad \forall j$$

for the out vacuum state. Carefully use $a_j = (\phi, f_j)$ and $b_j = (\phi, F_j)$ and you can expand the two sets of creation and annihilation operators in terms each other, resulting in the famous Bogolubov transformation:

$$a_j = \sum_k (\alpha^*_{jk} b_k - \beta^*_{jk} b_k^\dagger)$$
$$b_k = \sum_j (\alpha_{jk} a_j + \beta^*_{jk} a_j^\dagger)$$

(after Nikolai Bogolubov who laid the basis for the microscopic theory of superfluidity in Helium II). This construction allows us to describe particle creation by a time-dependent gravitational field. The interesting thing about this is that even if there are no particles to begin with, and the gravitational field is then turned on, we can see that the average number of particles in existence in the future will be

$$\langle N_k \rangle = \langle 0 | b_k^\dagger b_k | 0 \rangle_{in} = \sum_j |\beta_{jk}|^2$$

Note the use of the sandwiching $N_k = b_k^\dagger b_k$ between the $|0\rangle_i n$ states. This is using the Heisenberg picture to describe the quantum dynamics, so that $|0\rangle_i n$ is the state of the system for all time. So all we need for particles to exist is if any of the β_{jk} coefficients are non-zero. That is, we are looking for mixing of positive and negatives frequency solutions because as you remember $f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*)$. We are now in a great position to describe the creation of particles from gravitational fields. For instance the creation of partiles created from an expanding universe has been explored using this analysis.

***Note on the Moving Mirror**

The moving mirror example does not generalize to massive fields or to four dimensional spacetime. Only special trajectories will give exact solutions. Also, for general trajectories the solutions are not exact but only approximate. For more on this look at Ford and Vilenkin 1982.

2 Black Holes Create Particles

Let's look at the case of a massless scalar field in Schwarzschild spacetime with the understanding that the idea that this construction can be carried out with any quantum field in any spacetime geometry that describes black holes.

Assuming no particles were around before the collapse of the black hole began we have:

$$|\psi\rangle = |0\rangle_{in}$$

where the in-modes are pure positive frequency:

$$f_{\omega lm} \propto e^{-i\omega v} \quad asv \to -\infty$$

with $v = t + r^*$ as the advanced time coordinate and we note the modes are located on $\mathcal{I}^-$.

The out-modes are also to be considered pure positive:

$$F_{\omega lm} \propto e^{-i\omega u} \quad au \to \infty$$

where $u = t - r^*$ as the retardred time coordinate and we not the modes are located on $\mathcal{I}^+$.

We want the relationship between these sets of modes to calculate the Bogolubov coefficients and thus the average number of particles created. We don't solve the wave equation for the modes everywhere to determine the Bogolubov coefficients, in fact, to obtain the creation of particles all we look at is the particle emission long after the collapse occured. At late times, the modes that dominate are those coming from $\mathcal{I}^-$ with high frequency that propagated through the collapsing body right before the horizon formed. They redshifted greatly on the way to $\mathcal{I}^+$. Geometric optics are used to describe the propagation of these modes because of their high frequency. Using the geometric optics approximation our modes become

$$f_{\omega lm} \propto \frac{Y_{lm}(\theta,\phi)}{\sqrt{4\pi\omega}r} \times e^{-i\omega v[G(u)]} \quad on \quad \mathcal{I}^-[\mathcal{I}^-]$$

$$F_{\omega lm} \propto \frac{Y_{lm}(\theta,\phi)}{\sqrt{4\pi\omega}r} \times e^{-i\omega u[g(v)]} \quad on \quad \mathcal{I}^+[\mathcal{I}^-]$$

Here $u = g(v)$ is the constant outgoing ray and $v = g^{-1}(u) \equiv G(u)$ is the constant ingoing ray. $Y_{lm}(\theta,\phi)$ is a spherical harmonic. The ray tracing argument given by Hawking's 1975 work results in:

$$u = g(v) = -4M \ln\left(\frac{v_0 - v}{C}\right)$$

$$v = G(u) = v_0 - Ce^{-u/4M}$$

Here C is constant, v_0 is the limiting value of v for rays that pass through the body before the horizon forms, M is the black hole mass. This is the general result and essentially independent of interior geometry. We must become more familiar with this result so let us look at the specific example of a thin shell. After going over this explicitly we will be more familiar with how the lograthmic solution is in fact a general solution.

Thin Shell Example of Log Result

The spacetime inside a thin shell is flat and has the metric:

$$ds^2 = dT^2 - dr^2 - r^2 d\Omega^2$$

Interior region null coordinates for ingoing rays:

$$V = T + r$$

and for outgoing rays:

$$U = T - r$$

The spacetime outside the thin shell is Schwarzschild with the metric:

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2 d\Omega^2$$

Exterior region null coordinates for ingoing rays:

$$v = t + r^*$$

and for outgoing rays:

$$u = t - r^*$$

Allow us to recall the 'tortoise coordinate' r^* as:

$$r^* = r + 2M \ln\left(\frac{r - 2M}{2M}\right)$$

As spacetime here will be treated classically, we must have continuity across geometries, so the metric of this 3D hypersurface must be the same as seen from both sides of the shell. (i.e. match intrinsic geometries). The history of the shell itself is described by $r = R(t)$. Set the metrics equal to each other at $r = R$ and divide by dT^2 and we get the condition:

$$1-\left(\frac{dR}{dT}\right)^2=\left(\frac{R-2M}{R}\right)\left(\frac{dt}{dT}\right)^2-\left(\frac{R-2M}{R}\right)^{-1}\left(\frac{dR}{dT}\right)^2$$

We can ignore the extrinsic curvature matching because it will only determine the $R(t)$ in terms of the stress-energy of the shell and we are only trying to obtain the general lograthmic result in this example. So assume an arbitrary $R(t)$.

At this point, we have two ingoing rays, v and V which are outside and inside the shell respectively, and two outgoing rays U and u which are inside and outside the shell, respectively. The relationship between these are needed. The matchings must be made between inside and outside, thus v and V, then V and U and finally U and u.

The match of V and v is assumed to be a linear relationship. $V(v)=av+b$, where a and b are constants. For the matching of $V=T+r$ and $U=T-r$ we note that we are at the center of the collapsing shell where $r=0$ so we get the simple relationship, $U(V)=V$. The last matching considers the exit of the shell and is the most involved. We are only concerned with those rays that exit when the radius is close to the event horizon, i.e. $R\ 2M$. With T_0 being the time that $R=2M$.(this occurs at a finite time as seen by observers inside the shell). Near $T=T_0$, then $R\approx 2M+A(T_0-T)$. where A is constant. Approximate the above condition by ignoring the first term below (slow infinity approaching term),

$$(1-\left(\frac{dR}{dT}\right)^2)\left(\frac{R-2M}{R}\right)^{-1}+\left(\frac{R-2M}{R}\right)^{-2}\left(\frac{dR}{dT}\right)^2=\left(\frac{dt}{dT}\right)^2$$

$$\left(\frac{R-2M}{R}\right)^{-2}\left(\frac{dR}{dT}\right)^2\approx\left(\frac{dt}{dT}\right)^2$$

Plugging in $R\approx 2M+A(T_0-T)$,

$$\left(\frac{dt}{dT}\right)^2\approx\left(\frac{2M}{T_0-T}\right)^2$$

Take the square root, integrate and get as $T\to T_0$

$$t\approx -2M\ln\left(\frac{T_0-T}{B}\right)$$

For the r^* we also see, with $r=R(t)\approx 2M+A(T_0-T)$

$$r^*=r+2M\ln\frac{r-2M}{2M}\approx 2M\ln\left(\frac{A(T_0-T)}{2M}\right)$$

So as $u=t-r^*$ then

$$u\approx -2M\ln\frac{T_0-T}{B}-2M\ln\frac{A(T_0-T)}{2M}\approx -4M\ln\left(\frac{T_0-T}{B'}\right)$$

We still have B and B' as constants and in our limit, we also have

$$U=T-r=T-R(T)\approx(T-(2M+A(T_0-T)))\approx T(1+A)-2M-AT_0$$

Combine these results and obtain the general answer, $u=-4M\ln[(v_0-v)/C]$. Here are the details:

results:

$$U=(1+A)T-2M-AT_0$$

$$U = V$$

$$V = av + b$$

so

$$av = (1 + A)T - 2M - AT_0 - b$$

and

$$av_0 = T_0 - 2M - b$$

therefore

$$a(v_0 - v) = (1 + A)(T_0 - T)$$

So as

$$u \approx -4M \ln(\frac{T_0 - T}{B})$$

thus

$$u \approx -4M \ln(\frac{v_0 - v}{C})$$

The crucial step to notice is that the natural log comes from the last step of matching. The outgoing rays experience a huge redshift after passing through the collasping body. The interior geometry doesn't matter, what matters is the exit of the rays. The out-modes thus have the form for $v < v_0$:

$$F_\omega = i(4\pi\omega)^{-1/2} e^{-i\omega u}$$

and the modes higher than v_0 meet the singularity and never make it out, so $F_\omega = 0$ for $v > v_0$. Solving for the coefficients, using $\alpha = (f_{in}, F_{out})$ and $\beta = -(f_{in}, F^*_{out})$, I get:

$$\alpha^*_{\omega'\omega} = Z \int_{-\infty}^{v_0} dv e^{i\omega' v} e^{-i\omega u}$$

where, for notational convenience $Z = (2\pi)^{-1}(\omega'/\omega)^{1/2}$ and $u = -4M \ln[(v_0 - v)/C]$. Also,

$$\beta_{\omega'\omega} = Z \int_{-\infty}^{v_0} dv e^{-i\omega' v} e^{-i\omega u}$$

Let me make the substitution of $v' = v_0 - v$ to get, all over again:

$$\alpha^*_{\omega'\omega} = Z e^{i\omega v_0} \int_0^\infty dv' e^{-i\omega' v'} e^{-i\omega u}$$

and

$$\beta_{\omega'\omega} = Z e^{-i\omega v_0} \int_0^\infty dv' e^{i\omega' v'} e^{-i\omega u}$$

where $u = -4M \ln[v'/C]$.

Because the integrands are not analytic on the negative real axis, but analytic everywhere else, then $\int_C dv' e^{-i\omega' v'} e^{4Mi\omega \ln v'/C} = 0$ and making a change of variables $v' \to -v'$ we write

$$\int_0^\infty dv' e^{-i\omega' v'} e^{4Mi\omega \ln(v'/C)} = -\int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln(-v'/C - i\epsilon)} = -e^{4\pi M\omega} \int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln(v'/C)}$$

where in the second equality I've used Euler's formula, to see $\ln(-v'/C - i\epsilon) = -i\pi + \ln(v'/C)$. Put in compact notation, with the integrals represented now by q and Q for α^* and β respectively, we see that

$$\alpha^* = Ze^{i\omega' v_0} q$$

$$\beta = Ze^{-i\omega' v_0} Q$$

and

$$q = -e^{4\pi M\omega} Q$$

so therefore

$$|q|^2 = e^{8\pi M\omega} |Q|^2$$

and

$$|\alpha|^2 = e^{8\pi M\omega} |\beta|^2$$

using the Bogolubov property,

$$\sum_{\omega'} (|\alpha_{\omega'\omega}|^2 - |\beta_{\omega'\omega}|^2) = 1$$

you can see

$$\sum_{\omega'} (e^{8\pi M\omega} - 1) |\beta_{\omega'\omega}|^2 = 1$$

so since, $\sum |\beta|^2 = \langle N \rangle$ we have

$$\langle N \rangle = \frac{1}{e^{8\pi M\omega} - 1}$$

where again, since $\hbar = c = G = k_B = 1$ then the mean number of particles is a Planck spectrum with the argument of the exponent ω/T indicating the temperature of the particles escaping the black hole is

$$T = \frac{1}{8\pi M}$$