Cylindrical 2D Spacetime: A locally flat, non-trivial topological structure that generalizes Minkowksi space quantum field theory

Michael R.R. Good

June 27, 2007

This is the easiest example of such a generalization. Flat spacetime does not imply Minkowski space quantum field theory. The discussion of $\langle T_{\mu\nu}\rangle$ becomes possible without having to employ the full theory of curved space regularization and renormalization, and I will emphasize that $\langle T_{\mu\nu}\rangle$ is nonzero even for the vacuum. This recalls to one's mind the essence of the Casimir effect, an experimentally verified prediction.

Imagine a cylinder with cirumference L whose height is the time direction. The $R^1 \times S^1$ 2D spaetime has compactified or closed spatial sections, where the points x and $x + L$ are identified. The metric is still the 2D Minkowski space line element $ds^2 = dt^2 - dx^2$ or with null coordinates $u = t - x$ and $v = t + x$, $ds^2 = dudv$. The continuous modes of the field, $u_k = [2\omega(2\pi)^{n-1}]^{-1/2} e^{ik\cdot x - i\omega t}$ become descrete, from $u_k = (2L^{n-1}\omega)^{-1/2} e^{ik\cdot x - i\omega t}$ to

$$u_k = (2L\omega)^{-1/2} e^{i(kx-\omega t)} \tag{1}$$

where

$$k = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \pm 3, ... \tag{2}$$

The massless case, $\omega = |k|$, means that the modes with positive values of n have form

$$e^{ik(x-t)} = e^{-iku} \quad rightmoving\ waves \tag{3}$$

and those modes with negative values of n give

$$e^{ik(x+t)} = e^{ikv} \quad leftmoving\ waves \tag{4}$$

The periodic boundary conditions

$$u_k(t, x) = u_k(t, x + nL) \tag{5}$$

where imposed to get (1). If we consider imposing antiperiodic boundary conditions,

$$u_k(t, x) = (-1)^n u_k(t, x + nL) \tag{6}$$

where the modes are still given by (1) but now

$$k = \frac{2\pi(n + \frac{1}{2})}{L}, \quad n = 0, \pm 1, \pm 2, \pm 3, .. \tag{7}$$

we get what is called a 'twisted' field. Twisted fields are just as important as those fields where periodic boundary conditions apply. The reason for this is argued 'that for non-trivial topology, if the vacuum generating functional $Z(0, 0)$ is to invariant under Lorentz transformations of the vierbein one must include both twisted and untwisted spinors in the calculations'. (For twisted, the scalar field is regarded as a section through a non-product bundle.)

The interesting thing about this example spacetime is that since the field modes are forced into a discrete set, the field energy will be disturbed, just like the Casimir energy between the two plates. Using the stress tensor,

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\partial_\lambda\phi\partial_\delta\phi + \frac{1}{2}m^2\phi^2\eta_{\alpha\beta} \tag{8}$$

and setting $m = 0$, we can calculate the components,

$$T_{tt} = T_{xx} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 \tag{9}$$

$$T_{tx} = T_{xt} = \partial_t\phi\partial_x\phi \tag{10}$$

We are most interested in evaluating $\langle 0_L|T_{\mu\nu}|0_L\rangle$ where the state $|0_L\rangle$ is the vacuum associated with the discrete modes (1). Note that

$$|0_L\rangle \to |0\rangle \quad as\ L \to \infty \tag{11}$$

Using the general property that

$$\langle 0|T_{\alpha\beta}|0\rangle = \sum_k T_{\alpha\beta}[u_k, u_k^*] \tag{12}$$

where $T_{\alpha\beta}[u_k, u_k^*]$ denotes the bilinear expression (8) and using our modes (1) and vacuum $|0_L\rangle$ we get

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{1}{2L}\sum_{n=-\infty}^{\infty}|k| = \frac{2\pi}{L^2}\sum_{n=0}^{\infty}n \tag{13}$$

which is, once again, as in the Minkowski space, infinite. This is okay, its just the same ultraviolet divergence property shared by Minkowski space. Note that the compactified spatial sections can modify the long wavelength modes while the ultraviolet behaviour remains. For the Minkowski space vacuum energy the ultraviolet divergence is removed via normal ordering of creation and annihilation operators. For a general state $|\psi\rangle$, normal ordering reduces to

$$\langle\psi| : T_{\alpha\beta} : |\psi\rangle = \langle\psi|T_{\alpha\beta}|\psi\rangle - \langle 0|T_{\alpha\beta}|0\rangle \tag{14}$$

which guarantees that

$$\langle 0| : T_{\alpha\beta} : |0\rangle = 0 \tag{15}$$

So in our case, the state $|0_L\rangle$, we have

$$\langle 0_L| : T_{tt} : |0_L\rangle = \langle 0_L|T_{tt}|0_L\rangle - \langle 0|T_{tt}|0\rangle \tag{16}$$

where the last term on the right side may be expressed as $lim_{L'\to\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle$. Just as can be done to simplfy analysis in the Casimir calculation, an 'ad hoc' procedure of tacking on a cut-off factor can be justifably used. We'll use the cut off factor:

$$e^{-\alpha|k|} \tag{17}$$

This can be physically due to the absense of ultra high freqency being confined to the boundary imposed on the spacetime. The sum (13) is now finite:

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n e^{-2\pi\alpha n/L} = \frac{2\pi}{L^2}\frac{e^{2\pi\alpha/L}}{(e^{2\pi\alpha/L}-1)^2} \tag{18}$$

expand around $\alpha = 0$:

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{1}{2\pi\alpha^2} - \frac{\pi}{6L^2} + O(\alpha^3) \tag{19}$$

and obtain a similiar expression for $\langle 0_{L'}|T_{tt}|0_{L'}\rangle$ so that

$$lim_{L'\to\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle = \frac{1}{2\pi\alpha^2} \tag{20}$$

So the normal ordering result is, as $\alpha \to 0$:

$$\langle 0_L| : T_{tt} : |0_L\rangle = \langle 0_L|T_{tt}|0_L\rangle - \langle 0|T_{tt}|0\rangle = -\frac{\pi}{6L^2} \tag{21}$$

The same procedure will result in $\langle 0_L| : T_{tx} : |0_L\rangle = 0$.

So although $\langle T_{\alpha\beta}\rangle$ diverges for both states $|0\rangle$ and $|0_L\rangle$ the difference between the two is finite. Requiring normal ordering, and thus that $\langle 0| : T_{\alpha\beta} : |0\rangle = 0$ then the state $|0_L\rangle$ has in it a finite, negative energy density, as well a pressure

$$p = \langle 0_L| : T_{xx} : |0_L\rangle = \rho = \langle 0_L| : T_{tt} : |0_L\rangle = -\frac{\pi}{6L^2} \tag{22}$$

For the total energy, we see that throughout the $R^1 \times S^1$ universe, the cloud of negative vacuum energy is uniform and equal to

$$-\frac{\pi}{6L} \tag{23}$$

For twisted fields, a similiar procedure results in $-1/2$ for that of the untwisted fields:$\rho_{twisted} = \frac{\pi}{12L^2}$. For a massless spin 1/2 field the results for twisted and untwisted are -4 times the respective scalar field result.(Birrell and Davies).