An introduction to QED in lowest order: lepton pair production

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In any collision process in QED, to lowest non-vanishing order of perturbation theory, the cross section is derived as an application of the theory. Here I provide the calculation for the most simple QED process, lepton pair production. As an example of calculating processes to lowest non-vanishing order of perturbatin theory, lets consider when an electron-positron pair produces either a pair of muons or tauons. This is called lepton pair production. It is the simplest of all QED processes, but one of the most important. This process is used to calibrate e^+e^- colliders and is also similiar to the process that creates a quark-antiquark pair. It's highly relativistic cross section is arrogantly quoted as

$$\boxed{\sigma = \tfrac{\pi\alpha^2}{3E^2}}$$

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Allow me to get ready for the calculation by giving an overview first.

Preliminaries and Setup: An outline of the calculation is as follows:

1. Draw the Feynman diagram.
2. Use Feynman rules to find the amplitude $\mathcal{M}$.
3. Perform spin sums and find $|\mathcal{M}|^2$. Evaluate $|\mathcal{M}|^2$ using energy projection operators and trace properties.
4. Draw a kinematic diagram. Solve for all the 4-momentum vectors and plug them into $|\mathcal{M}|^2$.
5. Substitue the final form of $|\mathcal{M}|^2$ into the cross section formula.

I will state the general cross section formula in the center of mass frame. I will also assume the Feynman rules for deriving the amplitude. The center of mass differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(E_1+E_2)^2}\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\left(\prod_l (2m_l)\right)|\mathcal{M}|^2 \tag{1}$$

Allow me to state the energy projection operators for future reference. These are invaluable for the calculation.

$$\Lambda^+_{\alpha\beta}(\mathbf{p}) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} = \sum_{r=1}^{2} u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) \tag{2}$$

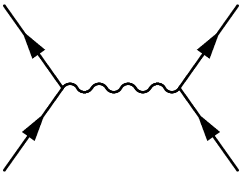
$$\Lambda^-_{\alpha\beta}(\mathbf{p}) = -\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} = -\sum_{r=1}^{2} v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p}) \tag{3}$$

The calculation. The process $e^+e^- \to l^+l^-$ may be written more precisely as

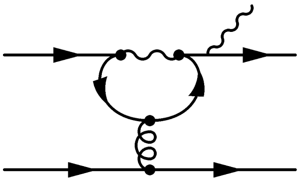
$$e^+(\mathbf{p}_1, r_1) + e^-(\mathbf{p}_2, r_2) \to l^+(\mathbf{p}_1', s_1) + l^-(\mathbf{p}_2', s_2) \tag{4}$$

Where the r is the initial spin index and s is the final spin index. Here $l = \mu$ or τ. Let's proceed with step 1. Draw the Feynman graph:

The Feynman diagram for lepton pair production is:



It can be hard drawing the right diagrams, and you may be tempted to draw



This is not right. Also, you may have trouble getting labels on your diagrams. You are not alone. On to step 2.

Let's construct the Feynman amplitude from the Feynman graph. Allow me to define $q = p_1 + p_2$. It goes like: electron, $u(p_2)$, vertex, $ie\gamma^\alpha$, positron, $\bar{v}(p_1)$, virtual photon, $iD_{F\alpha\beta}$, vertex, $ie\gamma^\beta$, anti-muon, $v(p_1')$, and muon, $\bar{u}(p_2')$. You can write down the Feynman amplitude from any Feynman graph by just following the Feynman rules. In practice, amplitudes are not derived from basic principles, although they can be.

$$\mathcal{M}(r_1, r_2, s_1, s_2) = -ie^2\bar{u}(p_2')\gamma^\alpha v(p_1')D_{F\alpha\beta}(q)\bar{v}(p_1)\gamma^\beta u(p_2) \tag{5}$$

I have suppressed the spin indices r and s. The Feynman photon propagator is

$$D_{F\alpha\beta}(k) = \frac{-g_{\alpha\beta}}{k^2 + i\epsilon} \tag{6}$$

Here the $i\epsilon$ term is only important at the pole of the propagator, but since $k = p_1 + p_2 = q > m_e + m_e = 2m_e$ I know that $k^2 > 4m_e^2$ will never vanish, so I may negate the $i\epsilon$ term. This leaves me with

$$\mathcal{M}(r_1, r_2, s_1, s_2) = \frac{ie^2}{q^2}[\bar{u}(p_2')\gamma_\alpha v(p_1')][\bar{v}(p_1)\gamma^\alpha u(p_2)] \tag{7}$$

With the spin index explicit, this is:

$$\mathcal{M}(r_1, r_2, s_1, s_2) = \frac{ie^2}{q^2}[\bar{u}_{s_2}(p_2')\gamma_\alpha v_{s_1}(p_1')][\bar{v}_{r_1}(p_1)\gamma^\alpha u_{r_2}(p_2)] \tag{8}$$

This is pretty ugly, so I won't include spin indicies. As we have successfully found the Feynman amplitude, lets move on to spin sums, or step 3. The idea here is that in this experiment, the initial electrons will be unpolarized and no one will care to detect the polarization of the final lepton pair that is produced. To find the unpolarized cross section we must perform spin sums. You average over the initial spin states and you sum over the final spin states.

$$\begin{cases} \frac{1}{2}\sum_r & \textit{initial spins} \\ \sum_s & \textit{final spins} \end{cases} \tag{9}$$

This then, brings in a $\frac{1}{4}$

$$X = \frac{1}{4}\sum_{r_1,r_2,s_1,s_2} |\mathcal{M}(r_1,r_2,s_1,s_2)|^2 \tag{10}$$

This X is just what I'm going to stick into the cross section formula for $|\mathcal{M}|^2$. Taking the complex conjugate of

$$\mathcal{M} = \frac{ie^2}{q^2}[\bar{u}(p_2')\gamma_\alpha v(p_1')][\bar{v}(p_1)\gamma^\alpha u(p_2)] \tag{11}$$

yields

$$\mathcal{M}^* = -\frac{ie^2}{q^2}[\bar{v}(p_1')\gamma_\beta u(p_2')][\bar{u}(p_2)\gamma^\beta v(p_1)] \tag{12}$$

This is a super shortcut due to the bar notation, $\bar{u} = u^\dagger\gamma^0$, because:

$$\begin{aligned} (\bar{v}\gamma^\alpha u)^* &= (v^\dagger\gamma^0\gamma^\alpha u)^* \\ &= (\gamma^\alpha u)^*(v^\dagger\gamma^0)^* \\ &= u^\dagger\gamma^{\alpha\dagger}\gamma^{0\dagger}v \\ &= u^\dagger\gamma^0\gamma^\alpha\gamma^0\gamma^{0\dagger}v \\ &= \bar{u}\gamma^\alpha v \end{aligned}$$

where I used the hermiticity condition $\gamma^{\alpha\dagger} = \gamma^0\gamma^\alpha\gamma^0$ in the second to last step. So, we may remember $(\bar{v}\gamma^\alpha u)^* = \bar{u}\gamma^\alpha v$. I now have

$$X = \frac{e^4}{4q^4}A_{\alpha\beta}B^{\alpha\beta} \tag{13}$$

where

$$A_{\alpha\beta} = \sum_{s_1,s_2}[\bar{u}(p_2')\gamma_\alpha v(p_1')\bar{v}(p_1')\gamma_\beta u(p_2')] \tag{14}$$

and

$$B^{\alpha\beta} = \sum_{r_1,r_2}\left[\bar{v}(p_1)\gamma^\alpha u(p_2)\bar{u}(p_2)\gamma^\beta v(p_1)\right] \tag{15}$$

Applying the energy projection operators, equations (2) and (3) to $A_{\alpha\beta}$ and $B_{\alpha\beta}$ will allow simplification in terms of traces. Assuming spinor indices allows you to freely move the u next to the $\bar{u}$ for $A_{\alpha\beta}$ and allows you to freely move the v next to the $\bar{v}$ for $B^{\alpha\beta}$. Here is what $A_{\alpha\beta}$ looks like with spinor indices:

$$A_{\alpha\beta} = \sum_{s_1,s_2}\bar{u}_a(p_2')\gamma_{\alpha ab}v_b(p_1')\bar{v}_d(p_1')\gamma_{\beta dc}u_c(p_2') \tag{16}$$

This yields

$$A_{\alpha\beta} = Tr\left[\frac{\not{p}_2' + m_l}{2m_l}\gamma_\alpha\frac{\not{p}_1' - m_l}{2m_l}\gamma_\beta\right] \tag{17}$$

and

$$B^{\alpha\beta} = Tr\left[\frac{\not{p}_1 - m_e}{2m_e}\gamma^\alpha \frac{\not{p}_2 + m_e}{2m_e}\gamma^\beta\right] \tag{18}$$

Using some trace technology, we know that the trace of an odd number of γ matrices vanishes. Applying this knowledge lends:

$$A_{\alpha\beta} = \frac{1}{4m_l^2}[Tr(\not{p}'_2\gamma_\alpha\not{p}'_1\gamma_\beta) - m_l^2 Tr(\gamma_\alpha\gamma_\beta)] \tag{19}$$

and

$$B^{\alpha\beta} = \frac{1}{4m_e^2}[Tr(\not{p}_1\gamma^\alpha\not{p}_2\gamma^\beta) - m_e^2 Tr(\gamma^\alpha\gamma^\beta)] \tag{20}$$

Applying the trace properties: $Tr(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$ and $Tr(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$ we get:

$$A_{\alpha\beta} = \frac{1}{m_l^2}[p'_{1\alpha}p'_{2\beta} + p'_{2\alpha}p'_{1\beta} - (m_l^2 + p'_1p'_2)g_{\alpha\beta}]. \tag{21}$$

and

$$B^{\alpha\beta} = \frac{1}{m_e^2}[p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - (m_e^2 + p_1p_2)g^{\alpha\beta}]. \tag{22}$$

Plugging these values into $X = \frac{e^4}{4q^4}A_{\alpha\beta}B^{\alpha\beta}$ yields:

$$\frac{e^4}{4q^4}A_{\alpha\beta}B^{\alpha\beta} = \frac{e^4}{4q^4}\frac{1}{m_e^2m_l^2}[p'_{1\alpha}p'_{2\beta} + p'_{2\alpha}p'_{1\beta} - (m_l^2 + p'_1p'_2)g_{\alpha\beta}][p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - (m_e^2 + p_1p_2)g^{\alpha\beta}] \tag{23}$$

This is :

$$\begin{aligned} X = \frac{e^4}{4q^4m_e^2m_l^2}&[(p'_1p_1)(p'_2p_2) + (p'_1p_2)(p'_2p_1) - p'_1p'_2(m_e^2 + p_1p_2) \\ &+(p'_2p_1)(p'_1p_2) + (p'_2p_2)(p'_1p_1) - p'_2p'_1(m_e^2 + p_1p_2) \\ &-(m_l^2 + p'_1p'_2)(p_1p_2) - (m_l^2 + p'_1p'_2)(p_2p_1) + 4(m_l^2 + p'_1p'_2)(m_e^2 + p_1p_2)] \end{aligned}$$

$$X = \frac{e^4}{4q^4m_l^2m_e^2}[2(p'_1p_1)(p'_2p_2) + 2(p'_1p_2)(p'_2p_1) - 2m_e^2(p'_1p'_2) - 2m_l^2(p_1p_2) + 4m_l^2m_e^2 + 4m_l^2p_lp_2 + 4m_e^2p'_1p'_2] \tag{24}$$

$$X = \frac{e^4}{2q^4m_l^2m_e^2}[(p_1p'_1)(p_2p'_2) + (p_2p'_1)(p_1p'_2) + 2m_l^2m_e^2 + m_e^2(p'_1p'_2) + m_l^2(p_1p_2)] \tag{25}$$

Let's move on to step 4. Now we examine the kinematics. This result is for an arbituary reference frame. Let us move to the CoM frame and find out what to use for all the four momentums. Drawing the diagram and solving for the factors we get:

$$p_1p'_1 = p_2p'_2 = E^2 - pp'\cos\theta \tag{26}$$

$$p_1p'_2 = p_2p'_1 = E^2 + pp'\cos\theta \tag{27}$$

$$p_1p_2 = E^2 + p^2 \tag{28}$$

$$p'_1p'_2 = E^2 + p'^2 \tag{29}$$

$$q^2 = (p_1 + p_2)^2 = 4E^2 \tag{30}$$

Here I have taken $p \equiv |\mathbf{p}|$ and $p' \equiv |\mathbf{p}'|$. I will ignore terms proportional to m_e^2 in X and take $p = E$ because $E \geq m_\mu \approx 207 m_e$. Plugging these kinematic relations into X and making these approximations yields:

$$X = \frac{e^4}{2m_l^2 m_e^2 (16E^4)}[(E^2 - pp'\cos\theta)^2 + (E^2 + pp'\cos\theta)^2 + 2m_l^2 E^2] \tag{31}$$

This is

$$X = \frac{e^4}{16 m_l^2 m_e^2 E^2}[E^2 + m_l^2 + p'^2 \cos^2\theta] \tag{32}$$

Using the cross section formula, equation (1) and $\alpha = e^2/4\pi$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 p'}{16E^5}(E^2 + m_l^2 + p'^2\cos^2\theta) \tag{33}$$

for the total cross section:

$$\sigma = \frac{\alpha^2 p'}{16E^5}\int_0^{2\pi}\int_{-1}^{1}(E^2 + m_l^2 + p'^2\cos^2\theta)d(\cos\theta)d\phi \tag{34}$$

This is

$$\sigma = \frac{\pi\alpha^2 p'}{4E^5}[E^2 + m_l^2 + \frac{p'^2}{3}] \tag{35}$$

In the relativistic limit, $E >> m_l$, we have $E \approx p'$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E}{16E^5}(E^2 + E^2\cos^2\theta) \tag{36}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta) \tag{37}$$

and for the highly relativistic total cross section we have, (using equation (35) so you don't have to do any integration),

$$\sigma_{tot} = \frac{\pi\alpha^2 E}{4E^5}[E^2 + \frac{E^2}{3}] \tag{38}$$

$$\boxed{\sigma_{tot} = \tfrac{\pi\alpha^2}{3E^2}} \tag{39}$$