Maxwell's Equations in one unified expression

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May 21, 2006

Maxwell's equations may be written as

$$\Box F = J$$

This reveals in a simple way, the physics of electrodynamics. The source of the electromagnetic field is simply the charged current density. Here F is a bivector field and J is a vector. We are working in the Dirac algebra, that is, the algebra where γ_0,γ_1,γ_2,γ_3 are orthonormal vectors that obey the familiar multiplication rules of the Dirac matrices, it is often called the space-time algebra.

Let's take this expression and obtain the four familiar Maxwell equations. Note that $\Box F = J$ is not just a trick of notation, but a consequence of the utility of geometric algebra.

This can be done by concerning ourselves with only the space algebra, or the Pauli algebra. So the idea is to single out a partiuclar time-like direction, γ_0. A vector in the Dirac algebra can be represented by a vector in the Pauli algebra by:

$$p\gamma_0 = p_0 + \vec{p}$$

where

$$p_0 \equiv p \cdot \gamma_0 \quad \vec{p} = p \wedge \gamma_0$$

Now we have:

$$F = \vec{E} + i\vec{B}$$

with

$$\vec{E} = \frac{1}{2}(F - F^*) \qquad i\vec{B} = \frac{1}{2}(F + F^*)$$

Here $F^* = \gamma_0 F \gamma_0$. Now since $\gamma_0\gamma_0 = 1$ we may write J in the form:

$$J = (J\gamma_0)\gamma_0 = (J \cdot \gamma_0 + J \wedge \gamma_0)\gamma_0 \equiv (\rho + \vec{J})\gamma_0$$

where I used the fundamental multiplication rule of geometric algebra, $AB = A \cdot B + A \wedge B$. If I single out a particular time-like direction, I will multiply $\Box F = J$ by γ_0.

$$\gamma_0 \Box F = \gamma_0 J$$

Using Clifford's rule, or the fundamental law of multiplication,

$$(\gamma_0 \cdot \Box + \gamma_0 \wedge \Box)(\vec{E} + i\vec{B}) = \gamma_0(\rho + \vec{J})\gamma_0$$

This is

$$(\partial_0 + \vec{\nabla})(\vec{E} + i\vec{B}) = \gamma_0\rho\gamma_0 + \gamma_0\vec{J}\gamma_0$$

Here I used $\vec{\nabla} = \gamma_0 \wedge \Box$, and $\partial_0 = \gamma_0 \cdot \Box$ thus $\gamma_0\Box = \partial_0 + \vec{\nabla}$. Notice that because ρ is a scalar we have $\gamma_0\rho\gamma_0 = \rho$ and since $\vec{J}^* = \gamma_0\vec{J}\gamma_0 = -\vec{J}$ as it is just an example of spacial inversion. We have now

$$(\partial_0 + \vec{\nabla})(\vec{E} + i\vec{B}) = \rho - \vec{J}$$

$$\partial_0\vec{E} + \vec{\nabla}\vec{E} + i(\partial_0\vec{B} + \vec{\nabla}\vec{B}) = \rho - \vec{J}$$

Using Clifford's rule,

$$\partial_0\vec{E} + \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \wedge \vec{E} + i(\partial_0\vec{B} + \vec{\nabla} \cdot \vec{B} + \vec{\nabla} \wedge \vec{B}) = \rho - \vec{J}$$

Using the familiar cross product, where $\vec{a} \times \vec{b} \equiv -i\vec{a} \wedge \vec{b}$, our expression reads

$$\partial_0\vec{E} + \vec{\nabla} \cdot \vec{E} + i\vec{\nabla} \times \vec{E} + i(\partial_0\vec{B} + \vec{\nabla} \cdot \vec{B} + i\vec{\nabla} \times \vec{B}) = \rho - \vec{J}$$

This is

$$\vec{\nabla} \cdot \vec{E} + \partial_0\vec{E} - \vec{\nabla} \times \vec{B} + i\vec{\nabla} \times \vec{E} + i\partial_0\vec{B} + i\vec{\nabla} \cdot \vec{B} = \rho - \vec{J}$$

To grasp the four original equations of Maxwell, we simply match scalar, vector, bivector and pseudoscalar parts in the Pauli algebra. The scalar parts are the first terms on the left and right sides,

$$\vec{\nabla} \cdot \vec{E} = \rho$$

The vector parts are

$$\partial_0\vec{E} - \vec{\nabla} \times \vec{B} = -\vec{J}$$

The bivector parts are

$$i\vec{\nabla} \times \vec{E} + i\partial_0\vec{B} = 0$$

The pseudoscalar part is

$$i\vec{\nabla} \cdot \vec{B} = 0$$

These are Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J}$$