Creation of Particles via Cosmic Expansion

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This example shows that due to an expanding spacetime, particles are created. I will follow Birrell and Davies (1982), Bernard and Duncan (1977) and use some mathematical help from Abramowitz and Stegun (1972).

1 Cosmological Particle Creation

Let's see how particles are created using Minkowskian in and out regions. The Robertson-Walker spacetime element for 2D is

$$ds^2 = dt^2 - a^2(t)dx^2 \tag{1}$$

This allows for spatial expansion via the scalar function $a(t)$. Let's simplify this into a new form. Feel free to use a new time coordinate, η, (called conformal time because our new element will be conformal to the orginal spacetime). The conformal time will be defined by $d\eta = dt/a$ and thus $t = \int^t dt' = \int^\eta a(\eta')d\eta'$. So our simplified line element is

$$ds^2 = C(\eta)(d\eta^2 - dx^2) \tag{2}$$

where $C(\eta) = a^2(\eta)$. Allow

$$C(\eta) \equiv A + B\tanh(\rho\eta) \tag{3}$$

where A,B,ρ are constants so that in the past and future we obtain Minkowskian spacetimes

$$C(\eta) \to A \pm B \quad \eta \to \pm\infty \tag{4}$$

as $\tanh(\pm\infty) = \pm 1$. Let's look at the formation of massive, minimally coupled ($\xi = 0$) scalar particles. It's interesting to note that in 2D minimal coupling and conformal coupling are the same things. (conformal coupling being definied as $\xi = \frac{n-2}{4(n-1)}$).

As $C(\eta)$ is not a function of the spatial coordinate x, spatial translation invariance is a symmetry, therefore we can separate the variables in the modes:

$$u_k(\eta, x) = (2\pi)^{-1/2}e^{ikx}\chi_k(\eta) \tag{5}$$

Recall the field in terms of the modes

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] \tag{6}$$

Using the metric (2), place (5) into the wave equation

$$[\Box + m^2]\phi(x) = 0 \tag{7}$$

and solve, (I suggest using the 'test' state, $|0\rangle$) and you'll get the condition:

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)e^{-ikx}\chi_i(\eta) = 0 \tag{8}$$

which resolves to an ordinary differential equation for $\chi_k(\eta)$:

$$d^2_\eta\chi_k(\eta) + (k^2 + C(\eta)m^2)\chi_k(\eta) = 0 \tag{9}$$

This can be solved using hypergeometric functions. The normalized modes, that correspond to the positive frequency Minkowski space modes in the far past are:

$$u_k^{in} = (4\pi\omega_{in})^{-1/2}e^{D(x,\eta)}{}_2F_1(a,b,c,z) \to (4\pi\omega_{in})^{-1/2}e^{ikx-i\omega_{in}\eta} \tag{10}$$

and in the far future

$$u_k^{out} = (4\pi\omega_{out})^{-1/2}e^{D(x,\eta)}{}_2F_1(a,b,c',1-z) \to (4\pi\omega_{out})^{-1/2}e^{ikx-i\omega_{out}\eta} \tag{11}$$

where I've defined several symbols to make notation easier:

$$\omega_{in}^2 = k^2 + m^2(A-B)$$

$$\omega_{out}^2 = k^2 + m^2(A+B)$$

$$\omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in})$$

$$D(x,\eta) \equiv ikx - i\omega_+\eta - (i\omega_-/\rho)\ln[2\cosh(\rho\eta)]$$

$$a \equiv 1 + i\omega_-/\rho$$
$$b \equiv i\omega_-/\rho$$
$$c \equiv 1 - i\omega_{in}/\rho$$
$$c' \equiv 1 + i\omega_{out}/\rho$$
$$z \equiv \frac{1}{2}(1+\tanh\rho\eta)$$
$$1-z = \frac{1}{2}(1-\tanh\rho\eta)$$

.

At this point we can see the creation of particles must occur. Since we see that u_k^{in} and u_k^{out} are not equal, the β coefficient cannot be zero in

$$u_j^{in} = \sum_i(\alpha_{ji}u_i^{out} + \beta_{ji}u_i^{out*}) \tag{12}$$

But let us be more explicit, and use the properties $\alpha_{kk'} = \alpha_k\delta_{kk'}$, $\beta_{kk'} = \beta_k\delta_{-kk'}$ so that we can write

$$u_k^{in} = \alpha_k u_k^{out} + \beta_k u_{-k}^{out*} \tag{13}$$

If we use the linear tranformation properties of hypergeometric functions,

$$
{}_2F_1(a,b,c,z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b,a+b+1-c,1-z)
$$

$$
+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b,1+c-a-b,1-z)
$$

and

$$
{}_2F_1(a,b,c,z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c,z) \tag{14}
$$

then we can write the in-modes in terms of the out-modes as,

$$
u_k^{in}(\eta,x)=\alpha_k u_k^{out}(\eta,x)+\beta_k u_{-k}^{out*}(\eta,x) \tag{15}
$$

where

$$
\alpha_k=\left(\frac{\omega_{out}}{\omega_{in}}\right)^{\frac{1}{2}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{16}
$$

$$
\beta_k=\left(\frac{\omega_{out}}{\omega_{in}}\right)^{\frac{1}{2}}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{17}
$$

using our defined abc's, etc. Let us square $|\beta_k|^2$, using $|\Gamma(ix)|^2=\pi/x\sinh(\pi x)$, and $\Gamma(1+z)=z\Gamma(z)$, so that

$$
|\beta_k|^2=\frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{in}/\rho)\sinh(\pi\omega_{out}/\rho)} \tag{18}
$$

If the quantum field resides in the vacuum state, $|0,in\rangle$ defined by $a_i|0\rangle=0$, then in the remote past with a Minkowskian spacetime, inertial observers(detectors) will observe(register) no particles, i.e. an empty physical vacuum state. Now, working in the Heisenberg picture, we see that in the far future, the spacetime is also Minkowskian and the quantum field resides in the same state, $|0,in\rangle$. But here the role of the physical vacuum is assumed not by $|0,in\rangle$ but by $|0,out\rangle$. Therefore, particle detectors will register the presence of quanta in the out-region. In the mode k, the expected number of particles are given by (18). The creation of particles thus occurs as a consequence of cosmic expansion.