

Units, Renormalization, Vacuum Divergence and the Harmonic Potential

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Dimensional Analysis Problem

Consider a d -dimensional scalar field theory

$$S = \int d^d x \left(\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \lambda\phi^4 + \dots + \lambda_n \phi^n + \dots \right)$$

Show that $Dim(\phi) = (d - 2)/2$ and $Dim(\lambda_n) = n(2 - d)/2 + d$. Note that ϕ is dimensionless for $d = 2$.

Solution:

Time and space have inverse dimensions as mass, energy and momentum. Then all the pieces in the integrand have to have dimension d . So the Dim of the Lagrangian density has to be d , and in this notation $Dim(x) = -1$, $Dim(\partial) = 1$.

So to find $Dim(\phi)$ we add up the pieces in the first term, that is for $(\partial\phi)^2$ we have the pieces $2 * Dim(\partial) + 2 * Dim(\phi) = d$, so we've got

$$Dim(\phi) = (d - 2)/2$$

For $Dim(\lambda_n)$ we need to look at the pieces $Dim(\lambda_n) + nDim(\phi) = d$, so we've got $Dim(\lambda_n) = d - n(d - 2)/2$. That is

$$Dim(\lambda_n) = n(2 - d)/2 + d$$

Coupling Constant Related to Cutoff Scale Problem

Computing the actual scattering amplitude, experimentalists do not want an equation with the cutoff scale, Λ in it. If you want to change the threshold of our ignorance, ie, Λ , then you have to shift the coupling constant, λ in a way that does not change the scattering amplitude M .

Say you want to change the cut off scale from Λ to $e^\epsilon \Lambda$, effectively changing the threshold of ignorance, and the energy scale. Show that for M to maintain invariance to order $O(\lambda^3)$, then λ must change by $\delta\lambda = 6\epsilon C\lambda^2 + O(\lambda^3)$, that is

$$\Lambda \frac{d\lambda}{d\Lambda} = 6C\lambda^2 + O(\lambda^3)$$

Solution:

As we wish for $\Lambda \rightarrow e^\epsilon \Lambda$, it is best to note that M , the scattering amplitude, for meson-meson scattering is:

$$M = -i\lambda + iC\lambda^2[\log(\Lambda^2/s) + \log(\Lambda^2/t) + \log(\Lambda^2/u)] + O(\lambda^3)$$

and can be written as a more clear function of Λ as

$$M = -i\lambda + iC\lambda^2 2(3\log(\Lambda)) - o.t. + O(\lambda^3)$$

Where the *o.t.* are the negative log functions. As

$$\log \Lambda \rightarrow \log \Lambda + \delta(\log \Lambda)$$

must go to

$$\log \Lambda \rightarrow \log(e^\epsilon \Lambda) = \log \Lambda + \epsilon$$

noting we have $\delta(\log \Lambda) = \epsilon$, we have for M :

$$M = -i\lambda + iC\lambda^2 6\log(\Lambda) + iC\lambda^2 6\epsilon - o.t. + O(\lambda^3)$$

For M not to change, λ must change accordingly so that

$$\lambda \rightarrow \lambda + \delta\lambda$$

$$-i\delta\lambda = -iC\lambda^2 6\epsilon + O(\lambda^3)$$

this means that

$$\delta\lambda = 6\epsilon C\lambda^2 + O(\lambda^3)$$

or, as remembering from above that $\delta(\log \Lambda) = \epsilon$ we have

$$\delta\lambda = 6\delta(\log \Lambda)C\lambda^2 + O(\lambda^3)$$

$$\delta\lambda = 6C\lambda^2\delta\Lambda/\Lambda + O(\lambda^3)$$

$$\Lambda \frac{d\lambda}{d\Lambda} = 6C\lambda^2 + O(\lambda^3)$$

A fun example of renormalization:

The meson-meson scattering amplitude is

$$M = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(K - k)^2 - m^2 + i\epsilon}$$

with $K \equiv k_1 + k_2$ and diverges logarithmically. This gives us infinity! For large values of k this infinity really blows up and is given the name ‘ultraviolet divergence’. Evaluating this integral gives

$$M = -i\lambda + iC\lambda^2 \left(\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right) \right) + O(\lambda^3)$$

where $s \equiv K^2 = (k_1 + k_2)^2$, $t \equiv (k_1 - k_3)^2$, and $u \equiv (k_1 - k_4)^2$.

Renormalize this amplitude now that its been regularized and it depends on cutoff Λ instead of a divergent k .

Solution:

Here, λ is actually a function of Λ . That is, the coupling constant depends on the cutoff quantity. The measured coupling constant, λ_P is related to the theoretical λ by

$$-i\lambda_P = -i\lambda + iC\lambda^2 \left(\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right) \right) + O(\lambda^3)$$

Rewriting M and $-i\lambda_P$ by grouping the sum of logs into L and L_0 respectively, we get a simpler way to see:

$$M = i\lambda + iC\lambda^2 L + O(\lambda^3)$$

$$-i\lambda_P = -i\lambda + iC\lambda^2 L_0 + O(\lambda^3)$$

Now we need to put λ in terms of λ_P , something that the experimentalist can use. Solving for λ ,

$$-i\lambda = -i\lambda_P - iC\lambda^2 L_0 + O(\lambda^3)$$

To the order of approximation given, this is (with $\lambda = \lambda_P$ for the second term):

$$-i\lambda = -i\lambda_P - iC\lambda_P^2 L_0 + O(\lambda_P^3)$$

Plugging this into M yields an amplitude that an experimentalist may start to be happy with... where

$$M = -i\lambda + iC\lambda^2 L + O(\lambda^3)$$

becomes

$$M = -i\lambda_P - iC\lambda_P^2 L_0 + iC\lambda_P^2 L + O(\lambda_P^3)$$

This is the great thing about renormalization, as you can see, $L - L_0$ destroys the Λ .

$$L - L_0 = [\log(s_0/s) + \log(t_0/t) + \log(u_0/u)]$$

So we have, a renormalized amplitude, completely in terms of the physically measured coupling constant λ_P with no cutoff Λ ,

$$M = -i\lambda_P + iC\lambda_P^2 [\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)] + O(\lambda_P^3)$$

Vacuum energy expectation value divergence

Take $\langle 0|H|0\rangle$ where H is the standard quantized Hamiltonian of the Minkowski scalar field:

$$H = \frac{1}{2} \sum_k (a_k^\dagger a_k + a_k a_k^\dagger) \omega = \sum_k (a_k^\dagger a_k + \frac{1}{2}) \omega$$

Solution:

$$\langle 0|H|0\rangle = \langle 0|0\rangle \sum_k \frac{1}{2} \omega = \sum_k \frac{1}{2} \omega$$

Replacing \sum_k with a continuous integral, that is going from box discrete normalization to a continuum:

$$\sum_k = \left(\frac{L}{2\pi}\right)^{n-1} \int d^{n-1}k$$

we get

$$\sum_k \frac{1}{2} \omega = \frac{1}{2} \left(\frac{L}{2\pi}\right)^{n-1} \int \omega d^{n-1}k$$

This is

$$\sum_k \frac{1}{2}\omega = \left(\frac{L^2}{4\pi}\right)^{\frac{n-1}{2}} \frac{1}{\Gamma(\frac{n-1}{2})} \int_0^\infty (k^2 + m^2)^{\frac{1}{2}} k^{n-2} dk$$

When n is continued away from integral values, $-L^{n-1}2^{-n-1}\pi^{-n/2}m^n\Gamma(-n/2)$ is obtained. Poles are found at all even integral values of $n \geq 0$. So, by temporarily making divergent quantities finite by continuing the dimension of spacetime away from integer values we learn an important taste of the basis of dimensional regularization procedure. Check Davies and Birrell for more info.

A more simplistic way, may be to note, that $k^2 + m^2 = \omega^2$ where $\hbar = 1$ and put this system in a box of volume L^3 and let the box cover all of infinity. As we impose periodic boundary conditions, forcing the wavelength to be $\lambda_i = L/n_i$ for some integer n_i , then since $k_i = 2\pi/\lambda_i$ there are $\frac{L}{2\pi}dk_i$ discrete values of k_i in the range from k_i to $k_i + dk_i$. So we get

$$\sum_k \frac{1}{2}\omega = \frac{1}{2}\left(\frac{L}{2\pi}\right)^3 \int \omega_k d^3k$$

This is

$$\frac{1}{2}\left(\frac{L}{2\pi}\right)^3 \int (k^2 + m^2)^{\frac{1}{2}} k^2 4\pi dk$$

You can see now that this certainly gives infinity as we have no cutoff value of k . The rhetoric is called ‘ultraviolet divergence’ and it makes sense that our theory is some low energy approximation, where when we have high energies, we have new particles, new forces, and new physics come into play. Useful: Mandl and Shaw page 13.

Any oscillatory motion is approximately simple harmonic:

The potential energy of a simple harmonic oscillator is just the parabola:

$$V(x) = \frac{1}{2}kx^2$$

Show that any oscillatory motion in the universe, (for that matter basically any potential energy) is approximately simple harmonic.

Solution:

Expanding some potential energy $V(x)$ in a Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

around the minimum, $a = x_0$ gives:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \dots$$

Drop $V(x_0)$, as adding a constant to $V(x)$ doesn't change the force, and only differences in energy are what matters. Also, note that $V'(x_0) = 0$ as x_0 is a minimum. Also, we will neglect the higher order terms (as long as $x - x_0$ remains small). So we have

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2$$

where every potential energy in the universe is essentially approximated by simple harmonic motion.