Collapse of a Ball of Dust

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When looking at the interior of a collapsing star and the world line that its surface follows in Schwarzschild geometry, complicated mathematics arise. The simplest case is to treat the star as one with uniform density and zero pressure. This was done in the classic 'ball of dust' paper by Oppenheimer-Snyder in 1939. As treated in 'Gravitation', I will follow the approach of Beckedorff-Misner.

Particles on the surface of any ball of dust must move along radial geodesics in the exterior Schwarzschild geometry because there are no pressure gradients to deflect their motion. Assume the ball of dust begins at rest with a radius, $R = R_i$ at time $t = 0$. The geodesic motion of its surface is then:

$$R = \frac{R_i}{2}(1 + \cos\eta)$$

$$t = 2Mln\left|\frac{\sqrt{(\frac{R_i}{2M}-1)} + tan(\frac{\eta}{2})}{\sqrt{(\frac{R_i}{2M}-1)} - tan(\frac{\eta}{2})}\right| + 2M\sqrt{(\frac{R_i}{2M}-1)}[\eta + \frac{R_i}{4M}(\eta + \sin\eta)]$$

where R is the Schwarzschild radial coordinate at Schwarzschild time t. The surface area of the star is $4\pi R^2$. The proper time read by a clock on the surface of the collapsing star is given by

$$\tau = (\frac{R_i^3}{8M})^{1/2}(\eta + \sin\eta)$$

The start of collapse occurs when $\eta = 0$, that is, when $R = R_i$ and $t = \tau = 0$. The end of the collapse occurs when $\eta = \pi$, that is when $R = 0$, the singularity. The lapse of proper time as measured by the test particle falling with the dust is $\Delta\tau = \pi(\frac{R_i^3}{8M})^{1/2}$.

Turn your attention now to the interior of the ball of dust. The simplest of interiors is that which is homogeneous and isotropic everywhere except at the surface. We will then be looking at an interior locally identical to a dust-filled Friedmann cosmological model. We will only look at the closed $k = +1$ case as the star is at rest initially. (the intial rate of change of density equals zero, 'moment of maximum expansion'). With comoving hyperspherical coordinates, ξ, θ, ϕ for the interior, along with the origin of coordinates at the center of the star, the line element in the interior is in the Friedmann form:

$$ds^2 = -d\tau^2 + a^2(\tau)[d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta d\phi^2)]$$

$a(\tau)$ is given by the cycloidal relation,

$$a = \frac{1}{2}a_m(1 + \cos\eta)$$

$$\tau = \frac{1}{2}a_m(\eta + \sin\eta)$$

The density is given by

$$\rho = \frac{3a_m}{8\pi}a^{-3} = \frac{3}{8\pi a_m^2}[\frac{1}{2}(1+\cos\eta)]^{-3}$$

Homogeneity and isotropy are broken at the star's surface, which lies at some radius $\xi = \xi_0$ for all τ as measured in terms of the hyperspherical polar angle ξ, a comoving coordinate. At that surface, the interior must match the exterior. That is, the interior Friedmann geometry must match smoothly onto the exterior Schwarzschild geometry. This match is possible with zero pressure. Let us breifly look at a verification of this by examining the predications of the geometry for the stars circumference as a function of proper time.

The external Schwarzschild case:

$$C = 2\pi R = 2\pi\frac{R_i}{2}(1+\cos\eta)$$

$$\tau = (\frac{R_i^3}{8M})^{1/2}(\eta + \sin\eta)$$

The interior Friedmann case:

$$C = 2\pi R = 2\pi a\sin\xi_0 = 2\pi(\frac{1}{2}a_m\sin\xi_0)(1+\cos\eta)$$

$$\tau = \frac{1}{2}a_m(\eta + \sin\eta)$$

The two geometries agree perfectly for all time if and only if:

$$R_i = a_m\sin\xi_0$$

$$M = \frac{1}{2}a_m\sin^3\xi_0$$

1 Matching of Interior and Exterior Geometries

The gravitational collapse of a spherically symmetric perfect fluid star of zero pressure and uniform density is treated in Lightman Problem 16.30. I will follow closely that of the Problem Book in Relativity. Observers comoving with the fluid will measure uniform density throughout the star. I want to look more seriously at how the Friedmann and Schwarzschild metrics match together smoothly at the surface of the star. For closed $k = +1$ interior, let us start with the Einstein field equations for zero pressure:

$$\left(\frac{a_{,\tau}}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi\rho}{3}$$

$$\rho a^3 = constant$$

Set $k = +1$ and $a = a_m$ for the maximum value of a when $a_{,\tau} = 0$ and you will get

$$(a_{,\tau})^2 = \frac{a_m}{a} - 1$$

Introducing a convenient time parameter, $d\tau = ad\eta$, this will yield the famous cycloidal relations when integrated.

$$a = \frac{1}{2}a_m(1 + \cos\eta)$$

$$\tau = \frac{1}{2}a_m(\eta + \sin\eta)$$

The constants of integration are such that at $\eta = 0$, then $a = a_m$, $\tau = 0$, and at $\eta = \pi$, then $a = 0$, $\tau = \pi\frac{a_m}{2}$.

The Friedmann solution inside the star describes the intrinsic geometry when we put $\xi = \xi_0$. Note that this is a 3-geometry now.

$$ds^2 = -d\tau^2 + a^2(\tau)\sin^2\xi_0 d\Omega^2$$

while recalling that $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and with our time parameter $d\tau = a d\eta$ we have

$$ds^2 = a^2(\eta)(-d\eta^2 + \sin^2\xi_0 d\Omega^2)$$

The exterior Schwarzschild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2$$

As the surface of the star is at $r = R(\tau)$ and the using the equations for radial geodesics

$$R = \frac{1}{2}R_i(1 + \cos\eta)$$

$$\tau = (\frac{R_i^3}{8M})^{1/2}(\eta + \sin\eta)$$

$$u^t = \frac{dt}{dr} = \frac{(1 - 2M/R_i)^{1/2}}{1 - 2M/R}$$

Therefore the geometry of the surface as measured from the outside

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega^2$$

is

$$ds^2 = -\frac{R_i^3}{8M}(1 + \cos\eta)^2 d\eta^2 + \frac{R_i^2}{4}(1 + \cos\eta)^2 d\Omega^2$$

compare this with the inside:

$$ds^2 = a^2(\eta)(-d\eta^2 + \sin^2\xi_0 d\Omega^2)$$

and you see that the outside and inside will match only if

$$R_i = a_m \sin\xi_0$$

$$2M = a_m \sin^3\xi_0$$

Now our task is to calculate the extrinsic curvature inside and outside, and see if they are equal. The extrinsic curvature K_{ij}^- is denoted with a negative superscript to indicate the interior. The indices i, j range over τ, θ, ϕ. The extrinsic curvature is definied by

$$K_{ij} = -\mathbf{e}_i \cdot \nabla_j \mathbf{n}$$

The normal to the surface is

$$\mathbf{n} = \frac{1}{a}\frac{\partial}{\partial \xi}$$

Remember that $\mathbf{n}\cdot\mathbf{n} = 1$. The vectors $\mathbf{u} = \partial_\tau, \partial_\theta, \partial_\phi$ lie in the surface.

$$K_{ij} = -e_i \cdot \Gamma^{\alpha}_{\ nj}\mathbf{e}_\alpha = -g_{i\alpha}\Gamma^{\alpha}_{\ nj} = -\Gamma_{inj} = -\frac{1}{2}(g_{in,j} + g_{ij,n} - g_{nj,i}) = -\frac{1}{2}g_{ij,n}$$

where in the last step, you see that $g_{in} = a^{-1}g_{i\xi} = 0$ as there is no ξ part. Therefore

$$K_{ij} = -\frac{1}{2}g_{ij,n}$$

and looking at the metric we have

$$K^-_{\theta\theta} = K^-_{\phi\phi}\sin^{-2}\theta = -\frac{1}{2}a_m(1+\cos\eta)\sin\xi_0\cos\xi_0$$

while all the other curvature terms are zero, as their are no metric crossterms and the $\tau\tau$ has no ξ dependence.

$$K^-_{\tau\tau} = K^-_{\tau\theta} = K^-_{\tau\phi} = K^-_{\theta\phi} = 0$$

For the extrinsic curvature K^+_{ij} calculated in the exterior, we want to look at the four-velocity.

$$\mathbf{u} = u^t\mathbf{e}_t + u^r\mathbf{e}_r$$

As the normal vector $\mathbf{n} = n^t\mathbf{e}_t + n^r\mathbf{e}_r$ satisfies

$$\mathbf{n}\cdot\mathbf{n} = 1 = g_{tt}(n^t)^2 + g_{rr}(n^r)^2$$

$$\mathbf{n}\cdot\mathbf{u} = 0 = n^t u_t + n^r u_r$$

Now as

$$\mathbf{u}\cdot\mathbf{u} = -1 = g^{tt}(u_t)^2 + g^{rr}(u_r)^2$$

as well as

$$g^{rr} = (g_{rr})^{-1} = -(g^{tt})^{-1} = -g_{tt} = 1 - \frac{2M}{r}$$

Adding the non-zero products together, these equations imply

$$n^t = u_r$$

$$n^r = -u_t$$

Since $g_{in} = \mathbf{n}\cdot\mathbf{e}_i = 0$ as i, j range over τ, θ, ϕ, then

$$K^+_{ij} = -\frac{1}{2}g_{ij,n}$$

just like the calculation for the interior. The crossterms are all zero like before and the $\tau\tau$ term vanishes as the derivative of a constant is zero. So we are left with

$$K^+_{\theta\theta} = K^+_{\phi\phi} = -\frac{1}{2}(r^2)_{,n} = -\frac{1}{2}(r^2), rn^r$$

Using $n^r = -u_t$

We have then

$$K^+_{\theta\theta} = K^+_{\phi\phi} = ru_t = -R(1-\frac{2M}{R_i})^{1/2} = -\frac{1}{2}R_i(1+\cos\eta)(1-2M/R_i)^{1/2} = -\frac{1}{2}a_m(1+\cos\eta)\sin\xi_0\cos\xi_0$$

using $u_t = (1-\frac{2M}{R_i})^{1/2}$ and the identities of $R_i = a_m \sin\xi_0$ and $2M = a_m \sin^3\xi_0$. So since the extrinsic curvature calculated on the interior and exterior are equal,

$$K^+_{ij} = K^-_{ij}$$

then the metrics match smoothly. It was necessary and sufficient to show that both the intrinsic 3-geometry of the surface and the extrinsic curvature were the same whether measured in the exterior or interior.

2 Geodesic Motion of Particles on the Surface

Let us derive the motion of particles on the ball's surface as they move along radial geodesics in the exterior Schwarzschild geometry. For radial fall we have two constants of motion

$$u_0 = -\tilde{E}$$

$$\mathbf{u}\cdot\mathbf{u} = -1 = g^{00}u_0^2 + g_{rr}(u^r)^2$$

So this leads to

$$u^0 = \frac{dt}{d\tau} = g^{00}u_0 = \frac{\tilde{E}}{1-\frac{2M}{r}}$$

$$u^r = \frac{dr}{d\tau} = -(\tilde{E}^2 - 1 + \frac{2M}{r})^{1/2}$$

Where the use of the minus sign is applied from the square root to correspond to the infalling particle. The particle is released from rest so $\frac{dr}{d\tau} = 0$ at $r = R_i$ so from the equation above, this implies

$$\frac{2M}{R_i} = 1 - \tilde{E}^2$$

Therefore $\tilde{E} < 1$ otherwise we would have $2M/R_i$ be negative. Now we take the equation for u^r and re-write it in preperation for integration

$$d\tau = \frac{dr}{(\frac{2M}{r} - \frac{2M}{R_i})^{1/2}}$$

Once integrated with the constant chosen so that $\tau = 0$ at $r = R_i$, yields

$$\tau = (\frac{R_i^3}{8M})^{1/2}[2(\frac{r}{R_i} - \frac{r^2}{R_i^2})^{1/2} + \cos^{-1}(\frac{2r}{R_i} - 1)]$$

A much nicer form is obtained by using our familiar cycloidal parameter η.

$$\eta = \cos^{-1}(\frac{2r}{R_i} - 1)$$

This yields

$$r = \frac{1}{2}R_i(1+\cos\eta) \qquad \tau = (\frac{R_i^3}{8M})^{1/2}(\eta + \sin\eta)$$

Don't forget to be aware that at $r = R_i$, $\eta = 0$ and that at the singularity $\eta = \pi$. Lasenby, Doran and Gull have detailed more about how the proper time equals $\tau = \pi\frac{R_i^3}{8M})^{1/2}$ (when η reaches π). This is the same in Newtonian dynamics and comes as no surprise using geometric algebra.

Integrate the equation for u^0

$$t = \int \frac{\tilde{E}d\tau}{(1-\frac{2M}{r})} = \tilde{E}\int \frac{d\tau}{d\eta}\frac{1}{1-\frac{2M}{r}}d\eta$$

Take the simple derivative, substitute for r in terms of η and finally use the fact that the particle started at rest, where $2M/R_i = 1-\tilde{E}^2$:

$$t = (1-\frac{2M}{R_i})^{1/2}\int \frac{(\frac{R_i^3}{8M})^{1/2}(1+\cos\eta)d\eta}{1-4M[R_i(1+\cos\eta)]^{-1}}$$

This is a challenging integral and a consultation with a mathematician or simply a table of integrals is advised (Khuri 1957):

$$t = 2Mln\left|\frac{(\frac{R_i}{2M}-1)^{1/2}+tan(\frac{\eta}{2})}{(\frac{R_i}{2M}-1)^{1/2}-tan(\frac{\eta}{2})}\right| + 2M(\frac{R_i}{2M}-1)^{1/2}[\eta + \frac{R_i}{4M}(\eta+\sin\eta)]$$

Here the constant of integration is such that the time is zero when the particle is at the finite radius. $t = 0$ at $\eta = 0$, $r = R_i$. It's interesting to be aware that as the particle approaches the event horizon, $r \to 2M$, it take infinite time to do so, i.e. $t \to \infty$.

This can be seen as

$$\tan(\frac{\eta}{2}) \to (R_i/2M-1)^{1/2}$$

then

$$t \to \infty$$

where if $r \to 2M$, then $\eta \to \cos^{-1}(4M/R_i - 1)$ and using the half angle formula

$$\tan(\frac{\eta}{2}) = \pm\sqrt{\frac{1-\cos\eta}{1+\cos\eta}}$$

then

$$\tan(\frac{1}{2}\cos^{-1}(4M/R_i-1)) = \pm\sqrt{R_i/2M-1}$$

Therefore as $r \to 2M$ then $t \to \infty$.

3 Collapsing Ball of Dust Creates Particles

The gravitational disturbance produced by the collapsing ball of dust induces the creation of an outgoing thermal flux of radiation. This is treated by Birrell and Davies in 'Quantum fields in curved space', and I will follow their treatment here. A famous result of a collapsing star is that the wavelength of radiation leaving the surface increases exponentially. We should expect that that standard incoming complex exponential field modes should also be exponentially redshifted (after traveling through the interior and exiting on the other side).

The idea is to show that the Bogolubov transformation between the exponentially redshifted modes and standard outgoing complex exponential modes has a Planck structure. Thus the 'in vacuum' state contains a thermal flux of outgoing particles.

The problem is that writing solutions of the wave equation in the background metric is no simple matter. The exponential redshift in the outgoing modes is thus contructed using a nice simple 2D model of gravitational collapse and another simplified model where in 4D we ignore backscattering. The 2D model is great because we can write down the full solution of the renormalized stress-tensor at all events.

Let us focus on only the exterior metric and modes of quantum fields that pass through the interior of the ball that will change form, indicating particle production. The Minkowksi vacuum can define the quantum vacuum state long before the collapse. After the collapse, Schwarzschild describes the spacetime. Particle production is derived though the Bogolubov transformation from the in and out vacuum states.

The massless scalar field, (with no concern for the diference in conformal/minimal coupling as $R = 0$), has mode solutions for the wave equation

$$\Box\phi = 0$$

with the form

$$\phi \sim r^{-1} R_{\omega l}(r) Y_{lm}(\theta, \phi) e^{-i\omega t}$$

The radial function satisfies

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + (\omega^2 - [\frac{l(l+1)}{r^2} + \frac{2M}{r^3}][1 - \frac{2M}{r}]) R_{\omega l} = 0$$

where

$$r^* = r + 2M \ln|\frac{r}{2M} - 1|$$

As $r \to \infty$, the solution for the radial equations simplify to $R_{\omega l} = e^{\pm i\omega r^*}$ and therefore the modes simplify to

$$\phi \sim r^{-1} Y_{lm} e^{-i\omega u}$$

$$\phi \sim r^{-1} Y_{lm} e^{-i\omega v}$$

with $u = t - r^*$ and $v = t + r^*$, the null coordinates. Decomposing the solutions into a complete set of positive frequency modes denoted $f_{\omega lm}$:

$$\phi = \sigma_{l,m} \int d\omega (a_{\omega lm} f_{\omega lm} + a^{\dagger}_{\omega lm} f^*_{\omega lm})$$

where the positive frequency modes are normalized

$$(f_{\omega' l' m'}, f_{\omega lm}) = \delta(\omega' - \omega)\delta_{l' l}\delta_{m' m}$$

The in vacuum is

$$a_{\omega lm}|0\rangle = 0 \quad \forall\, \omega, l, m$$

This state indicates the absence of incoming (advanced) radiation from past null infinity, $\mathcal{I}^-$.

We are after the remote future positive frequency modes. Finding the form of the positive frequency modes, $f_{\omega lm}$, in the remote future will require us to observe several things. First, the incoming waves

$$\phi \sim r^{-1} Y_{lm} e^{-i\omega v}$$

will go through the center of the ball and become outgoing spherical waves. As they come in, they blueshift, and as they go out, they redshift. If they ball isn't collapsing but is static, then the two effects will lead only to the outgoing form of

$$\phi \sim r^{-1} Y_{lm} e^{-i\omega v}$$

But with a collapsing ball of dust, the shrinkage during transit time will raise the surface gravity of the ball causing a stronger redshift for the emerging waves. As we go to the extreme where the ball collapses to a mature black hole then the increase in gravity is enough to make an appreciable redshift because the shrinkage timesecale is comparable to the timescale it takes light to travel through the ball. Acutally, the redshift increases exponentially. (the e-folding time will be comparable to the transit time).

Let us ignore any direct interaction between the quantum field and the collapsing matter. The gravitational field is all that matters. The redshifted modes will be found by using this two dimensional model where the angular variables are supressed. Let us look at the spacetime outside and inside the collapsing ball.

Outside:

$$ds^2 = C(r)dudv$$

where

$$u = t - r^* + R_0^*$$
$$v = t + r^* - R_0^*$$
$$r* = \int C^{-1} dr$$
$$R*_0 = constant$$

Inside:

$$ds^2 = A(U, V)dUdV$$

where A is smooth, arbitrary, non-singular, and

$$U = \tau - r + R_0$$
$$V = \tau + r - R_0$$
$$R_0^* = \int C^{-1} dR_0$$

We have a ball of dust, sitting at rest at $\tau = 0$. Its surface has a radius $r = R_0$. Once time starts going, $\tau > 0$ we assume the ball begins collapsing and the surface shrinks along $r = R(\tau)$, the world line. We will discover that the forms of $A(U, V)$ and $R(\tau)$ are irrelevant for late times at $\mathcal{I}^-$, that is, large u. These coordinates are nice because at the start of collapse, that is $\tau = t = 0$, for the surface of the ball, $u = v = U = V = 0$.

There can be two ways to model the spherical symmetry of the 4D situation. We can reflect the two metrics in the origin of spatial coordinates ($r = 0$), or we can think only of the $r \geq 0$ region and reflect the null rays at $r = 0$. Doing this will reproduce the effect of radially incoming rays propagating through the centre of the ball and out again. This can be done by setting the boundary condition $\phi = 0$ at $r = 0$.

Interior and exterior transformation equations are $U = \alpha(u)$ and $v = \beta(V)$, ignoring all reflection at the surface of the ball. The radial coordinate center is the line $V = U - 2R_0$. Solving for the solutions that vanish along this line and reduce to standard expoentials from the infinite past of the 2D wave equation,

$$\Box\phi = 0$$

yields mode solutions

$$\frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]})$$

This is because at $r = 0$, $v = \beta(V) = \beta(U - 2R_0) = \beta[\alpha(u) - 2R_0]$. This physically shows that the incoming wave, $e^{-i\omega v}$, turns into the ugly outgoing wave, $e^{-i\omega\beta[\alpha(u)-2R_0]}$. That is, the simple left moving waves becomes a complicated right moving wave.

As the surface of the ball gets nearer to the event horizon, we should look for the phase factor, $\beta[\alpha(u) - 2R_0]$ to reduce to a steady escalating redshift. Match interior and exterior metrics across the collapsing surface, $r = R(\tau)$. Near the event horizon, the relation

$$\kappa u = -\ln|U + R_h - R_0 - \tau_h| + constant$$

holds. This shows thats as $U \to \tau_h + R_0 - R_h$ then $u \to \infty$. So we have $U \propto e^{-\kappa u} + constant$ for late times $\mathcal{I}^+$. The surface gravity is defined by

$$\kappa = \frac{1}{2}\frac{\partial C}{\partial r}\bigg|_{r=R_h}$$

The asymptotic observer at late times has the modes

$$\frac{i}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{i\omega(ce^{-\kappa u}+d)})$$

where c and d are just constants. This shows that the outgoing null rays undergo an exponetially increasing redshift, with e-folding time of κ^{-1}. These asymptotic modes have the same form as those for a moving mirror an a receding trajectory. The geometric optic calculation is the same for both so we have the same Bogolubov transformations. For the mirror, the Doppler shift acts the same as the gravitational redshift does for the collapsing star. The particle detector at $\mathcal{I}^-$ will detect nothing but at $\mathcal{I}^+$ where we have the complicated modes, the detector will register particles. These detected particles are right moving in the 2D model, that is, an outgoing flux of particles will be leaving the black hole.

Performing the Bogolubov transformations between these new complicated modes and the standard expontial modes yields a spectrum that is Planckian with temperature,

$$T = \frac{\kappa}{2\pi k_B}$$

where $c = 1$, and κ is the surface gravity.

Let's take a look at the 4D model. The results are the same, a flux of particles from the black hole with temperature spectrum of $T = \kappa/2\pi k_B$, but there are some technical problems associated with the inability to write the solutions to the radial equation, $R_{\omega l}(r)$ in terms of known functions.//

The way to deal with the complication is to ignore backscattering. In the one dimensional wave equation,

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \{\omega^2 - [l(l+1)r^{-2} + 2Mr^{-3}][1 - 2Mr^{-1}]\}R_{\omega l} = 0$$

The term in the square brackets acts as the potential term, physically representing reflected waves. The waves can be interpreted as backscattering of field modes from the spacetime curvature. The thermal effects that are of most interest are due to the field disturbance from passage through the interior of the collasping ball. So let us remove the term artifically and examine the results:

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \omega^2 R_{\omega l} = 0$$

We have reduced radial functions that are now just exponentials and so our normalized field modes are:

$$\frac{Y_{lm}(\theta, \phi)}{(8\pi^2\omega)^{1/2} r} \times e^{-i\omega(t \pm r^*)}$$

At large r, this reduces to the flat space form, $u \equiv t - r^* \to t - r$, $v \equiv t + r^* \to t + r$. The linear combination of modes ($e^{-i\omega v}$) that corresponds to standard modes at $\mathcal{I}^+$ are of interest. Going backwards in time and tracing the modes through the collapsing ball and out along the advanced null ray to $\mathcal{I}^-$, the mode that has the form $e^{-i\omega u}$ looks like, for $v < v_0$

$$\frac{Y_{lm}(\theta, \phi)}{(8\pi^2\omega)^{1/2} r} \times \exp\{4Mi\omega \ln[(v_0 - v)/c]\}$$

and zero for $v > v_0$.

The ordinary in vacuum is defined with respect to modes ($e^{-i\omega v}$). The Bogolubov coefficients relating these and the above modes with the natural log argument in the exponent are given by:

$$\alpha_{\omega\omega'}[\beta_{\omega\omega'}] = (1/2\pi)\int_{-\infty}^{v_0} dv(\omega'/\omega)^{1/2} e^{\pm i\omega' v} \exp\{4Mi\omega \ln[(v_0 - v)/c]\}$$

This is evaluated in terms of Γ functions. The answer will have a factor $(\omega')^{-1/2}$ which implies that $\int |\beta_{\omega\omega'}|^2 d\omega'$ diverges logarithmically. Infinite particles exist in each mode at $\mathcal{I}^+$. The divergence is connected with the normalization of the continuous modes. So basically the total flux for all time is infinite as the collapsing ball produces a steady flux of radiation to $\mathcal{I}^+$. We are then concerned instead with the number of particles emitted per time. If you want to find this out, you localize the modes, i.e. put them in a box.

Using the properties of the Bogolubov coefficients, particularly the Wronskian condition, we have for these modes:

$$\sum_{\omega}(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = 1$$

And the coefficeients themselves, including an analyticity argument, we obtain

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi M\omega}|\beta_{\omega\omega'}|^2$$

Computing the particle flux that goes to $\mathcal{I}^+$ at late times, take note of the density of states inside a sphere of radius R centered on the collapsing ball is $Rd\omega/2\pi$. The particle number per modes is

$$N_{\omega lm} = \sum_{\omega'} |\beta_{\omega\omega'}|^2 = 1/(e^{8\pi M\omega} - 1)$$

Assuming the particle takes a time R to reach the surface of the sphere, the number of particles per unit time in the range ω to $\omega + d\omega$ passing outward through the surface of the sphere is

$$(d\omega/2\pi)(e^{8\pi M\omega} - 1)^{-1}$$

This is a black body (Planck) spectrum, with temperature

$$T = \kappa/2\pi k_B$$

Neglect of backscattering made the spectrum precisely Planckian. The conservation of probability equation, $\sum_\omega(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = 1$ will be altered because backscattering depletes the outgoing flux by a factor of $1 - \Gamma_\omega$. A factor of Γ_ω is introduced into the number of particles per unit time in the select frequency range. It may not be exactly Planckian but its okay, because we can still call it thermal. As the black hole is immersed in a heat bath, the fraction of incoming radiation that will be backscattered by the hole will be the same as that removed from the outoing flux back down the hole. Only the fraction Γ_ω of incoming radiation will be absorbed. The ratio of emission to absorption per mode ω is independent of Γ_ω and thus it's only the ratio of the black body replacing the black hole. A thermal equilibrium is reached and despite the Planckian spectrum distortion, the spectrum can be regarded as thermal.