

Strategic Party Placement with a Dynamic Electorate

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December 8, 2008

*An earlier version of this paper was presented at the 2008 summer conference of the Society for Political Methodology, Ann Arbor, MI, July 9-12. For helpful commentary, I would like to thank George Rabinowitz, Georg Vanberg, Virginia Gray, Tom Carsey, John Aldrich, and the participants in UNC's state politics working group. Any errors remaining in this paper are my own.

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Abstract

This article expands upon formal research on elections by considering competition in a dynamic environment of multiple elections. The key assumptions are that the ideology of the electorate is changing in a known way, parties cannot change their position from one election to the next, and one party has a non-ideological advantage in the first election. A deterministic version of this game shows that with a large valence advantage for one party, both parties will converge to the median for a future election. With a small valence advantage, there is no pure-strategy equilibrium, so a stochastic version of the game is considered. With probabilistic voting, parties place themselves closer to the present median, but move towards the future median the more highly they value winning in the future and the less uncertain they are about election outcomes.

1 Introduction

How do political parties place themselves ideologically if their behavior has contradictory short- and long-term electoral consequences? Strategic politicians are likely to consider how the world around them is changing and what their behavior means for future electoral prospects, but most of the formal literature on party competition in elections focuses on single-shot elections. Of the research that does consider repeated elections, much of it focuses on the policies politicians enact once in office, showing that in some cases policy-motivated parties or candidates will take divergent strategies.¹ Additionally, some studies consider how candidates should place themselves if they need to win a primary and a general election with a sticky issue position. Aranson and Ordeshook (1972) consider

¹For a review of the articles on policy-motivated candidates in repeated elections, see Osborne (1995: 285-288).

a version of this game that is soluble if the candidate has a prior belief about his or her opponent's issue position, while Coleman (1972) shows results when conditions occur that lead the candidate to worry only about winning the primary. This article adds a new perspective to strategic placement by considering how parties should place themselves when their positions are sticky and they compete in multiple general elections. Placing political parties in this context of multiple elections would facilitate understanding of the actions of forward-looking parties that simply want to win elections. This paper reports how rational parties should behave in this situation.

The model that I investigate has three essential features. First, the median voter in the electorate changes positions in a predictable way over two elections. Second, in the first of the two elections one of the parties has an advantage based on valence factors.² Third, the position the party adopts is fixed across the two elections. The model has two features that are common to spatial voting models. I assume that any issue based advantage that accrues to one party over the other is based on the relative proximity of the parties (Hotelling, 1929; Downs, 1957). Voters prefer parties that are closer.³ The model also allows for a discounting of the second election compared to the first. Utility in all cases derives from winning office rather than any intrinsic reward based on ideology.

The three potentially controversial features are all motivated by an interest in understanding rational behavior in a dynamic environment. The fact that the median changes in a predictable way is the driving element of the model. For this question to be interesting, parties must necessarily be constrained by past position. Certainly there is ample evidence that parties do not change positions easily, at least in mass perception, and when they do it is often costly.⁴ The final feature, valence advantage in the first

²Stokes defines a valence advantage as an issue that involves “the linking of the parties with some condition that is positively or negatively valued by the electorate” (1963: 373).

³See appendix B for similar results from the directional model of voter utility (Rabinowitz and Macdonald, 1989). The results are worth considering because much research shows that the proximity model fails to meet empirical expectations (Macdonald et al., 1998; Rabinowitz, 1978) or that empirical models cannot discern between the proximity and directional model (Lewis and King, 2000).

⁴Noteworthy is that the American National Election Studies have been collecting data on the perceived

election, is introduced to give the model empirical credibility and provide a natural rationale for party divergence: a significant body of research demonstrates how valence advantages can promote divergent strategies in party ideological self-placement (Adams et al., 2005; Ansolabehere and Snyder, 2000; Groseclose, 2001; Macdonald and Rabinowitz, 1998; Schofield, 2003). Thus, the goal is to present a meaningful, yet reasonable and tractable model for understanding the ideological motivation of parties in dynamic environments.

What is the best strategy for a party to take in these situations? Does the optimal strategy differ between a party with a valence advantage and one with a disadvantage? This article explains how rational parties should react to the strategic considerations of dynamic issues like these. Section 2 takes the assumptions just described, formally defines a deterministic model, and explains the results. Because a hard-to-interpret result emerges with the deterministic game, section 3 makes the case for a stochastic model and reports equilibrium behavior and comparative statics found through a simulation-based analysis. Finally, the substantive importance of the findings is discussed.

2 A Deterministic Model

To formalize the assumptions, for the first and second election the median voter, m_t , evaluates each of the parties as follows:

$$\begin{aligned} U_{m_t}(A) &= -(m_t - \theta_A)^2 + V_t \text{ where } [V_1 = V > 0, V_2 = 0] \\ U_{m_t}(D) &= -(m_t - \theta_D)^2 \end{aligned} \tag{1}$$

Here, U represents the utility a voter gets from a certain party winning the election. t is the time of the election (1 for present, 2 for future). The two parties the voter

positions of the Democratic and Republican parties on a limited number of issues since 1972, and these positions have been remarkably stable.

evaluates are an advantaged party (A) and a disadvantaged party (D). Because A is the advantaged party, the median voter in the first election (m_1) adds V , a positive and constant non-ideological utility, to his or her evaluation of A . Again, the assumption is that a valence advantage will not persist over time, so V is zero and the valence term drops out in the second election. Lastly, the utility of ideological proximity is the negative squared distance between the voter's ideal point and the position taken by each party; θ_A & θ_D represent the issue positions staked-out by the advantaged and disadvantaged party, respectively. In each of the elections, the median voter will elect the party from which it receives a higher utility.

The principal result is that there is only a pure strategy equilibrium if the valence advantage is so large in the first election that the advantaged party can adopt position m_2 , the median voter's position in the second election, while still being assured of winning the first election. Figure 1 shows the intuition as to why this is the case. In each panel, the horizontal line represents issue space from liberal at left to conservative at right. θ_A & θ_D are the positions taken by each party and m_1 & m_2 are the ideal points of the median voters in the first and second election.⁵ Panel 1(a) shows how voters in this space will make their decisions in the first election if there is no valence advantage. Since the bisector lies halfway between the two parties, anyone on the liberal side votes for A , and anyone on the conservative side votes for D . This is the case where vote choice is driven purely by proximity. Panel 1(b) shows the case when there is a valence advantage that is small relative to the difference between median ideal points. Because voters receive a constant, non-ideological utility from choosing A , some voters more proximate to D will actually choose A . In panel 1(b), any voter left of the vertical, dashed line chooses A over D .⁶ As the next panel illustrates, holding the parties' positions constant, an increase in

⁵We can assume that zero is the midpoint between the two parties without loss of generality. Wherever the parties place themselves, simply adding a constant to the scale will not affect relative proximity. Having zero to be the midpoint simplifies the math.

⁶Any voter with an ideal point less than $\frac{V}{2(\theta_D - \theta_A)}$ will choose A . Hence, $\frac{V}{2(\theta_D - \theta_A)}$ defines the vertical,

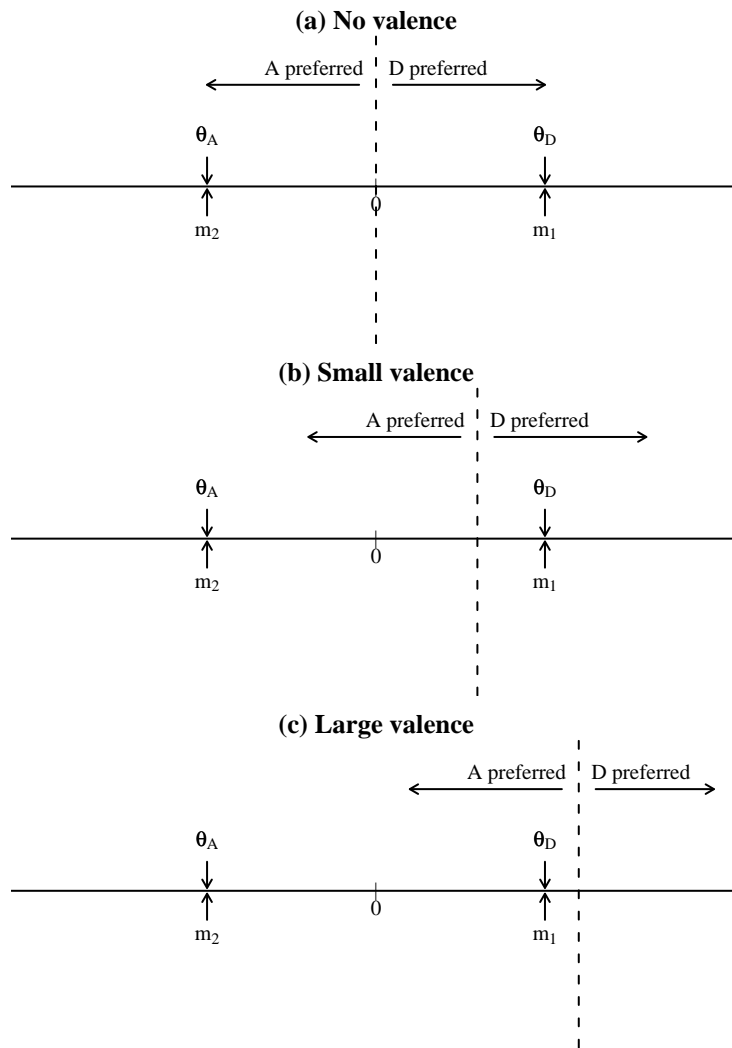


Figure 1: Division of voters by valence advantage

valence increases the advantaged party's vote share. In fact, panel 1(c) shows how voters divide in the first election when the valence is large relative to the movement over time of the median voter.⁷ In this case the advantaged party can adopt the position of the second median voter, guaranteeing victory in the first election and no worse than a 50-50 chance in the second election. The disadvantaged party, with no chance in the first election, will also adopt the second median to ensure a 50-50 chance in that election. Neither party can do anything to improve its result in either election, which leads to observation 1:

Observation 1 *Whenever the valence advantage is large relative to movement in the median voter's ideal point, both parties will place themselves at the future median voter's ideal point.*⁸

Alternatively, in the case with a small valence relative to the movement in the median, party D can win the first election if party A positions itself to the optimal position for the second median. This is illustrated in panel 1(b). Party A will win the second election in this case, but by moving, it could win the first election. Since present elections are more valuable than future ones, A will move closer to m_1 . But if A places itself closer to the first median voter, then D would like to move closer to m_2 to at least win the future election. But if D does this, A would like to move back to m_2 because it can win both elections there if D is close to m_2 . D would be dissatisfied if both parties played close to m_2 , so it would move back to its original position (m_1) to win the first election, completing a cycle of strategies. This logic of instability motivates observation 2:

Observation 2 *Whenever valence is small relative to the ideological movement of the median, there is no stable set of party positions.*⁹

dashed line that separates the A and D voters.

⁷Specifically, when $V > (m_1 - m_2)^2$.

⁸Observations 1 & 2 are formally proved as propositions 1 & 2 in appendix A.

⁹Specifically, if $V \leq (m_1 - m_2)^2$.

3 A Stochastic Version of the Game

Since the deterministic game only has a pure strategy equilibrium if one party has a very large non-ideological advantage in the present election, the parties would have to take a mixed strategy to find an equilibrium. Since parties' issue positions are announced publicly to the voters, they are also highly visible to the opposing party. It therefore would be difficult to substantively interpret probabilistic party issue placement. This article will keep with the trend of other spatial research by focusing on pure strategy equilibria.¹⁰

In particular, this article looks for pure party strategies in a model where voters vote probabilistically. In fact, a number of studies show more stable equilibria for probabilistic voting models than the often fragile results of deterministic voting models (Coughlin and Nitzan, 1981; Ledyard, 1984; McKelvey, 1975; Wittman, 1995). Probabilistic voting is possible if there is a stochastic element of utility. This notion is natural to politics because some last-minute news event or the way a candidate's campaign is perceived could influence voters' choices.¹¹ However, politicians cannot anticipate this effect ahead of time, so a random draw from a probability distribution is added to voters' utility from each party. The ultimate result is that the parties face uncertainty in the election outcome and can only hope to maximize their probability of winning an election. Nothing is guaranteed.

Under this model, the median voter in each election evaluates the parties as follows:

$$\begin{aligned} U_{m_t}(A) &= -(m_t - \theta_A)^2 + V_t + \epsilon_{At} \text{ where } [V_1 = V > 0, V_2 = 0] & (2) \\ U_{m_t}(D) &= -(m_t - \theta_D)^2 + \epsilon_{Dt} \end{aligned}$$

¹⁰To illustrate the norm in the spatial literature, Adams et al. (2005) describe cases without a fixed point equilibrium as lacking an equilibrium.

¹¹Several studies offer justifications for probabilistic voting, including Coughlin et al. (1990), Hinich (1977), & Hinich and Munger (1994). For a review, see Mueller (2003).

The equations of 2 are the same as those of equation 1, with the addition that each ϵ represents utility coming from a stochastic draw from a probability distribution.¹² With probabilistic election results, parties' expected utilities become the sum of utilities from winning, weighted by the probability of winning. Solving for a closed-form solution with such utility functions is impossible.¹³ Hence, the remainder of this article searches for equilibria by following a growing trend of using computer simulations to analyze insoluble games.¹⁴ In particular, this analysis searches for equilibria based on fixed values of the game's parameters.¹⁵

3.1 Results

The most surprising result of this stochastic model is that parties never diverge from each other. This contrasts from the aforementioned models of voting behavior with a valence advantage, which show that valence causes parties to take differing positions (Adams et al., 2005; Ansolabehere and Snyder, 2000; Groseclose, 2001; Macdonald and Rabinowitz, 1998; Schofield, 2003). It also is surprising because some of the literature on competition in repeated elections shows incentives for divergent positions (Alesina,

¹²Specifically, they are drawn from a standard Gumbel distribution. This distribution looks like a skewed normal and is chosen because the resulting probability that voter m_t will vote for a party in a given election has a standard logistic distribution.

¹³Specifically, party A 's expected utility is $\Lambda_1(\theta_A, \theta_D) + \delta\Lambda_2(\theta_A, \theta_D)$, where δ is the value of the future election, and Λ_1 & Λ_2 are different logistic distribution functions. Solving for the maximum of this equation would require solving for the parties' behavior parameters in a sum of transcendental functions. This is algebraically impossible, so the game does not yield a closed-form equation for a party's optimal response. This problem would emerge with several probability distributions besides logistic, including the normal distribution.

¹⁴For an accessible introduction to the utility of analyzing games without an analytic solution, see Miller and Page (2007).

¹⁵For treatments manipulating the values of δ (future discount), V (valence advantage), and s (the logistic scale parameter), the behavior of both parties is varied from -1 to 1 in increments of 0.001 . Considering this discrete strategy set, rather than a continuous strategy space, a program reports the Nash equilibrium for each treatment. Simulations were done in R 2.6.1 for Red Hat Linux on UNC's Emerald computing cluster (R Development Core Team, 2008). The simulation code is available from the author; at the time of writing, example code can be found at http://www.unc.edu/~monogan/research/variance_SIMULATIONS_1000.R. The program should work for any version of R that includes the S4 class of objects.

1988; Harrington, 1992). In fact, even though a general solution cannot be found to this game, it can be proved that the parties will match in any equilibrium. Intuitively, if the parties take different positions, then each party can improve its expected utility by moving towards the other party's position. Since there is an incentive for parties to change strategies when they do not match, we have the intuition behind observation 3:

Observation 3 *Any equilibrium is characterized by the feature that the parties will take the same issue stance, whatever that stance may be.*¹⁶

In addition to convergence, the simulations show that party behavior responds to the value of the future election and the amount of uncertainty parties face in electoral outcomes, but valence advantage only has a marginal effect. In particular, parties will converge to a position near the first election's median voter, but as the value that the parties place on the second election rises, approaching the value of the first election, the parties place themselves farther from the present median voter to be closer to the future median voter. Furthermore, the parties' position is more sensitive to the value of the future election when there is less variance in the stochastic element of voter utility, with the parties making bigger moves towards the future median voter for each increase in the discount parameter. Substantively, this means that the less uncertain parties are about election outcomes, the more they will play to win in the future. Lastly, a larger valence advantage leads parties to place themselves marginally closer to the future median voter, though the effect increases somewhat if the value of the future election (δ) is higher or if the variance of voters' stochastic utility is higher.

The Nash equilibria for all treatments are reported in table 1. Table 1 shows the equilibria as the variance of voters' stochastic utility terms, the discounting for the second election (δ), and the valence advantage (V) are varied. In all treatments, median voters'

¹⁶The formal proof in appendix A, called proposition 3, more specifically compares matching strategies to those in which a party diverges from the other party. It shows that whenever one party takes a position off of the other party's position, then the other party can force the first into a worse position than it would receive from matching.

ideal points in each of the two elections (m_1 & m_2) are held constant. Each cell is the Nash equilibrium of the treatment resulting from the corresponding values of variance, discount (δ), and valence (V), first reporting the behavior of party A , then the behavior of D .

Table 1: Nash Equilibria by Stochastic Variance, Discounting Factor, and Valence Advantage

		Valence (V)			
Variance	Discount (δ)	0.1	0.6	1.0	1.5
Large	0.0	0.700, 0.700	0.700, 0.700	0.700, 0.700	0.700, 0.700
	0.1	0.627, 0.627	0.621, 0.621	0.610, 0.610	0.585, 0.585
	0.5	0.423, 0.423	0.417, 0.417	0.389, 0.389	0.335, 0.335
	0.9	0.321, 0.321	0.303, 0.303	0.273, 0.273	0.219, 0.219
Medium	0.0	0.700, 0.700	0.700, 0.700	0.700, 0.700	0.700, 0.700
	0.1	0.515, 0.515	0.500, 0.500	0.479, 0.479	0.432, 0.432
	0.5	0.220, 0.220	0.203, 0.203	0.175, 0.175	0.128, 0.128
	0.9	0.116, 0.116	0.100, 0.100	0.080, 0.080	0.045, 0.045
Small	0.0	0.700, 0.700	0.700, 0.700	0.700, 0.700	0.700, 0.700
	0.1	0.400, 0.400	0.383, 0.383	0.354, 0.354	0.299, 0.299
	0.5	0.100, 0.100	0.087, 0.087	0.066, 0.066	0.033, 0.033
	0.9	0.025, 0.025	0.016, 0.016	0.002, 0.002	-0.020, -0.020

Note: Each cell entry lists θ_A, θ_D in the strategy profile of a simulated Nash equilibrium. $m_1 = 0.7$ & $m_2 = -0.1$. Large variance: $\sigma^2 = \pi^2/3$; medium variance: $\sigma^2 = \pi^2/27$; small variance: $\sigma^2 = \pi^2/108$.

Notice that as the value of the future election (δ) increases, parties place themselves closer to the future median. Equilibrium positions range from the present median voter's ideal point (0.700, 0.700) whenever the future election has no value to a position that is closer to the future median voter than it is the present median voter (-0.020, -0.020) when the variance of voters' stochastic utility term is small, the future election is 90% as valuable as the present election, and there is a large valence advantage. Notice too that valence has little impact on the equilibrium. In any case when the second election has any value ($\delta > 0$), the parties place themselves a little closer to the future median as valence increases, and this effect gets bigger as the value of the second election increases.

To a very slight degree, the effect of valence on equilibrium movement is greater in the higher variance case. Overall, though, valence does little to influence party behavior.

4 Conclusion

Although research has shown that some of this model's features, such as sequential elections and a valence advantage, lead parties to take different issue stances, this model with sticky party positions and probabilistic voting leads parties to adopt the same position. So for both the equilibrium case in the deterministic game and in all cases in the stochastic game, the traditional Hotelling-Downs result of party convergence in a single issue dimension holds, though the parties often match at an off-median position. The stochastic model also shows that the position at which parties match is closer to the future median voter the more certain the parties are about the outcome of the present election. When the variance of the stochastic utility term is large, leaving more to chance, the parties are more inclined to stay close to the present median. With a smaller stochastic variance or a larger valence advantage term, the outcome of the present election is more certain, so parties have more of an incentive to cheat towards the future median voter and influence the future election, which is decided strictly by ideology. This movement becomes even stronger when there is a greater reward for winning the future election, which emerges when the discounting factor increases.

So how might these results be applied to understanding real politics? Several issues are likely to fit the dynamic described in this game. One of these issues is immigration. Much anti-immigrant discourse has an anti-Latino feel to it. So even if some elites want to push a strict view on immigration without an anti-Latino message, prior discourse might make it difficult to separate ethnic considerations from the immigration issue. Taking a strong anti-immigrant stance might win over voters who hold an anti-Latino sentiment, and this strategy could make a substantial contribution to winning a plurality of voters

in the short term. However, the growing Latino population, which will surely overtake the number of voters with anti-Latino sentiments, likely will react unfavorably to such a position in the long term. Immigration, therefore, provides an example of an issue for which public ideology is changing on account of demographic change.

Another example is the environment, for which the problems of current policy are abundantly clear to experts. As the public learns more about the dangers of global warming, scarce potable water, and other environmental hazards, it can only become more pro-environmental. Rational elites might have short-term electoral success by arguing the merits of less green policies for jobs and businesses. However, they also ought to consider future electoral movement when they choose their environmental policy positions. For example, they might promote policies such as offshore drilling to fill petrol-based energy demands and thereby win popular support today. However, future voters who will be more aware of the consequences of carbon emissions likely will punish parties who have advocated such positions in the past. Therefore, the environment is an issue for which public ideology ought to change due to an information flow.

So what positions will candidates for office take on issues like immigration and the environment? To start, party institutions are more likely to be concerned with party prospects in future elections than individual candidates are. Therefore, in instances where parties have more levers of control over the issue stances adopted by their candidates, the positions ought to be more future-oriented than instances when parties have fewer control mechanisms over candidates. For example, places with stronger parties are more likely to have candidates who are welcoming on immigration and who favor environmental regulations. Since both parties ought to converge to the same position, we also would expect to see more policy action that is future-oriented in constituencies with stronger parties. Similarly, the equilibrium from the deterministic game showed that with a large valence advantage, parties ought to converge to a position favorable to the

future electorate, and the stochastic game showed a moderate increase in future-oriented position taking as valence advantage increases. Therefore, in elections where the leading candidate at the start of the election has a larger lead or when incumbents enter an election with an economy that is over- or underperforming to a great degree, we ought to see the candidates taking more future-oriented positions than in elections where the frontrunner has a smaller lead or economic growth is moderate.

A final use of this model would be to take a new look at optimal strategies ahead of primary elections. As the aforementioned literature on primaries notes, the structure of an election with primaries is similar to the game just considered: candidates want to win two elections, the median voter in a general election has a different ideal point from the median voter in a primary constituency, and candidates are likely stuck with the positions they take in the primaries because flip-flopping is not credible in the span of a few months.¹⁷ So whether analyzing issues with clear trends in the electorate or primary politics, understanding the placement of sticky party positions can be useful for several real political questions.

Of course, models are inherently limited. The formal assumptions at best only modestly match real world dynamics. What models accomplish, however, is a means to analyze the fundamental strategic logic of a particular context—in this case party and candidate placement on issues that change in known directions.

¹⁷The major change from this article's game would be that the winners of two separate elections would compete in a general election, but utility is only awarded for winning the general election.

A Appendix: Formal Derivation of the Results

Propositions 1 & 2 formally demonstrate the results of the deterministic game, while proposition 3 focuses on the stochastic game. Proposition 1 proves the presence of an equilibrium in the deterministic game whenever valence is large relative to the ideological movement of the median voter (see observation 1). Proposition 2 shows why there is no equilibrium in the deterministic game whenever valence is small relative to the movement of the median voter (see observation 2). Finally, proposition 3 demonstrates that in any equilibrium for the stochastic game, the parties will match positions (see observation 3).

Proposition 1 *Under proximity voter utilities, when $V > (m_1 - m_2)^2$, it is a Nash equilibrium if the parties converge to the second median voter's ideal point.*

Proof If both parties play $\theta = m_2$, then A earns $1 + \frac{\delta}{2}$ for winning the first election and tying the second. D earns $\frac{\delta}{2}$ for tying the second. If either unilaterally defects, then it will lose the second election outright, diminishing utility by $\frac{\delta}{2}$, without gaining anything. This is because A will win the first election for $\theta_A = m_2$ regardless of the value of θ_D . Since neither party will defect, $\theta_A = \theta_D = m_2$ is a Nash equilibrium. \square

Proposition 2 *Under proximity voter utilities, when $V \leq (m_1 - m_2)^2$, there is no pure strategy Nash equilibrium.*

Proof D will win the first election if:

$$\begin{aligned} -(m_1 - \theta_D)^2 &> -(m_1 - \theta_A)^2 + V \\ -\theta_D^2 + 2m_1\theta_D + (\theta_A^2 - 2m_1\theta_A - V) &> 0 \end{aligned} \tag{3}$$

Therefore, D 's best response function is:

$$B_D(\theta_A) : \quad \begin{array}{ll} m_2 \leq \theta_D < \theta_A & \text{if } \theta_A \geq m_1 - \sqrt{V} \\ m_1 - \sqrt{(m_1 - \theta_A)^2 - V} < \theta_D \leq m_1 & \text{if } \theta_A < m_1 - \sqrt{V} \end{array} \quad (4)$$

This is because whenever $\theta_A \geq m_1 - \sqrt{V}$ party A has placed itself sufficiently close to m_1 that D cannot win the first election, so D will play for the second election (m_2). Whenever $\theta_A < m_1 - \sqrt{V}$, though, party D can win the first election by taking a position very close to m_1 . Party A can win both elections, so its best response function is:

$$B_A(\theta_D) : \quad \begin{array}{ll} m_1 - \sqrt{(m_1 - \theta_D)^2 + V} < \theta_A < \theta_D & \text{if } m_1 - \sqrt{(m_1 - \theta_D)^2 + V} \geq m_2 \\ m_2 \leq \theta_A < \theta_D & \text{if } m_1 - \sqrt{(m_1 - \theta_D)^2 + V} < m_2 \end{array} \quad (5)$$

Given these best response functions, there is no Nash equilibrium for the game when valence does not exceed the square of ideological movement. First for the case when $\theta_A \geq m_1 - \sqrt{V}$, for an equilibrium to exist the parties would have to play in a way such that $\theta_D < \theta_A$ and $\theta_D > \theta_A$. This is logically impossible, so both players cannot play a best response. Second, for the case when $\theta_A < m_1 - \sqrt{V}$, it can be shown that the parties would have to play such that $(m_1 - \theta_D)^2 < (m_1 - \theta_A)^2 - V$ and $(m_1 - \theta_D)^2 > (m_1 - \theta_A)^2 - V$, which is also logically impossible. Hence, there is no pure-strategy equilibrium whenever $V \leq (m_1 - m_2)^2$. \square

Proposition 3 *For any Nash equilibrium of the stochastic game with proximity utilities, the two parties will match in their behavior ($\theta_A^* = \theta_D^*$).*

Proof Because this is a zerosum game, we can say that $(\theta_A^* = \theta_D, \theta_D^* = \theta_A)$ characterizes Nash equilibria if the following three **conditions** are true (Osborne and Rubinstein, 1994: proposition 22.2.c):

1. $\theta_A^* = \theta_D$ must be a maxminimizer for A .
2. $\theta_D^* = \theta_A$ must be a maxminimizer for D .
3. $\max_{\theta_A} \min_{\theta_D} EU_A(\theta_A, \theta_D) = \min_{\theta_D} \max_{\theta_A} EU_A(\theta_A, \theta_D)$

First, considering **condition 1**, is $\theta_A^* = \theta_D$ a maxminimizer for A ? A 's expected utility function is:

$$EU_A(\theta_A, \theta_D) = \Lambda\{-(m_1 - \theta_A)^2 + (m_1 - \theta_D)^2 + V\} + \delta\Lambda\{-(m_2 - \theta_A)^2 + (m_1 - \theta_D)^2\} \quad (6)$$

Where Λ represents the logistic distribution function. If $\theta_A^* = \theta_D$, then $EU_A(\theta_A^* = \theta_D, \theta_D) = \Lambda(V) + \delta\Lambda(0) = \Lambda(V) + \frac{\delta}{2}$ for any value of θ_D . If $\Lambda(V) + \delta\Lambda(0) \geq \min_{\theta_D} EU_A(\theta_A, \theta_D)$ for any value of θ_A , then $\theta_A^* = \theta_D$ is a maxminimizer for A .

To determine whether this is true, we will represent any other strategy that A can take as $\theta'_A = \theta_D + \eta$, where η is the difference between θ_A & θ_D . It can be shown that A 's expected utility for θ'_A is:

$$EU_A(\theta'_A = \theta_D + \eta, \theta_D) = \Lambda\{2m_1\eta - 2\theta_D\eta - \eta^2 + V\} + \delta\Lambda\{2m_2\eta - 2\theta_D\eta - \eta^2\} \quad (7)$$

So the key question is now whether $\Lambda(V) + \delta\Lambda(0) \geq \min_{\theta_D} EU_A(\theta'_A = \theta_D + \eta, \theta_D)$ for all η . If $\eta > 0$, then D can minimize A 's utility by choosing $\theta_D = 1$, but if $\eta < 0$, then D minimizes with $\theta_D = -1$.¹⁸ Consider first the case where A plays a position more conservative than D ($\eta > 0$). Substituting $\theta_D = 1$ yields:

$$\min_{\theta_D} EU_A(\theta'_A = \theta_D + \eta, \theta_D) = \Lambda\{2\eta(m_1 - 1) - \eta^2 + V\} + \delta\Lambda\{2\eta(m_2 - 1) - \eta^2\} \quad (8)$$

We know that this minimized utility is less than or equal to the utility when $\theta_A^* = \theta_D$ because each corresponding probability term is less than or equal to its counterpart from

¹⁸ $\eta = 0$ simply produces $\theta_A^* = \theta_D$, which all other strategies are being compared to.

the equation $EU_A(\theta_A^* = \theta_D, \theta_D)$. Specifically, it can be shown that $\Lambda(V) \geq \Lambda\{2\eta(m_1 - 1) - \eta^2 + V\}$ because $0 \geq 2\eta(m_1 - 1) - \eta^2$.¹⁹ Second, $\Lambda(0) \geq \Lambda\{2\eta(m_2 - 1) - \eta^2\}$ because the logistic distribution function on the right side of the equation contains a negative term.²⁰ Therefore, for any $\eta > 0$:

$$\Lambda(V) + \delta\Lambda(0) \geq \Lambda\{2\eta(m_1 - 1) - \eta^2 + V\} + \delta\Lambda\{2\eta(m_2 - 1) - \eta^2\} \quad (9)$$

So for no positive η will A earn a better minimized utility than it gets by matching party D 's position. Now consider the case when A plays a position more liberal than D ($\eta < 0$). Substituting $\theta_D = -1$:

$$\min_{\theta_D} EU_A(\theta'_A = \theta_D + \eta, \theta_D) = \Lambda\{2\eta(m_1 + 1) - \eta^2 + V\} + \delta\Lambda\{2\eta(m_2 + 1) - \eta^2\} \quad (10)$$

By a similar argument to the previous case, it can be shown that:

$$\Lambda(V) + \delta\Lambda(0) \geq \Lambda\{2\eta(m_1 + 1) - \eta^2 + V\} + \delta\Lambda\{2\eta(m_2 + 1) - \eta^2\} \quad (11)$$

Within each logistic distribution function on the right side of equation 11, every term other than V (which is also part of the corresponding left-side term) is negative.²¹ Between equations 9 & 11, we can conclude that $\Lambda(V) + \delta\Lambda(0) \geq \min_{\theta_D} EU_A(\theta'_A = \theta_D + \eta, \theta_D)$ for all η and therefore $\theta_A^* = \theta_D$ is a maxminimizer for party A .

Second, **condition 2** requires that $\theta_D^* = \theta_A$ be a maxminimizer for D . Given the zerosum nature of the game, this is equivalent to D choosing a strategy that minimizes the maximum utility that A can earn. If $\theta_D^* = \theta_A$, then party A earns $EU_A(\theta_A, \theta_D^* =$

¹⁹This relies on the fact that the policy space is defined as $\Theta = [-1, 1]$. Hence, $m_1 \leq 1$, which means that the first term is the product of a nonpositive number ($m_1 - 1$) and a positive number (2η). Since the second term subtracts a squared number, we know that both terms on the right are negative.

²⁰Again, $m_2 \leq 1$ makes the first term the product of a nonpositive number and a positive number.

²¹This is because η is now negative and each term of the form $m_t + 1$ is now nonnegative since $m_t \geq -1$ to be in the policy space.

$\theta_A) = \Lambda(V) + \delta\Lambda(0) = \Lambda(V) + \delta\Lambda(0)$ for any value of θ_A . If $\Lambda(V) + \frac{\delta}{2} \leq \max_{\theta_A} EU_A(\theta_A, \theta_D)$ for any value of θ_D , then $\theta_D^* = \theta_A$ is a maxminimizer for D .

Much as in the proof the the prior condition, we compare $\theta_D^* = \theta_A$ to $\tilde{\theta}_D = \theta_A + \zeta$ where ζ is the difference between θ_A & θ_D . Party A 's expected utility when D plays $\tilde{\theta}_D$ is:

$$EU_A(\theta_A, \tilde{\theta}_D = \theta_A + \zeta) = \Lambda\{-2m_1\zeta + 2\theta_A\zeta + \zeta^2 + V\} + \delta\Lambda\{-2m_2\zeta + 2\theta_A\zeta + \zeta^2\} \quad (12)$$

If $\zeta > 0$, party A can maximize its utility by playing $\theta_A = 1$, but if $\zeta < 0$ party A can maximize its utility by playing $\theta_A = -1$. Consider first whether party D would want to play a strategy more conservative than party A ($\zeta > 0$). Substituting $\theta_A = 1$:

$$\Lambda(V) + \delta\Lambda(0) \leq \max_{\theta_A} EU_A(\theta_A, \tilde{\theta}_D) = \Lambda\{2\zeta(1 - m_1) + \zeta^2 + V\} + \delta\Lambda\{2\zeta(1 - m_2) + \zeta^2\} \quad (13)$$

Equation 13 holds because each term within the logistic distribution functions on the far right side is nonnegative.²² Therefore, party D would never like to play a more conservative position than party A since matching θ_A reduces A 's maximum utility. Now consider whether party D would want to play a strategy more liberal than party A ($\zeta < 0$). Substituting $\theta_A = -1$:

$$\Lambda(V) + \delta\Lambda(0) \leq \max_{\theta_A} EU_A(\theta_A, \tilde{\theta}_D) = \Lambda\{-2\zeta(1 + m_1) + \zeta^2 + V\} + \delta\Lambda\{-2\zeta(1 + m_2) + \zeta^2\} \quad (14)$$

Again, all of the terms in the logistic distribution functions on the far right side are nonnegative because ζ is now negative and is subtracted from the equation. Therefore, the inequality relationship holds. Between equations 13 & 14, we can conclude that $\Lambda(V) + \delta\Lambda(0) \leq \max_{\theta_A} EU_A(\theta_A, \tilde{\theta}_D = \theta_A + \zeta)$ for all ζ and therefore $\theta_D^* = \theta_A$ is a maxminimizer

²²Note again that $m_t \leq 1$.

for party D .

Finally, **condition 3** is trivial to demonstrate at this point. In proving the previous two conditions, we demonstrated first that $\max_{\theta_A} \min_{\theta_D} EU_A(\theta_A, \theta_D) = \Lambda(V) + \frac{\delta}{2}$, then that $\min_{\theta_D} \max_{\theta_A} EU_A(\theta_A, \theta_D) = \Lambda(V) + \frac{\delta}{2}$. Since these two quantities are the same, condition 3 is true. Therefore, all three conditions are satisfied for $\theta_A^* = \theta_D^*$ to characterize all Nash equilibria. \square

B Appendix: The Model with Directional Theory

In the directional model of voter utility, voters vote for candidates who are on the same side of policy space as themselves before they will vote for any candidate on the opposite side of the space (Rabinowitz and Macdonald, 1989). Additionally, the more a politician emphasizes the symbolism of one side, the higher he or she will be rated by voters on that side and the lower he or she will be rated by voters on the other side. The unidimensional result with two parties is convergence to the outside bounds of the policy space, though candidates stop short of taking a position for which they could be labeled an extremist.²³

Equation 15 formalizes the way voters directionally evaluate parties:

$$\begin{aligned} U_{m_t}(A) &= m_t \times \theta_A + V_t \text{ where } [V_1 = V > 0, V_2 = 0] \\ U_{m_t}(D) &= m_t \times \theta_D \end{aligned} \tag{15}$$

Just as in equation 1, U is the voter's utility from a party, m_t is the median voter's ideal point at time $t \in \{1, 2\}$, θ is the party's position, and V is a constant non-ideological utility that the first median voter (m_1) adds to its evaluation of party A in the first

²³Several researchers have tried to combine the proximity and directional theories of electoral competition, typically by adding a new consideration to the proximity model. Namely, Adams, Merrill & Grofman add a discounting factor (2005), and Kedar adds a strategic compromise consideration (2005a; 2005b).

election. The parameters are the same as with the proximity model. The main difference is that the voter's utility is a function of the product of his or her ideal point with the party's stated position, rather than the negative squared distance between the ideal point and party position.²⁴ For simplicity, party positions are restricted to the policy space ($\theta \in \Theta = [-1, 1]$), and I assume that no position in this space would be considered extreme by voters. To keep the game interesting, the game assumes that the present median voter and the future median voter are on different sides of the issue. Without loss of generality, the present median voter is conservative ($m_1 > 0$) and the future median voter is liberal ($m_2 < 0$).

Much like the deterministic proximity-based game in the paper, there is only an equilibrium in a special case. Interestingly, the condition of the special case depends not on movement of the median, but whether the valence advantage is large relative to the present median voter's ideological extremity. If the advantaged party has a non-ideological advantage more than twice the ideal point of the present median voter, then party *A* will win the present election even if it takes a very liberal position that appeals to the future median voter. Therefore, both parties will take the most liberal possible positions when the valence advantage is large relative to the present median voter's ideal point, so they will tie in the future election. When valence is not big enough relative to the present median voter's ideal point, there is no pure strategy equilibrium because for any set of positions that the parties might take, one party would change its behavior in order to win or tie an additional election. The remainder of this subsection proves propositions 4 & 5, which formally demonstrate these ideas. Then subsection B.1 considers a stochastic version of the directional game.

²⁴We can use the simple product to capture ideological utility if we assume that zero is ideologically neutral, negative numbers represent liberalism, and positive numbers represent conservatism. This product gives the voter positive utility for any party on the same side, negative utility for any party on the opposing side, and zero utility if either the voter or party is neutral on the issue. As either the voter or the party move further to an extreme, the absolute size of the voter's utility increases.

Proposition 4 *Under directional utilities, when $V > 2m_1$, it is a Nash equilibrium if the parties converge to the most liberal point in the policy space.*

Proof If both parties play $\theta = -1$, then A earns $1 + \frac{\delta}{2}$ for winning the first election and tying the second. D earns $\frac{\delta}{2}$ for tying the second. If either unilaterally defects, then it will lose the second election outright, diminishing utility by $\frac{\delta}{2}$, without gaining anything. This is because A will win the first election for $\theta_A = -1$ against any position D can take in the policy range $\theta_D \in [-1, 1]$. Since neither party will defect, $\theta_A = \theta_D = -1$ is a Nash equilibrium. \square

Proposition 5 *Under directional voter utilities, when $V \leq 2m_1$, there is no pure strategy Nash equilibrium.*

Proof Under these circumstances, D will win the first election if:

$$\begin{aligned} m_1\theta_A + V &< m_1\theta_D \\ \theta_A + \frac{V}{m_1} &< \theta_D \end{aligned} \tag{16}$$

Otherwise, A will win. Therefore, D 's best response function is:

$$\begin{aligned} B_D(\theta_A) : \quad & -1 \leq \theta_D < \theta_A \quad \text{if } \theta_A + \frac{V}{m_1} \geq 1 \\ & \theta_A + \frac{V}{m_1} < \theta_D \leq 1 \quad \text{if } \theta_A + \frac{V}{m_1} < 1 \end{aligned} \tag{17}$$

This is because whenever $\theta_A + \frac{V}{m_1} \geq 1$, party A has taken a sufficiently conservative position that the valence advantage will exceed any ideological advantage with voter m_1 that party D can edge out. In this case, D should play for the voter in the second election (m_2). Whenever $\theta_A + \frac{V}{m_1} < 1$, though, party D can win the more valuable first election by taking a position conservative enough to stimulate an ideological advantage with the first voter that exceeds the valence advantage.

On the other hand, party A can win both elections, so its best response function is:

$$B_A(\theta_D) : \begin{cases} \theta_D - \frac{V}{m_1} < \theta_A < \theta_D & \text{if } \theta_D - \frac{V}{m_1} \geq -1 \\ -1 \leq \theta_A < \theta_D & \text{if } \theta_D - \frac{V}{m_1} < -1 \end{cases} \quad (18)$$

In order to win the second election, party A has to play something lower than θ_D . In order to simultaneously win the first election, though, A needs to be close enough to D that party D 's ideological advantage with the first voter is smaller than A 's valence advantage.

Given these best response functions, there is no Nash equilibrium for the game when valence does not exceed the ideological difference. First, in the case when $\theta_A + \frac{V}{m_1} \geq 1$, for an equilibrium to exist the parties would have to play in such a way that $\theta_D < \theta_A$ and $\theta_D > \theta_A$, which is impossible. Second, for an equilibrium to exist in the case when $\theta_A + \frac{V}{m_1} < 1$, the parties would have to play such that $\theta_A + \frac{V}{m_1} < \theta_D$ (i.e., $\theta_D - \frac{V}{m_1} > \theta_A$) and $\theta_D - \frac{V}{m_1} < \theta_A$, which is also impossible. Hence, there is no pure strategy equilibrium whenever $V \leq 2m_1$. \square

B.1 The Directional Model with Probabilistic Voting

Similar to the stochastic model with proximity utilities, this subsection presents a version of the game in which voters evaluate parties' issue positions directionally, but vote probabilistically. This is again created by adding a shock to voter utilities in the form of a random draw from a probability distribution:

$$\begin{aligned} U_{m_t}(A) &= m_t \times \theta_A + V_t + \epsilon_{At} \text{ where } [V_1 = V > 0, V_2 = 0] \\ U_{m_t}(D) &= m_t \times \theta_D + \epsilon_{Dt} \end{aligned} \quad (19)$$

Equation 19 resembles equation 15, with ϵ as a stochastic utility term. This game does not yield a closed-form solution, so I again ran simulations to determine the equilibria for fixed parameters.²⁵ In all treatments, both parties converge to the same extreme of the issue space, be it liberal or conservative. If the present median voter is more extreme in its position than the future median voter, then the parties always take the most extreme position on the side of the present median voter. If the future median voter is more extreme than the present median voter, then the results vary. For low values of the future election, the parties will take the most extreme position on the side of the present median voter. When the value of the future election reaches a certain level, the parties switch to taking the most extreme position on the side of the future median voter. Where the parties switch strategies depends on their level of uncertainty: as the variance of the voters' stochastic utility term decreases, the value of the future election that draws parties to the other side of the issue decreases. Hence, the effect that the value of the future has on party strategies is conditional on uncertainty and relative extremity of the present and future electorates.

²⁵The ϵ terms still have a Gumbel distribution, which means parties' expected utilities are a sum in which each term includes a logistic distribution. This creates the same insolubility problem from the stochastic proximity game.

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