M/P
QA274
.H6
c.3
AO/Acc

Introduction to Stochastic Processes

10000596017

Paul G. Hoel
Sidney C. Port
Charles J. Stone
University of California, Los Angeles

HOUGHTON MIFFLIN COMPANY BOSTON
Atlanta Dallas Geneva, Illinois Hopewell, New Jersey
Palo Alto London

MATH-PHYSICS LIBRARY
UNIVERSITY OF NORTH CAROLINA

3 Markov Pure Jump Processes

Consider again a system that at any time can be in one of a finite or countably infinite set $\mathscr{S}$ of states. We call $\mathscr{S}$ the *state space* of the system. In Chapters 1 and 2 we studied the behavior of such systems at integer times. In this chapter we will study the behavior of such systems over all times $t \geq 0$.

3.1. Construction of jump processes

Consider a system starting in state x_0 at time 0. We suppose that the system remains in state x_0 until some positive time τ_1, at which time the system jumps to a new state $x_1 \neq x_0$. We allow the possibility that the system remains permanently in state x_0, in which case we set $\tau_1 = \infty$. If τ_1 is finite, upon reaching x_1 the system remains there until some time $\tau_2 > \tau_1$ when it jumps to state $x_2 \neq x_1$. If the system never leaves x_1, we set $\tau_2 = \infty$. This procedure is repeated indefinitely. If some $\tau_m = \infty$, we set $\tau_n = \infty$ for $n > m$.

Let $X(t)$ denote the state of the system at time t, defined by

$$(1) \qquad X(t) = \begin{cases} x_0, & 0 \leq t < \tau_1, \\ x_1, & \tau_1 \leq t < \tau_2, \\ x_2, & \tau_2 \leq t < \tau_3, \\ \vdots \end{cases}$$

The process defined by (1) is called a *jump process*. At first glance it might appear that (1) defines $X(t)$ for all $t \geq 0$. But this is not necessarily the case.

Consider, for example, a ball bouncing on the floor. Let the state of the system be the number of bounces it has made. We make the physically reasonable assumption that the time in seconds between the nth bounce and the $(n+1)$th bounce is 2^{-n}. Then $x_n = n$ and

$$\tau_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

MATH-PHYSICS LIBRARY
UNIVERSITY OF NORTH CAROLINA

We see that $\tau_n < 2$ and $\tau_n \to 2$ as $n \to \infty$. Thus (1) defines $X(t)$ only for $0 \le t < 2$. By the time $t = 2$ the ball will have made an infinite number of bounces. In this case it would be appropriate to define $X(t) = \infty$ for $t \ge 2$.

In general, if

$$\lim_{n\to\infty} \tau_n < \infty, \tag{2}$$

we say that the $X(t)$ process *explodes*. If the $X(t)$ process does not explode, i.e., if

$$\lim_{n\to\infty} \tau_n = \infty, \tag{3}$$

then (1) *does* define $X(t)$ for all $t \ge 0$.

We will now specify a probability structure for such a jump process. We suppose that all states are of one of two types, *absorbing* or *non-absorbing*. Once the process reaches an absorbing state, it remains there permanently. With each non-absorbing state x, there is associated a distribution function $F_x(t)$, $-\infty < t < \infty$, which vanishes for $t \le 0$, and *transition probabilities* Q_{xy}, $y \in \mathscr{S}$, which are nonnegative and such that $Q_{xx} = 0$ and

$$\sum_y Q_{xy} = 1. \tag{4}$$

A process starting at x remains there for a random length of time τ_1 having distribution function F_x and then jumps to state $X(\tau_1) = y$ with probability Q_{xy}, $y \in \mathscr{S}$. We assume that τ_1 and $X(\tau_1)$ are chosen independently of each other, i.e., that

$$P_x(\tau_1 \le t,\ X(\tau_1) = y) = F_x(t)Q_{xy}.$$

Here, as in the previous chapters, we use the notation $P_x(\ )$ and $E_x(\ )$ to denote probabilities of events and expectations of random variables defined in terms of a process initially in state x. Whenever and however the process jumps to a state y, it acts just as a process starting initially at y. For example, if x and y are both non-absorbing states,

$$P_x(\tau_1 \le s,\ X(\tau_1) = y,\ \tau_2 - \tau_1 \le t,\ X(\tau_2) = z) = F_x(s)Q_{xy}F_y(t)Q_{yz}.$$

Similar formulas hold for events defined in terms of three or more jumps. If x is an absorbing state, we set $Q_{xy} = \delta_{xy}$, where

$$\delta_{xy} = \begin{cases} 1, & y = x, \\ 0, & y \ne x. \end{cases}$$

Equation (4) now holds for all $x \in \mathscr{S}$.

We say that the jump process is *pure* or *non-explosive* if (3) holds with probability one regardless of the starting point. Otherwise we say the process is *explosive*. If the state space $\mathscr{S}$ is finite, the jump process is necessarily non-explosive. It is easy to construct examples having an infinite state space which are explosive. Such processes, however, are unlikely to arise in practical applications. At any rate, to keep matters simple we assume that our process is non-explosive. The set of probability zero where (3) fails to hold can safely be ignored. We see from (1) that $X(t)$ is then defined for all $t \geq 0$.

Let $P_{xy}(t)$ denote the probability that a process starting in state x will be in state y at time t. Then

$$P_{xy}(t) = P_x(X(t) = y)$$

and

$$\sum_y P_{xy}(t) = 1.$$

In particular, $P_{xy}(0) = \delta_{xy}$. We can also choose the initial state x according to an *initial distribution* $\pi_0(x)$, $x \in \mathscr{S}$, where $\pi_0(x) \geq 0$ and

$$\sum_x \pi_0(x) = 1.$$

In this case,

$$P(X(t) = y) = \sum_x \pi_0(x) P_{xy}(t).$$

The *transition function* $P_{xy}(t)$ cannot be used directly to obtain such probabilities as

$$P(X(t_1) = x_1, \ldots, X(t_n) = x_n)$$

unless the jump process satisfies the *Markov property*, which states that for $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ and $x_1, \ldots, x_n, x, y \in \mathscr{S}$,

$$P(X(t) = y \mid X(s_1) = x_1, \ldots, X(s_n) = x_n, X(s) = x) = P_{xy}(t - s).$$

By a *Markov pure jump process* we mean a pure jump process that satisfies the Markov property. It can be shown, although not at the level of this book, that a pure jump process is Markovian if and only if all non-absorbing states x are such that

$$P_x(\tau_1 > t + s \mid \tau_1 > s) = P_x(\tau_1 > t), \qquad s, t \geq 0,$$

i.e., such that

$$\frac{1 - F_x(t + s)}{1 - F_x(s)} = 1 - F_x(t), \qquad s, t \geq 0. \tag{5}$$

Now a distribution function F_x satisfies (5) if and only if it is an exponential distribution function (see Chapter 5 of *Introduction to Probability Theory*). We conclude that a pure jump process is Markovian if and only if F_x is an exponential distribution for all non-absorbing states x.

Let $X(t)$, $0 \le t < \infty$, be a Markov pure jump process. If x is a non-absorbing state, then F_x has an exponential density f_x. Let q_x denote the parameter of this density. Then $q_x = 1/E_x(\tau_1) > 0$ and

$$f_x(t) = \begin{cases} q_x e^{-q_x t}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

Observe that

$$P_x(\tau_1 \ge t) = \int_t^\infty q_x e^{-q_x s}\, ds = e^{-q_x t}, \qquad t \ge 0.$$

If x is an absorbing state, we set $q_x = 0$.

It follows from the Markov property that for $0 \le t_1 \le \cdots \le t_n$ and $x_1, \ldots, x_n$ in $\mathscr{S}$,

$$\begin{aligned}(6) \qquad P(X(t_1) = x_1, \ldots, X(t_n) = x_n) \\ = P(X(t_1) = x_1)P_{x_1x_2}(t_2 - t_1) \cdots P_{x_{n-1}x_n}(t_n - t_{n-1}).\end{aligned}$$

In particular, for $s \ge 0$ and $t \ge 0$

$$P_x(X(t) = z, X(t + s) = y) = P_{xz}(t)P_{zy}(s).$$

Since

$$P_{xy}(t + s) = \sum_z P_x(X(t) = z, X(t + s) = y),$$

we conclude that

$$(7) \qquad P_{xy}(t + s) = \sum_z P_{xz}(t)P_{zy}(s), \qquad s \ge 0 \text{ and } t \ge 0.$$

Equation (7) is known as the *Chapman-Kolmogorov* equation.

The transition function $P_{xy}(t)$ satisfies the integral equation

$$(8) \qquad P_{xy}(t) = \delta_{xy}e^{-q_x t} + \int_0^t q_x e^{-q_x s}\left(\sum_{z \ne x} Q_{xz}P_{zy}(t - s)\right) ds, \qquad t \ge 0,$$

which we will now verify. If x is an absorbing state, (8) reduces to the obvious fact that

$$P_{xy}(t) = \delta_{xy}, \qquad t \ge 0.$$

Suppose x is not an absorbing state. Then for a process starting at x, the event $\{\tau_1 \le t, X(\tau_1) = z \text{ and } X(t) = y\}$ occurs if and only if the first jump occurs at some time $s \le t$ and takes the process to z, and the process goes from z to y in the remaining $t - s$ units of time. Thus

$$P_x(\tau_1 \le t, X(\tau_1) = z \text{ and } X(t) = y) = \int_0^t q_x e^{-q_x s}Q_{xz}P_{zy}(t - s)\, ds,$$

so

$$P_x(\tau_1 \le t \text{ and } X(t) = y) = \sum_{z \ne x} P_x(\tau_1 \le t,\ X(\tau_1) = z \text{ and } X(t) = y)$$
$$= \int_0^t q_x e^{-q_x s} \left(\sum_{z \ne x} Q_{xz} P_{zy}(t - s) \right) ds.$$

Also

$$P_x(\tau_1 > t \text{ and } X(t) = y) = \delta_{xy} P_x(\tau_1 > t)$$
$$= \delta_{xy} e^{-q_x t}.$$

Consequently,

$$\begin{aligned} P_{xy}(t) &= P_x(X(t) = y) \\ &= P_x(\tau_1 > t \text{ and } X(t) = y) + P_x(\tau_1 \le t \text{ and } X(t) = y) \\ &= \delta_{xy} e^{-q_x t} + \int_0^t q_x e^{-q_x s} \left(\sum_{z \ne x} Q_{xz} P_{zy}(t - s) \right) ds, \end{aligned}$$

as claimed. Replacing s by $t - s$ in the integral in (8), we can rewrite (8) as

(9) $$P_{xy}(t) = \delta_{xy} e^{-q_x t} + q_x e^{-q_x t} \int_0^t e^{q_x s} \left(\sum_{z \ne x} Q_{xz} P_{zy}(s) \right) ds, \qquad t \ge 0.$$

It follows from (9) that $P_{xy}(t)$ is continuous in t for $t \ge 0$. Therefore the integrand in (9) is a continuous function, so we can differentiate the right side. We obtain

(10) $$P'_{xy}(t) = -q_x P_{xy}(t) + q_x \sum_{z \ne x} Q_{xz} P_{zy}(t), \qquad t \ge 0.$$

In particular,

$$\begin{aligned} P'_{xy}(0) &= -q_x P_{xy}(0) + q_x \sum_{z \ne x} Q_{xz} P_{zy}(0) \\ &= -q_x \delta_{xy} + q_x \sum_{z \ne x} Q_{xz} \delta_{zy} \\ &= -q_x \delta_{xy} + q_x Q_{xy}. \end{aligned}$$

Set

(11) $$q_{xy} = P'_{xy}(0), \qquad x, y \in \mathscr{S}.$$

Then

(12) $$q_{xy} = \begin{cases} -q_x, & y = x, \\ q_x Q_{xy}, & y \ne x. \end{cases}$$

It follows from (12) that

(13) $$\sum_{y \ne x} q_{xy} = q_x = -q_{xx}.$$

The quantities q_{xy}, $x \in \mathscr{S}$ and $y \in \mathscr{S}$, are called the *infinitesimal parameters* of the process. These parameters determine q_x and Q_{xy}, and thus by our construction determine a unique Markov pure jump process. We can rewrite (10) in terms of the infinitesimal parameters as

$$P'_{xy}(t) = \sum_z q_{xz} P_{zy}(t), \qquad t \geq 0. \tag{14}$$

This equation is known as the *backward equation.*

If $\mathscr{S}$ is finite, we can differentiate the Chapman-Kolmogorov equation with respect to s, obtaining

$$P'_{xy}(t + s) = \sum_z P_{xz}(t) P'_{zy}(s), \qquad s \geq 0 \text{ and } t \geq 0. \tag{15}$$

In particular,

$$P'_{xy}(t) = \sum_z P_{xz}(t) P'_{zy}(0), \qquad t \geq 0,$$

or equivalently,

$$P'_{xy}(t) = \sum_z P_{xz}(t) q_{zy}, \qquad t \geq 0. \tag{16}$$

Formula (16) is known as the *forward equation.* It can be shown that (15) and (16) hold even if $\mathscr{S}$ is infinite, but the proofs are not easy and will be omitted.

In Section 3.2 we will describe some examples in which the backward or forward equation can be used to find explicit formulas for $P_{xy}(t)$.

3.2. Birth and death processes

Let $\mathscr{S} = \{0, 1, \ldots, d\}$ or $\mathscr{S} = \{0, 1, 2, \ldots\}$. By a *birth and death process* on $\mathscr{S}$ we mean a Markov pure jump process on $\mathscr{S}$ having infinitesimal parameters q_{xy} such that

$$q_{xy} = 0, \qquad |y - x| > 1.$$

Thus a birth and death process starting at x can in one jump go only to the states $x - 1$ or $x + 1$.

The parameters $\lambda_x = q_{x,x+1}$, $x \in \mathscr{S}$, and $\mu_x = q_{x,x-1}$, $x \in \mathscr{S}$, are called respectively the *birth rates* and *death rates* of the process. The parameters q_x and Q_{xy} of the process can be expressed simply in terms of the birth and death rates. By (13)

$$-q_{xx} = q_x = q_{x,x+1} + q_{x,x-1},$$

so that

$$q_{xx} = -(\lambda_x + \mu_x) \qquad \text{and} \qquad q_x = \lambda_x + \mu_x. \tag{17}$$

Thus x is an absorbing state if and only if $\lambda_x = \mu_x = 0$. If x is a non-absorbing state, then by (12)

$$Q_{xy} = \begin{cases} \dfrac{\mu_x}{\lambda_x + \mu_x}, & y = x - 1, \\ \dfrac{\lambda_x}{\lambda_x + \mu_x}, & y = x + 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{18}$$

A birth and death process is called a *pure birth process* if $\mu_x = 0$, $x \in \mathscr{S}$, and a *pure death process* if $\lambda_x = 0$, $x \in \mathscr{S}$. A pure birth process can move only to the right, and a pure death process can move only to the left.

Given nonnegative numbers λ_x, $x \in \mathscr{S}$, and μ_x, $x \in \mathscr{S}$, it is natural to ask whether there is a birth and death process corresponding to these parameters. Of course, $\mu_0 = 0$ is a necessary requirement, as is $\lambda_d = 0$ if $\mathscr{S}$ is finite. The only additional problem is that explosions must be ruled out if $\mathscr{S}$ is infinite. It is not difficult to derive a necessary and sufficient condition for the process to be non-explosive. A simple sufficient condition for the process to be non-explosive is that for some positive numbers A and B

$$\lambda_x \leq A + Bx, \qquad x \geq 0.$$

This condition holds in all the examples we will consider.

In finding the birth and death rates of specific processes, we will use some standard properties of independent exponentially distributed random variables. Let $\xi_1, \ldots, \xi_n$ be independent random variables having exponential distributions with respective parameters $\alpha_1, \ldots, \alpha_n$. Then $\min(\xi_1, \ldots, \xi_n)$ has an exponential distribution with parameter $\alpha_1 + \cdots + \alpha_n$ and

$$P(\xi_k = \min(\xi_1, \ldots, \xi_n)) = \frac{\alpha_k}{\alpha_1 + \cdots + \alpha_n}, \qquad k = 1, \ldots, n. \tag{19}$$

Moreover, with probability one, the random variables $\xi_1, \ldots, \xi_n$ take on n distinct values.

To verify these results we observe first that

$$\begin{aligned} P(\min(\xi_1, \ldots, \xi_n) > t) &= P(\xi_1 > t, \ldots, \xi_n > t) \\ &= P(\xi_1 > t) \cdots P(\xi_n > t) \\ &= e^{-\alpha_1 t} \cdots e^{-\alpha_n t} \\ &= e^{-(\alpha_1 + \cdots + \alpha_n)t}, \end{aligned}$$

and hence that $\min(\xi_1, \ldots, \xi_n)$ has the indicated exponential distribution.

Set

$$\eta_k = \min(\xi_j : j \neq k).$$

Then η_k has an exponential distribution with parameter

$$\beta_k = \sum_{j \neq k} \alpha_j,$$

and ξ_k and η_k are independent. Thus

$$\begin{aligned} P(\xi_k = \min(\xi_1, \ldots, \xi_n)) &= P(\xi_k \leq \eta_k) \\ &= \int_0^\infty \left(\int_x^\infty \alpha_k e^{-\alpha_k x} \beta_k e^{-\beta_k y}\, dy \right) dx \\ &= \int_0^\infty \alpha_k e^{-\alpha_k x} e^{-\beta_k x}\, dx \\ &= \frac{\alpha_k}{\alpha_k + \beta_k} = \frac{\alpha_k}{\alpha_1 + \cdots + \alpha_n}. \end{aligned}$$

In order to show that the random variables $\xi_1, \ldots, \xi_n$ take on n distinct values with probability one, it is enough to show that $P(\xi_i \neq \xi_j) = 1$ for $i \neq j$. But since ξ_i and ξ_j have a joint density f, it follows that

$$P(\xi_i = \xi_j) = \iint_{\{(x,y):\, x=y\}} f(x, y)\, dx\, dy = 0,$$

as desired.

Example 1. Branching process. Consider a collection of particles which act independently in giving rise to succeeding generations of particles. Suppose that each particle, from the time it appears, waits a random length of time having an exponential distribution with parameter q and then splits into two identical particles with probability p and disappears with probability $1 - p$. Let $X(t)$, $0 \leq t < \infty$, denote the number of particles present at time t. This branching process is a birth and death process. Find the birth and death rates.

Consider a branching process starting out with x particles. Let $\xi_1, \ldots, \xi_x$ be the times until these particles split apart or disappear. Then $\xi_1, \ldots, \xi_x$ each has an exponential distribution with parameter q, and hence $\tau_1 = \min(\xi_1, \ldots, \xi_x)$ has an exponential distribution with parameter $q_x = xq$. Whichever particle acts first has probability p of splitting into two particles and probability $1 - p$ of disappearing. Thus for $x \geq 1$

$$Q_{x,x+1} = p \qquad \text{and} \qquad Q_{x,x-1} = 1 - p.$$

State 0 is an absorbing state. Since $\lambda_x = q_x Q_{x,x+1}$ and $\mu_x = q_x Q_{x,x-1}$, we conclude that

$$\lambda_x = xqp \qquad \text{and} \qquad \mu_x = xq(1-p), \qquad x \geq 0.$$

In the preceding example we did not actually prove that the process is a birth and death process, i.e., that it "starts from scratch" after making a jump. This intuitively reasonable property basically depends on the fact that an exponentially distributed random variable ξ satisfies the formula

$$P(\xi > t + s \mid \xi > s) = P(\xi > t), \qquad s, t \geq 0,$$

but a rigorous proof is complicated.

By (17) and the definition of λ_x and μ_x, the backward and forward equations for a birth and death process can be written respectively as

$$(20) \qquad P'_{xy}(t) = \mu_x P_{x-1,y}(t) - (\lambda_x + \mu_x)P_{xy}(t) + \lambda_x P_{x+1,y}(t), \qquad t \geq 0,$$

and

$$(21) \qquad P'_{xy}(t) = \lambda_{y-1}P_{x,y-1}(t) - (\lambda_y + \mu_y)P_{xy}(t) + \mu_{y+1}P_{x,y+1}(t), \qquad t \geq 0.$$

In (21) we set $\lambda_{-1} = 0$, and if $\mathscr{S} = \{0, \ldots, d\}$ for $d < \infty$, we set $\mu_{d+1} = 0$.

We will solve the backward and forward equations for a birth and death process in some special cases. To do so we will use the result that if

$$(22) \qquad f'(t) = -\alpha f(t) + g(t), \qquad t \geq 0,$$

then

$$(23) \qquad f(t) = f(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}g(s)\,ds, \qquad t \geq 0.$$

The proof of this standard result is very easy. We multiply (22) through by $e^{\alpha t}$ and rewrite the resulting equation as

$$\frac{d}{dt}(e^{\alpha t}f(t)) = e^{\alpha t}g(t).$$

Integrating from 0 to t we find that

$$e^{\alpha t}f(t) - f(0) = \int_0^t e^{\alpha s}g(s)\,ds,$$

and hence that (23) holds.

3.2.1. Two-state birth and death process. Consider a birth and death process having state space $\mathscr{S} = \{0, 1\}$, and suppose that 0 and 1 are both non-absorbing states. Since $\mu_0 = \lambda_1 = 0$, the process is determined by the parameters λ_0 and μ_1. For simplicity in notation we set $\lambda = \lambda_0$ and $\mu = \mu_1$. We can interpret such a process by thinking of state 1 as the system (e.g., telephone or machine) operating and state 0 as the system being idle. We suppose that starting from an idle state the system remains idle for a random length of time which is exponentially distributed with parameter λ, and that starting in an operating state the system continues operating for a random length of time which is exponentially distributed with parameter μ.

We will find the transition function of the process by solving the backward equation. It is left as an exercise for the reader to obtain the same results by solving the forward equation.

Setting $y = 0$ in (20), we see that

$$P'_{00}(t) = -\lambda P_{00}(t) + \lambda P_{10}(t), \qquad t \geq 0, \tag{24}$$

and

$$P'_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t), \qquad t \geq 0. \tag{25}$$

Subtracting the second equation from the first,

$$\frac{d}{dt}(P_{00}(t) - P_{10}(t)) = -(\lambda + \mu)(P_{00}(t) - P_{10}(t)).$$

Applying (23),

$$\begin{aligned} P_{00}(t) - P_{10}(t) &= (P_{00}(0) - P_{10}(0))e^{-(\lambda+\mu)t} \\ &= e^{-(\lambda+\mu)t}. \end{aligned} \tag{26}$$

Here we have used the formulas $P_{00}(0) = 1$ and $P_{10}(0) = 0$. It now follows from (24) that

$$\begin{aligned} P'_{00}(t) &= -\lambda(P_{00}(t) - P_{10}(t)) \\ &= -\lambda e^{-(\lambda+\mu)t}. \end{aligned}$$

Thus

$$\begin{aligned} P_{00}(t) &= P_{00}(0) + \int_0^t P'_{00}(s)\, ds \\ &= 1 - \int_0^t \lambda e^{-(\lambda+\mu)s}\, ds \\ &= 1 - \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda+\mu)t}), \end{aligned}$$

or equivalently,

$$P_{00}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0. \tag{27}$$

Now, by (26), $P_{10}(t) = P_{00}(t) - e^{-(\lambda+\mu)t}$, and therefore

$$P_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0. \tag{28}$$

By setting $y = 1$ in the backward equation, or by subtracting $P_{00}(t)$ and $P_{10}(t)$ from one, we conclude that

$$P_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0, \tag{29}$$

and

$$P_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0. \tag{30}$$

From (27)–(30) we see that

$$\lim_{t \to +\infty} P_{xy}(t) = \pi(y), \tag{31}$$

where

$$\pi(0) = \frac{\mu}{\lambda+\mu} \quad \text{and} \quad \pi(1) = \frac{\lambda}{\lambda+\mu}. \tag{32}$$

If π_0 is the initial distribution of the process, then by (27) and (28)

$$\begin{aligned} P(X(t) = 0) &= \pi_0(0)P_{00}(t) + (1 - \pi_0(0))P_{10}(t) \\ &= \frac{\mu}{\lambda+\mu} + \left(\pi_0(0) - \frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t}, \qquad t \geq 0. \end{aligned}$$

Similarly,

$$P(X(t) = 1) = \frac{\lambda}{\lambda+\mu} + \left(\pi_0(1) - \frac{\lambda}{\lambda+\mu}\right) e^{-(\lambda+\mu)t}, \qquad t \geq 0.$$

Thus $P(X(t) = 0)$ and $P(X(t) = 1)$ are independent of t if and only if π_0 is the distribution π given by (32).

3.2.2. Poisson process. Consider a pure birth process $X(t)$, $0 \leq t < \infty$, on the nonnegative integers such that

$$\lambda_x = \lambda > 0, \qquad x \geq 0.$$

Since a pure birth process can move only to the right,

$$P_{xy}(t) = 0, \qquad y < x \text{ and } t \geq 0. \tag{33}$$

Also $P_{xx}(t) = P_x(\tau_1 > t)$ and hence

$$P_{xx}(t) = e^{-\lambda t}, \qquad t \geq 0. \tag{34}$$

The forward equation for $y \neq 0$ is

$$P'_{xy}(t) = \lambda P_{x,y-1}(t) - \lambda P_{xy}(t), \qquad t \geq 0.$$

From (23) we see that

$$P_{xy}(t) = e^{-\lambda t} P_{xy}(0) + \lambda \int_0^t e^{-\lambda(t-s)} P_{x,y-1}(s)\, ds, \qquad t \geq 0.$$

Since $P_{xy}(0) = \delta_{xy}$, we conclude that for $y > x$

(35) $$P_{xy}(t) = \lambda \int_0^t e^{-\lambda(t-s)} P_{x,y-1}(s)\, ds, \qquad t \geq 0.$$

It follows from (34) and (35) that

$$P_{x,x+1}(t) = \lambda \int_0^t e^{-\lambda(t-s)} e^{-\lambda s}\, ds = \lambda e^{-\lambda t} \int_0^t ds = \lambda t e^{-\lambda t}$$

and hence by using (35) once more that

$$P_{x,x+2}(t) = \lambda \int_0^t e^{-\lambda(t-s)} \lambda s e^{-\lambda s}\, ds = \lambda^2 e^{-\lambda t} \int_0^t s\, ds = \frac{(\lambda t)^2}{2} e^{-\lambda t}.$$

By induction

(36) $$P_{xy}(t) = \frac{(\lambda t)^{y-x} e^{-\lambda t}}{(y-x)!}, \qquad 0 \leq x \leq y \text{ and } t \geq 0.$$

Formulas (33) and (36) imply that

(37) $$P_{xy}(t) = P_{0,y-x}(t), \qquad t \geq 0,$$

and that if $X(0) = x$, then $X(t) - x$ has a Poisson distribution with parameter λt.

In general, for $0 \leq s \leq t$, $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t - s)$. For if $0 \leq s \leq t$ and y is a nonnegative integer, then

$$\begin{aligned} P(X(t) - X(s) = y) &= \sum_x P(X(s) = x \text{ and } X(t) = x + y) \\ &= \sum_x P(X(s) = x) P_{x,x+y}(t - s) \\ &= \sum_x P(X(s) = x) P_{0y}(t - s) \\ &= P_{0y}(t - s) \\ &= \frac{(\lambda(t - s))^y e^{-\lambda(t-s)}}{y!}. \end{aligned}$$

If $0 \leq t_1 \leq \cdots \leq t_n$, the random variables

$$X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent. For we observe that if $z_1, \ldots, z_{n-1}$ are arbitrary integers, then by (6) and (37)

$$\begin{aligned}
P(X(t_2) - X(t_1) = z_1, \ldots, X(t_n) - X(t_{n-1}) = z_{n-1})
&= \sum_x P(X(t_1) = x) P_{0z_1}(t_2 - t_1) \cdots P_{0z_{n-1}}(t_n - t_{n-1}) \\
&= P_{0z_1}(t_2 - t_1) \cdots P_{0z_{n-1}}(t_n - t_{n-1}) \\
&= P(X(t_2) - X(t_1) = z_1) \cdots P(X(t_n) - X(t_{n-1}) = z_{n-1}).
\end{aligned}$$

By a *Poisson process with parameter* λ on $0 \le t < \infty$, we mean a pure birth process $X(t)$, $0 \le t < \infty$, having state space $\{0, 1, 2, \ldots\}$, constant birth rate $\lambda_x = \lambda > 0$, and initial value $X(0) = 0$. According to the above discussion the Poisson process satisfies the following three properties:

(i) $X(0) = 0$.
(ii) $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t - s)$ for $0 \le s \le t$.
(iii) $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ are independent for $0 \le t_1 \le t_2 \le \cdots \le t_n$.

The Poisson process can be used to model events occurring in time, such as calls coming into a telephone exchange, customers arriving at a queue, and radioactive disintegrations. Let $X(t)$, $0 \le t < \infty$, denote the number of events occurring in the time interval $(0, t]$. For $0 \le s \le t$ the random variable $X(t) - X(s)$ denotes the number of events in the time interval $(s, t]$. If the waiting times between successive events are independent and exponentially distributed with common parameter λ, then $X(t)$, $0 \le t < \infty$, is a Poisson process and properties (i)–(iii) hold. Property (ii) states that the number of events in any interval has a Poisson distribution. Property (iii) states that the numbers of events in disjoint time intervals are independent. Conversely, if $X(t)$, $0 \le t < \infty$, satisfies properties (i)–(iii), then the waiting times between successive events are independent and exponentially distributed with common parameter λ, and hence $X(t)$ is a pure birth process with constant birth rate λ. This result was proved in Chapter 9 of Volume I, but will not be needed.

Since the Poisson process is a pure birth process starting in state 0, it follows that for $n \ge 1$ the time τ_n of the nth jump equals the time T_n when the process hits state n. When the Poisson process is used to model events occurring in time as described above, the common time $\tau_n = T_n$ is the time when the nth event occurs.

The Poisson process can be used to construct a variety of other processes.

Example 2. Branching process with immigration. Consider the branching process introduced in Example 1. Suppose that new particles immigrate into the system at random times that form a Poisson process with parameter λ and then give rise to succeeding generations as described in Example 1. Find the birth and death rates of this birth and death process.

Suppose there are initially x particles present. Let $\xi_1, \ldots, \xi_x$ be the times at which these particles split apart or disappear, and let η be the first time a new particle enters the system. We interpret the description of the system as implying that η is independent of $\xi_1, \ldots, \xi_x$. Then $\xi_1, \ldots, \xi_x$, η are independent exponentially distributed random variables having respective parameters $q, \ldots, q, \lambda$. Thus

$$\tau_1 = \min(\xi_1, \ldots, \xi_x, \eta)$$

is exponentially distributed with parameter $q_x = xq + \lambda$, and by (19)

$$P(\tau_1 = \eta) = \frac{\lambda}{xq + \lambda}.$$

The event $\{X(\tau_1) = x + 1\}$ occurs if either $\tau_1 = \eta$ or

$$\tau_1 = \min(\xi_1, \ldots, \xi_x)$$

and a particle splits into two new particles at time τ_1. Thus

$$Q_{x,x+1} = \frac{\lambda}{xq + \lambda} + \frac{xq}{xq + \lambda} p.$$

Also,

$$Q_{x,x-1} = \frac{xq}{xq + \lambda}(1 - p).$$

We conclude that

$$\lambda_x = q_x Q_{x,x+1} = xqp + \lambda$$

and

$$\mu_x = q_x Q_{x,x-1} = xq(1 - p).$$

It is also possible to construct a Poisson process with parameter λ on $-\infty < t < \infty$. We first construct two independent Poisson processes $X_1(t)$, $0 \le t < \infty$, and $X_2(t)$, $0 \le t < \infty$, both having parameter λ. We then define $X(t)$, $-\infty < t < \infty$, by

$$X(t) = \begin{cases} -X_1(-t), & t < 0, \\ X_2(t), & t \ge 0. \end{cases}$$

It is easy to show that the process $X(t)$, $-\infty < t < \infty$, so constructed, satisfies the following three properties:

(i) $X(0) = 0$.
(ii) $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t - s)$ for $s \le t$.
(iii) $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent for $t_1 \le t_2 \le \cdots \le t_n$.

3.2.3. Pure birth process. Consider a pure birth process $X(t)$, $0 \le t < \infty$, on $\{0, 1, 2, \ldots\}$. The forward equation (21) reduces to

$$P'_{xy}(t) = \lambda_{y-1}P_{x,y-1}(t) - \lambda_y P_{xy}(t), \qquad t \ge 0. \tag{38}$$

Since the process moves only to the right,

$$P_{xy}(t) = 0, \qquad y < x \text{ and } t \ge 0. \tag{39}$$

It follows from (38) and (39) that

$$P'_{xx}(t) = -\lambda_x P_{xx}(t).$$

Since $P_{xx}(0) = 1$ and $P_{xy}(0) = 0$ for $y > x$, we conclude from (23) that

$$P_{xx}(t) = e^{-\lambda_x t}, \qquad t \ge 0, \tag{40}$$

and

$$P_{xy}(t) = \lambda_{y-1}\int_0^t e^{-\lambda_y(t-s)}P_{x,y-1}(s)\,ds, \qquad y > x \text{ and } t \ge 0. \tag{41}$$

We can use (40) and (41) to find $P_{xy}(t)$ recursively for $y > x$. In particular,

$$P_{x,x+1}(t) = \lambda_x \int_0^t e^{-\lambda_{x+1}(t-s)}e^{-\lambda_x s}\,ds,$$

and hence for $t \ge 0$

$$P_{x,x+1}(t) = \begin{cases} \dfrac{\lambda_x}{\lambda_{x+1} - \lambda_x}\left(e^{-\lambda_x t} - e^{-\lambda_{x+1} t}\right), & \lambda_{x+1} \ne \lambda_x, \\ \lambda_x t e^{-\lambda_x t}, & \lambda_{x+1} = \lambda_x. \end{cases} \tag{42}$$

Example 3. Linear birth process. Consider a pure birth process on $\{0, 1, 2, \ldots\}$ having birth rates

$$\lambda_x = x\lambda, \qquad x \ge 0,$$

for some positive constant λ (the branching process with $p = 1$ is of this form). Find $P_{xy}(t)$.

As noted above, $P_{xy}(t) = 0$ for $y < x$ and

$$P_{xx}(t) = e^{-\lambda_x t} = e^{-x\lambda t}.$$

We see from (42) that

$$P_{x,x+1}(t) = xe^{-x\lambda t}(1 - e^{-\lambda t}).$$

To compute $P_{x,x+2}(t)$ we set $y = x + 2$ in (41) and obtain

$$\begin{aligned} P_{x,x+2}(t) &= (x + 1)x\lambda \int_0^t e^{-(x+2)\lambda(t-s)} e^{-x\lambda s}(1 - e^{-\lambda s})\, ds \\ &= (x + 1)x\lambda e^{-(x+2)\lambda t} \int_0^t e^{2\lambda s}(1 - e^{-\lambda s})\, ds \\ &= (x + 1)x\lambda e^{-(x+2)\lambda t} \int_0^t e^{\lambda s}(e^{\lambda s} - 1)\, ds \\ &= (x + 1)x\lambda e^{-(x+2)\lambda t} \frac{(e^{\lambda t} - 1)^2}{2\lambda} \\ &= \binom{x + 1}{2} e^{-x\lambda t}(1 - e^{-\lambda t})^2. \end{aligned}$$

It is left as an exercise for the reader to show by induction that

$$(43) \qquad P_{xy}(t) = \binom{y - 1}{y - x} e^{-x\lambda t}(1 - e^{-\lambda t})^{y-x}, \qquad y \geq x \text{ and } t \geq 0.$$

3.2.4. Infinite server queue. Suppose that customers arrive for service according to a Poisson process with parameter λ and that each customer starts being served immediately upon his arrival (i.e., that there are an infinite number of servers). Suppose that the service times are independent and exponentially distributed with parameter μ. Let $X(t)$, $0 \leq t < \infty$, denote the number of customers in the process of being served at time t. This birth and death process, called an *infinite server queue*, is a special case of the branching process with immigration corresponding to $q = \mu$ and $p = 0$. We conclude that $\lambda_x = \lambda$ and $\mu_x = x\mu$, $x \geq 0$. The transition function $P_{xy}(t)$ will now be obtained by a probabilistic argument.

Let $Y(t)$ denote the number of customers who arrive in the time interval $(0, t]$. An interesting and useful result about the Poisson process is that conditioned on $Y(t) = k$, the times when the first k customers arrive are distributed as k independent random variables each uniformly distributed on $(0, t]$. In order to see intuitively why this should be true, consider an arbitrary partition $0 = t_0 < t_1 < \cdots < t_m = t$ of $[0, t]$ and let X_i denote the number of customers arriving between time t_{i-1} and time t_i. Then $X_1, \ldots, X_m$ are independent random variables having Poisson distributions with respective parameters

$$\lambda(t_1 - t_0), \ldots, \lambda(t_m - t_{m-1}),$$

and $X_1 + \cdots + X_m = Y(t)$ has a Poisson distribution with parameter λt. Thus for $x_1, \ldots, x_m$ nonnegative integers adding up to k,

$$\begin{aligned}
P(X_1 = x_1, \ldots, X_m = x_m \mid Y(t) = k) \\
&= P(X_1 = x_1, \ldots, X_m = x_m \mid X_1 + \cdots + X_m = k) \\
&= \frac{P(X_1 = x_1, \ldots, X_m = x_m, X_1 + \cdots + X_m = k)}{P(X_1 + \cdots + X_m = k)} \\
&= \frac{P(X_1 = x_1, \ldots, X_m = x_m)}{P(X_1 + \cdots + X_m = k)} \\
&= \frac{\displaystyle\prod_{i=1}^{m} \frac{[\lambda(t_i - t_{i-1})]^{x_i} e^{-\lambda(t_i - t_{i-1})}}{x_i!}}{\displaystyle\frac{(\lambda t)^k e^{-\lambda t}}{k!}} \\
&= \frac{k!}{\prod_{i=1}^m x_i!} \prod_{i=1}^{m} \left(\frac{t_i - t_{i-1}}{t}\right)^{x_i}.
\end{aligned}$$

But these multinomial probabilities are just those that would be obtained by choosing the k arrival times independently and uniformly distributed over $(0, t]$.

If a customer arrives at time $s \in (0, t]$, the probability that he is still in the process of being served at time t is $e^{-\mu(t-s)}$. Thus if a customer arrives at a time chosen uniformly from $(0, t]$, the probability that he is still in the process of being served at time t is

$$p_t = \frac{1}{t} \int_0^t e^{-\mu(t-s)}\, ds = \frac{1 - e^{-\mu t}}{\mu t}.$$

Let $X_1(t)$ denote the number of customers arriving in $(0, t]$ that are still in the process of being served at time t. It follows from the results of the previous two paragraphs that the conditional distribution of $X_1(t)$ given that $Y(t) = k$ is a binomial distribution with parameters k and p_t, i.e., that

$$P(X_1(t) = n \mid Y(t) = k) = \binom{k}{n} p_t^n (1 - p_t)^{k-n}.$$

Since $Y(t)$ has a Poisson distribution with parameter λt, we conclude that

$$\begin{aligned}
P(X_1(t) = n) &= \sum_{k=n}^{\infty} P(Y(t) = k, X_1(t) = n) \\
&= \sum_{k=n}^{\infty} P(Y(t) = k) P(X_1(t) = n \mid Y(t) = k) \\
&= \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{k!}{n!\,(k-n)!} p_t^n (1 - p_t)^{k-n} \\
&= \frac{(\lambda t p_t)^n e^{-\lambda t}}{n!} \sum_{k=n}^{\infty} \frac{(\lambda t(1 - p_t))^{k-n}}{(k-n)!} \\
&= \frac{(\lambda t p_t)^n e^{-\lambda t}}{n!} \sum_{m=0}^{\infty} \frac{(\lambda t(1 - p_t))^m}{m!} \\
&= \frac{(\lambda t p_t)^n e^{-\lambda t}}{n!} e^{\lambda t(1-p_t)} \\
&= \frac{(\lambda t p_t)^n e^{-\lambda t p_t}}{n!}.
\end{aligned}$$

Thus $X_1(t)$ has a Poisson distribution with parameter

$$\lambda t p_t = \frac{\lambda}{\mu}(1 - e^{-\mu t}).$$

Let x denote the number of customers present initially and let $X_2(t)$ denote the number of these customers still in the process of being served at time t. Then $X_2(t)$ is independent of $X_1(t)$ and has a binomial distribution with parameters x and $e^{-\mu t}$. Since $X(t) = X_1(t) + X_2(t)$, we conclude that

$$P_{xy}(t) = P_x(X(t) = y) = \sum_{k=0}^{\min(x,y)} P_x(X_2(t) = k) P(X_1(t) = y - k).$$

Therefore

$$(44) \qquad P_{xy}(t) = \sum_{k=0}^{\min(x,y)} \Bigg[\binom{x}{k} e^{-k\mu t}(1 - e^{-\mu t})^{x-k} \times \frac{\left(\frac{\lambda}{\mu}(1 - e^{-\mu t})\right)^{y-k}}{(y-k)!} \exp\left(-\frac{\lambda}{\mu}(1 - e^{-\mu t})\right)\Bigg].$$

As $t \to \infty$, $e^{-\mu t} \to 0$, and hence the terms in the sum in (44) all approach 0 except the term corresponding to $k = 0$. Consequently

$$(45) \qquad \lim_{t \to \infty} P_{xy}(t) = \frac{(\lambda/\mu)^y e^{-\lambda/\mu}}{y!}.$$

3.3. Properties of a Markov pure jump process

In this section we will discuss the notions of recurrence, transience, irreducibility, stationary distributions, and positive recurrence of Markov pure jump processes. The results will be described briefly and without proofs, as they are very similar to those for the Markov chains discussed in Chapters 1 and 2. In Section 3.3.1 we apply these results to birth and death processes.

Let $X(t)$, $0 \leq t < \infty$, be a Markov pure jump process having state space $\mathscr{S}$. For $y \in \mathscr{S}$ and $X(0) \neq y$, the first visit to y takes place at time

$$T_y = \min (t \geq 0 : X(t) = y).$$

If $X(0) = y$, then $\min (t \geq 0 : X(t) = y) = 0$. A more useful random variable in this case is the time T_y of the first return to y after the process leaves y. Both cases are covered by setting

$$T_y = \min (t \geq \tau_1 : X(t) = y).$$

Here τ_1 is the time of the first jump. If $\tau_1 = \infty$ or $X(t) \neq y$ for all $t \geq \tau_1$, we set $T_y = \infty$.

If x is an absorbing state, set $\rho_{xy} = \delta_{xy}$; and if x is a non-absorbing state, set

$$\rho_{xy} = P_x(T_y < \infty).$$

A state $y \in \mathscr{S}$ is called *recurrent* if $\rho_{yy} = 1$ and *transient* if $\rho_{yy} < 1$. The process is said to be a *recurrent process* if all of its states are recurrent and a *transient process* if all of its states are transient. The process is called *irreducible* if $\rho_{xy} > 0$ for all choices of $x \in \mathscr{S}$ and $y \in \mathscr{S}$.

The function $P(x, y) = Q_{xy}$, $x \in \mathscr{S}$ and $y \in \mathscr{S}$, is the transition function of a Markov chain called the *embedded chain*. The quantities ρ_{xy} for this Markov chain are equal to the corresponding quantities for the Markov pure jump process. This relationship shows that results of Chapter 1 involving only the numbers ρ_{xy} are also valid in the present context. In particular, an irreducible process is either a recurrent process or a transient process. It is recurrent if and only if the embedded chain is recurrent.

If $\pi(x)$, $x \in \mathscr{S}$, are nonnegative numbers summing to one and if

$$\sum_x \pi(x)P_{xy}(t) = \pi(y), \qquad y \in \mathscr{S} \text{ and } t \geq 0, \tag{46}$$

then π is called a *stationary distribution*. If $X(0)$ has a stationary distribution π for its initial distribution, then

$$P(X(t) = y) = \sum_x \pi(x)P_{xy}(t) = \pi(y),$$

so that $X(t)$ has distribution π for all $t \geq 0$.

If (46) holds and $\mathscr{S}$ is finite, we can differentiate this equation and obtain

$$\sum_x \pi(x)P'_{xy}(t) = 0, \qquad y \in \mathscr{S} \text{ and } t \geq 0. \tag{47}$$

In particular, by setting $t = 0$ in (47), we conclude from (11) that

$$\sum_x \pi(x)q_{xy} = 0, \qquad y \in \mathscr{S}. \tag{48}$$

It can be shown that (47) and (48) are valid even if $\mathscr{S}$ is an infinite set. Suppose, conversely, that (48) holds. If $\mathscr{S}$ is finite we conclude from the backward equation (14) that

$$\begin{aligned}\frac{d}{dt}\sum_x \pi(x)P_{xy}(t) &= \sum_x \pi(x)P'_{xy}(t)\\ &= \sum_x \pi(x)\left(\sum_z q_{xz}P_{zy}(t)\right)\\ &= \sum_z \left(\sum_x \pi(x)q_{xz}\right) P_{zy}(t)\\ &= 0.\end{aligned}$$

Thus

$$\sum_x \pi(x)P_{xy}(t)$$

is a constant in t and the constant value is given by

$$\sum_x \pi(x)P_{xy}(0) = \sum_x \pi(x)\delta_{xy} = \pi(y).$$

Consequently (46) holds. This conclusion is also valid if $\mathscr{S}$ is infinite, but the proof is much more complicated. In summary, (46) holds if and only if (48) holds.

A non-absorbing recurrent state x is called *positive recurrent* or *null recurrent* according as the *mean return time* $m_x = E_x(T_x)$ is finite or infinite. An absorbing state is considered to be positive recurrent. The process is said to be a *positive recurrent process* if all its states are positive recurrent and a *null recurrent process* if all its states are null recurrent. An irreducible recurrent process must be either a null recurrent process or a positive recurrent process. It can be shown that a stationary distribution is concentrated on the positive recurrent states, and hence a process that is transient or null recurrent has no stationary distribution. An irreducible positive recurrent process has a unique stationary distribution π, which, unless $\mathscr{S}$ consists of a single necessarily absorbing state, is given by

$$\pi(x) = \frac{1}{q_x m_x}, \qquad x \in \mathscr{S}. \tag{49}$$

Formula (49) is intuitively reasonable. For in a large time interval $[0, t]$, the process makes about t/m_x visits to x and the average time in x per visit is $1/q_x$. Thus the total time spent in state x during the time interval $[0, t]$ should be about $t/(q_x m_x)$ and the proportion of time spent in state x should be about $1/(q_x m_x)$. This argument can be made rigorous by using the strong law of large numbers as was done in Section 2.3.

Markov pure jump processes do not have any periodicities, and, in particular, for an irreducible positive recurrent process having stationary distribution π,

$$\lim_{t \to \infty} P_{xy}(t) = \pi(y), \qquad x, y \in \mathscr{S}. \tag{50}$$

If $X(0)$ has the initial distribution $\pi_0(x)$, $x \in \mathscr{S}$, then

$$P(X(t) = y) = \sum_x \pi_0(x) P_{xy}(t),$$

which, by (50) and the bounded convergence theorem, converges to

$$\sum_x \pi_0(x)\pi(y) = \pi(y)$$

as $t \to \infty$. In other words

$$\lim_{t \to \infty} P(X(t) = y) = \pi(y),$$

and hence the distribution of $X(t)$ converges to the stationary distribution π regardless of the initial distribution of the process.

3.3.1. Applications to birth and death processes. Let $X(t)$, $0 \le t < \infty$, be an irreducible birth and death process on $\{0, 1, 2, \ldots\}$. The process is transient if and only if the embedded birth and death chain having transition function $P(x, y) = Q_{xy}$, $x \ge 0$ and $y \ge 0$, is transient. From (18) in this chapter and the results in Section 1.7, we conclude that the birth and death process is transient if and only if

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} < \infty. \tag{51}$$

Equation (48) for a stationary distribution π becomes

$$\pi(1)\mu_1 - \pi(0)\lambda_0 = 0,$$

(52)

$$\pi(y + 1)\mu_{y+1} - \pi(y)\lambda_y = \pi(y)\mu_y - \pi(y - 1)\lambda_{y-1}, \qquad y \ge 1.$$

It follows easily from (52) and induction that

$$\pi(y + 1)\mu_{y+1} - \pi(y)\lambda_y = 0, \qquad y \ge 0,$$

and hence that

$$\pi(y+1) = \frac{\lambda_y}{\mu_{y+1}}\pi(y), \qquad y \geq 0.$$

Consequently,

$$\pi(x) = \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x}\pi(0), \qquad x \geq 1. \tag{53}$$

Set

$$\pi_x = \begin{cases} 1, & x = 0, \\ \dfrac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x}, & x \geq 1. \end{cases} \tag{54}$$

Then (53) can be written as

$$\pi(x) = \pi_x \pi(0), \qquad x \geq 0. \tag{55}$$

Conversely, (52) follows from (54) and (55).

Suppose now that $\sum_x \pi_x < \infty$, i.e., that

$$\sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} < \infty. \tag{56}$$

We conclude from (55) that the birth and death process has a unique stationary distribution π, given by

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{\infty} \pi_y}, \qquad x \geq 0. \tag{57}$$

If (56) fails to hold, the birth and death process has no stationary distribution.

In summary, an irreducible birth and death process on $\{0, 1, 2, \ldots\}$ is transient if and only if (51) holds, positive recurrent if and only if (56) holds, and null recurrent if and only if (51) and (56) each fail to hold, i.e., if and only if

$$\sum_{x=1}^{\infty} \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x} = \infty \quad \text{and} \quad \sum_{x=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{x-1}}{\mu_1 \cdots \mu_x} = \infty. \tag{58}$$

An irreducible birth and death process having finite state space $\{0, 1, \ldots, d\}$ is necessarily positive recurrent. It has a unique stationary distribution given by

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{d} \pi_y}, \qquad 0 \leq x \leq d, \tag{59}$$

where π_x, $0 \leq x \leq d$, is given by (54).

Example 4. Show that the infinite server queue is positive recurrent and find its stationary distribution.

The infinite server queue has state space $\{0, 1, 2, \ldots\}$ and birth and death rates

$$\lambda_x = \lambda \qquad \text{and} \qquad \mu_x = x\mu, \qquad x \geq 0.$$

This process is clearly irreducible. It follows from (54) that

$$\pi_x = \frac{\lambda^x}{x!\,\mu^x} = \frac{(\lambda/\mu)^x}{x!}, \qquad x \geq 0.$$

Since

$$\sum_{x=0}^{\infty} \frac{(\lambda/\mu)^x}{x!} = e^{\lambda/\mu}$$

is finite, we conclude that the process is positive recurrent and has the unique stationary distribution π given by

$$\pi(x) = \frac{(\lambda/\mu)^x}{x!} e^{-\lambda/\mu}, \qquad x \geq 0, \tag{60}$$

which we note is a Poisson distribution with parameter λ/μ. We also note that (50) holds for this process, a direct consequence of (45) and (60).

Example 5. *N* server queue. Suppose customers arrive according to a Poisson process with parameter $\lambda > 0$. They are served by N servers, where N is a finite positive number. Suppose the service times are exponentially distributed with parameter μ and that whenever there are more than N customers waiting for service the excess customers form a queue and wait until their turn at one of the N servers. This process is a birth and death process on $\{0, 1, 2, \ldots\}$ with birth rates $\lambda_x = \lambda$, $x \geq 0$, and death rates

$$\mu_x = \begin{cases} x\mu, & 0 \leq x < N, \\ N\mu, & x \geq N. \end{cases}$$

Determine when this process is transient, null recurrent, and positive recurrent; and find the stationary distribution in the positive recurrent case.

Condition (51) for transience reduces to

$$\sum_{x=0}^{\infty} \left(\frac{N\mu}{\lambda}\right)^x < \infty.$$

Thus the N server queue is transient if and only if $N\mu < \lambda$. Condition (56) for positive recurrence reduces to

$$\sum_{x=0}^{\infty} \left(\frac{\lambda}{N\mu}\right)^x < \infty.$$

The N server queue is therefore positive recurrent if and only if $\lambda < N\mu$. Consequently the N server queue is null recurrent if and only if $\lambda = N\mu$. These results naturally are similar to those for the 1 server queue discussed in Chapters 1 and 2.

In the positive recurrent case,

$$\pi_x = \begin{cases} \dfrac{(\lambda/\mu)^x}{x!}, & 0 \le x < N, \\ \dfrac{(\lambda/\mu)^x}{N!\,N^{x-N}}, & x \ge N. \end{cases}$$

Set

$$K = \sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{N-1} \frac{(\lambda/\mu)^x}{x!} + \frac{(\lambda/\mu)^N}{N!}\left(1 - \frac{\lambda}{N\mu}\right)^{-1}.$$

We conclude that if $\lambda < N\mu$, the stationary distribution is given by

$$\pi(x) = \begin{cases} \dfrac{1}{K}\dfrac{(\lambda/\mu)^x}{x!}, & 0 \le x < N, \\ \dfrac{1}{K}\dfrac{(\lambda/\mu)^x}{N!\,N^{x-N}}, & x \ge N. \end{cases}$$

Exercises

1 Find the transition function of the two-state birth and death process by solving the forward equation.

2 Consider a birth and death process having three states 0, 1, and 2, and birth and death rates such that $\lambda_0 = \mu_2$. Use the forward equation to find $P_{0y}(t)$, $y = 0, 1, 2$.

Exercises 3–8 all refer to events occurring in time according to a Poisson process with parameter λ on $0 \le t < \infty$. Here $X(t)$ denotes the number of events that occur in the time interval $(0, t]$.

3 Find the conditional probability that there are m events in the first s units of time, given that there are n events in the first t units of time, where $0 \le m \le n$ and $0 \le s \le t$.

4 Let T_m denote the time to the mth event. Find the distribution function of T_m. *Hint:* $\{T_m \le t\} = \{X(t) \ge m\}$.

5 Find the density of the random variable T_m in Exercise 4. *Hint:* First consider some specific cases, say, $m = 1, 2, 3$.

6 Find $P(T_1 \le s \mid X(t) = n)$ for $0 \le s \le t$ and n a positive integer.

7 Let T be a random variable that is independent of the times when events occur. Suppose that T has an exponential density with parameter v:

$$f_T(t) = \begin{cases} ve^{-vt}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Find the distribution of $X(T)$, which is the number of events occurring by time T. *Hint:* Use the formulas

$$P(X(T) = n) = \int_0^\infty f_T(t) P(X(T) = n \mid T = t)\, dt$$

and

$$P(X(T) = n \mid T = t) = P(X(t) = n).$$

8 Solve the previous exercise if T has the gamma density with parameters α and v:

$$f_T(t) = \begin{cases} v^\alpha t^{\alpha-1} e^{-vt}/\Gamma(\alpha), & t > 0, \\ 0, & t \le 0. \end{cases}$$

9 Verify Equation (43).

10 Consider a pure death process on $\{0, 1, 2, \ldots\}$.
(a) Write the forward equation.
(b) Find $P_{xx}(t)$.
(c) Solve for $P_{xy}(t)$ in terms of $P_{x,y+1}(t)$.
(d) Find $P_{x,x-1}(t)$.
(e) Show that if $\mu_x = x\mu$, $x \ge 0$, for some constant μ, then

$$P_{xy}(t) = \binom{x}{y} (e^{-\mu t})^y (1 - e^{-\mu t})^{x-y}, \qquad 0 \le y \le x.$$

11 Let $X(t)$, $t \ge 0$, be the infinite server queue and suppose that initially there are x customers present. Compute the mean and variance of $X(t)$.

12 Consider a birth and death process $X(t)$, $t \ge 0$, such as the branching process, that has state space $\{0, 1, 2, \ldots\}$ and birth and death rates of the form

$$\lambda_x = x\lambda \quad \text{and} \quad \mu_x = x\mu, \qquad x \ge 0,$$

where λ and μ are nonnegative constants. Set

$$m_x(t) = E_x(X(t)) = \sum_{y=0}^{\infty} y P_{xy}(t).$$

(a) Write the forward equation for the process.
(b) Use the forward equation to show that $m_x'(t) = (\lambda - \mu) m_x(t)$.
(c) Conclude that

$$m_x(t) = x e^{(\lambda-\mu)t}.$$

13 Let $X(t)$, $t \ge 0$, be as in Exercise 12. Set

$$s_x(t) = E_x(X^2(t)) = \sum_{y=0}^{\infty} y^2 P_{xy}(t).$$

(a) Use the forward equation to show that

$$s_x'(t) = 2(\lambda - \mu)s_x(t) + (\lambda + \mu)m_x(t).$$

(b) Find $s_x(t)$.
(c) Find Var $X(t)$ under the condition that $X(0) = x$.

14 Suppose d particles are distributed into two boxes. A particle in box 0 remains in that box for a random length of time that is exponentially distributed with parameter λ before going to box 1. A particle in box 1 remains there for an amount of time that is exponentially distributed with parameter μ before going to box 0. The particles act independently of each other. Let $X(t)$ denote the number of particles in box 1 at time $t \geq 0$. Then $X(t)$, $t \geq 0$, is a birth and death process on $\{0, \ldots, d\}$.
(a) Find the birth and death rates.
(b) Find $P_{xd}(t)$. *Hint:* Let $X_i(t)$, $i = 0$ or 1, denote the number of particles in box 1 at time $t \geq 0$ that started in box i at time 0, so that $X(t) = X_0(t) + X_1(t)$. If $X(0) = x$, then $X_0(t)$ and $X_1(t)$ are independent and binomially distributed with parameters defined in terms of x and the transition function of the two-state birth and death process.
(c) Find $E_x(X(t))$.

15 Consider the infinite server queue discussed in Section 3.2.4. Let $X_1(t)$ and $X_2(t)$ be as defined there. Suppose that the initial distribution π_0 is a Poisson distribution with parameter ν.
(a) Use the formula

$$P(X_2(t) = k) = \sum_{x=k}^{\infty} \pi_0(x)P_x(X_2(t) = k)$$

to show that $X_2(t)$ has a Poisson distribution with parameter $\nu e^{-\mu t}$.
(b) Use the result of (a) to show that $X(t) = X_1(t) + X_2(t)$ has a Poisson distribution with parameter

$$\frac{\lambda}{\mu} + \left(\nu - \frac{\lambda}{\mu}\right)e^{-\mu t}.$$

(c) Conclude that $X(t)$ has the same distribution as $X(0)$ if and only if $\nu = \lambda/\mu$.

16 Consider a birth and death process on the nonnegative integers whose death rates are given by $\mu_x = x$, $x \geq 0$. Determine whether the process is transient, null recurrent, or positive recurrent if the birth rates are

(a) $\lambda_x = x + 1$, $x \geq 0$;
(b) $\lambda_x = x + 2$, $x \geq 0$.

17 Let $X(t)$, $t \geq 0$, be a birth and death process on the nonnegative integers such that $\lambda_x > 0$ and $\mu_x > 0$ for $x \geq 1$. Set $\gamma_0 = 1$ and

$$\gamma_x = \frac{\mu_1 \cdots \mu_x}{\lambda_1 \cdots \lambda_x}, \qquad x \geq 1.$$

(a) Show that if $\sum_{y=0}^{\infty} \gamma_y = \infty$, then $\rho_{x0} = 1$, $x \geq 1$.
(b) Show that if $\sum_{y=0}^{\infty} \gamma_y < \infty$, then

$$\rho_{x0} = \frac{\sum_{y=x}^{\infty} \gamma_y}{\sum_{y=0}^{\infty} \gamma_y}, \qquad x \geq 1.$$

Hint: Use Exercise 26 of Chapter 1.

18 Let $X(t)$, $t \geq 0$, be a single server queue ($N = 1$ in Example 5).
(a) Show that if $\mu \geq \lambda > 0$, then $\rho_{x0} = 1$, $x \geq 1$.
(b) Show that if $\mu < \lambda$, then

$$\rho_{x0} = (\mu/\lambda)^x, \qquad x \geq 1.$$

19 Consider the branching process introduced in Example 1. Use Exercise 17 to show that if $p \leq \frac{1}{2}$, then $\rho_{x0} = 1$ for all x and that if $p > \frac{1}{2}$, then

$$\rho_{x0} = \left(\frac{1-p}{p}\right)^x, \qquad x \geq 1.$$

20 Find the stationary distribution for the process in Exercise 14.

21 Suppose d machines are subject to failures and repairs. The failure times are exponentially distributed with parameter μ, and the repair times are exponentially distributed with parameter λ. Let $X(t)$ denote the number of machines that are in satisfactory order at time t. If there is only one repairman, then under appropriate reasonable assumptions, $X(t)$, $t \geq 0$, is a birth and death process on $\{0, 1, \ldots, d\}$ with birth rates $\lambda_x = \lambda$, $0 \leq x < d$, and death rates $\mu_x = x\mu$, $0 \leq x \leq d$. Find the stationary distribution for this process.

22 Consider a positive recurrent irreducible birth and death process on $\mathscr{S} = \{0, 1, 2, \ldots\}$, and let $X(0)$ have the stationary distribution π for its initial distribution. Then $X(t)$ has distribution π for all $t \geq 0$. The quantities

$$E\lambda_{X(t)} = \sum_{x=0}^{\infty} \lambda_x \pi(x) \qquad \text{and} \qquad E\mu_{X(t)} = \sum_{x=0}^{\infty} \mu_x \pi(x)$$

can be interpreted, respectively, as the average birth rate and the average death rate of the process.
(a) Show that the average birth rate equals the average death rate.
(b) What does (a) imply about a positive recurrent N server queue?