4

Second Order Processes

A *stochastic process* can be defined quite generally as any collection of random variables $X(t)$, $t \in T$, defined on a common probability space, where T is a subset of $(-\infty, \infty)$ and is usually thought of as the time parameter set. The process is called a *continuous parameter process* if T is an interval having positive length and a *discrete parameter process* if T is a subset of the integers. If $T = \{0, 1, 2, \ldots\}$ it is usual to denote the process by X_n, $n \geq 0$. The Markov chains discussed in Chapters 1 and 2 are discrete parameter processes, while the pure jump processes discussed in Chapter 3 are continuous parameter processes.

A stochastic process $X(t)$, $t \in T$, is called a *second order process* if $EX^2(t) < \infty$ for each $t \in T$. Second order processes and random variables defined in terms of them by various "linear" operations including integration and differentiation are the subjects of this and the next two chapters. We will obtain formulas for the means, variances, and covariances of such random variables.

We will consider continuous parameter processes almost exclusively in these three chapters. Since no new techniques are needed for handling the analogous results for discrete parameter processes, little would be gained by treating such processes in detail.

4.1. Mean and covariance functions

Let $X(t)$, $t \in T$, be a second order process. The *mean function* $\mu_X(t)$, $t \in T$, of the process is defined by

$$\mu_X(t) = EX(t).$$

The *covariance function* $r_X(s, t)$, $s \in T$ and $t \in T$, is defined by

$$r_X(s, t) = \operatorname{cov}(X(s), X(t)) = EX(s)X(t) - EX(s)EX(t).$$

This function is also called the auto-covariance function to distinguish it from the cross-covariance function which will be defined later. Since

Var $X(t) = \operatorname{cov}(X(t), X(t))$, the variance of $X(t)$ can be expressed in terms of the covariance function as

$$\text{(1)} \qquad \operatorname{Var} X(t) = r_X(t, t), \qquad t \in T.$$

By a *finite linear combination* of the random variables $X(t)$, $t \in T$, we mean a random variable of the form

$$\sum_{j=1}^{n} b_j X(t_j),$$

where n is a positive integer, $t_1, \ldots, t_n$ are points in T, and $b_1, \ldots, b_n$ are real constants. The covariance between two such finite linear combinations is given by

$$\begin{aligned} \operatorname{cov}\left(\sum_{i=1}^{m} a_i X(s_i), \sum_{j=1}^{n} b_j X(t_j)\right) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \operatorname{cov}(X(s_i), X(t_j)) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j r_X(s_i, t_j). \end{aligned}$$

In particular,

$$\text{(2)} \qquad \operatorname{Var}\left(\sum_{j=1}^{n} b_j X(t_j)\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j r_X(t_i, t_j).$$

It follows immediately from the definition of the covariance function that it is *symmetric* in s and t, i.e., that

$$\text{(3)} \qquad r_X(s, t) = r_X(t, s), \qquad s, t \in T.$$

It is also *nonnegative definite*. That is, if n is a positive integer, $t_1, \ldots, t_n$ are in T, and $b_1, \ldots, b_n$ are real numbers, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j r_X(t_i, t_j) \geq 0.$$

This is an immediate consequence of (2).

We say that $X(t)$, $-\infty < t < \infty$, is a *second order stationary process* if for every number τ the second order process $Y(t)$, $-\infty < t < \infty$, defined by

$$Y(t) = X(t + \tau), \qquad -\infty < t < \infty,$$

has the same mean and covariance functions as the $X(t)$ process. It is left as an exercise for the reader to show that this is the case if and only if $\mu_X(t)$ is independent of t and $r_X(s, t)$ depends only on the difference between s and t.

Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process. Then

$$\mu_X(t) = \mu_X, \qquad -\infty < t < \infty,$$

where μ_X denotes the common mean of the random variables $X(t)$, $-\infty < t < \infty$. Since $r_X(s, t)$ depends only on the difference between s and t,

$$r_X(s, t) = r_X(0, t - s), \qquad -\infty < s, t < \infty. \tag{4}$$

The function $r_X(t)$, $-\infty < t < \infty$, defined by

$$r_X(t) = r_X(0, t), \qquad -\infty < t < \infty, \tag{5}$$

is also called the *covariance function* (or auto-covariance function) of the process. We see from (4) and (5) that

$$r_X(s, t) = r_X(t - s), \qquad -\infty < s, t < \infty.$$

It follows from (3) that $r_X(t)$ is symmetric about the origin, i.e., that

$$r_X(-t) = r_X(t), \qquad -\infty < t < \infty.$$

The random variables $X(t)$, $-\infty < t < \infty$, have a common variance given by

$$\operatorname{Var} X(t) = r_X(0), \qquad -\infty < t < \infty.$$

Recall Schwarz's inequality, which asserts that if X and Y are random variables having finite second moment, then $(EXY)^2 \leq EX^2EY^2$. Applying Schwarz's inequality to the random variables $X - EX$ and $Y - EY$, we see that $(\operatorname{cov}(X, Y))^2 \leq \operatorname{Var} X \operatorname{Var} Y$.

It follows from this last inequality that

$$|\operatorname{cov}(X(0), X(t))| \leq \sqrt{\operatorname{Var} X(0) \operatorname{Var} X(t)},$$

and hence that

$$|r_X(t)| \leq r_X(0), \qquad -\infty < t < \infty.$$

If $r_X(0) > 0$, the correlation between $X(s)$ and $X(s + t)$ is given independently of s by

$$\frac{\operatorname{cov}(X(s), X(s + t))}{\sqrt{\operatorname{Var} X(s)}\sqrt{\operatorname{Var} X(t)}} = \frac{r_X(t)}{r_X(0)}, \qquad -\infty < s, t < \infty.$$

Example 1. Let Z_1 and Z_2 be independent normally distributed random variables each having mean 0 and variance σ^2. Let λ be a real constant and set $X(t) = Z_1 \cos \lambda t + Z_2 \sin \lambda t$, $-\infty < t < \infty$. Find the mean and covariance functions of $X(t)$, $-\infty < t < \infty$, and show that it is a second order stationary process.

We observe first that

$$\mu_X(t) = EZ_1 \cos \lambda t + EZ_2 \sin \lambda t = 0, \qquad -\infty < t < \infty.$$

Next,

$$\begin{aligned} r_X(s, t) &= \operatorname{cov}(X(s), X(t)) \\ &= EX(s)X(t) - EX(s)EX(t) \\ &= EX(s)X(t) \\ &= E(Z_1 \cos \lambda s + Z_2 \sin \lambda s)(Z_1 \cos \lambda t + Z_2 \sin \lambda t) \\ &= EZ_1^2 \cos \lambda s \cos \lambda t + EZ_2^2 \sin \lambda s \sin \lambda t \\ &= \sigma^2(\cos \lambda s \cos \lambda t + \sin \lambda s \sin \lambda t) \\ &= \sigma^2 \cos \lambda(t - s). \end{aligned}$$

This shows that $X(t)$, $-\infty < t < \infty$, is a second order stationary process having mean zero and covariance function

$$r_X(t) = \sigma^2 \cos \lambda t, \qquad -\infty < t < \infty.$$

Example 2. Consider a two-state birth and death process as discussed in Section 3.2.1. It follows from that discussion that the transition probabilities of the process are given by

$$\begin{aligned} P_{00}(t) &= 1 - P_{01}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0, \\ P_{11}(t) &= 1 - P_{10}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t}, \qquad t \geq 0, \end{aligned} \tag{6}$$

where λ and μ are positive constants. The process has the stationary distribution defined by

$$\pi(0) = \frac{\mu}{\lambda + \mu} \quad \text{and} \quad \pi(1) = \frac{\lambda}{\lambda + \mu}. \tag{7}$$

In Chapter 3 we discussed birth and death processes defined on $0 \leq t < \infty$. Actually in the positive recurrent case it is possible to construct a corresponding process on $-\infty < t < \infty$ having the stationary distribution determined by (7). This process will be such that

$$P(X(t) = 0) = \frac{\mu}{\lambda + \mu} \quad \text{and} \quad P(X(t) = 1) = \frac{\lambda}{\lambda + \mu}, \qquad -\infty < t < \infty, \tag{8}$$

and such that the Markov property

$$P(X(t) = y \mid X(s) = x) = P_{xy}(t - s), \qquad -\infty < s \leq t < \infty, \tag{9}$$

holds, where $P_{xy}(t)$, $t \geq 0$, is given by (6). We will show that such a process is a second order stationary process and find its mean and covariance functions.

The mean function is given by

$$\begin{aligned}\mu_X(t) &= EX(t)\\ &= 0 \cdot P(X(t) = 0) + 1 \cdot P(X(t) = 1) = \frac{\lambda}{\lambda + \mu}.\end{aligned}$$

Let $-\infty < s \leq t < \infty$. Then

$$\begin{aligned}EX(s)X(t) &= P(X(s) = 1 \quad \text{and} \quad X(t) = 1)\\ &= P(X(s) = 1)P(X(t) = 1 \mid X(s) = 1)\\ &= P(X(s) = 1)P_{11}(t - s)\\ &= \frac{\lambda}{\lambda + \mu}\left(\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)(t-s)}\right)\\ &= \left(\frac{\lambda}{\lambda + \mu}\right)^2 + \frac{\lambda\mu}{(\lambda + \mu)^2} e^{-(\lambda+\mu)(t-s)}.\end{aligned}$$

It follows that

$$r_X(s, t) = \frac{\lambda\mu}{(\lambda + \mu)^2} e^{-(\lambda+\mu)(t-s)}, \qquad -\infty < s \leq t < \infty.$$

By symmetry we see that

$$r_X(s, t) = \frac{\lambda\mu}{(\lambda + \mu)^2} e^{-(\lambda+\mu)|t-s|}, \qquad -\infty < s, t < \infty.$$

Thus $X(t)$, $-\infty < t < \infty$, is a second order stationary process having mean $\lambda/(\lambda + \mu)$ and covariance function

$$r_X(t) = \frac{\lambda\mu}{(\lambda + \mu)^2} e^{-(\lambda+\mu)|t|}, \qquad -\infty < t < \infty.$$

Other interesting examples of second order processes can be obtained from Poisson processes.

Example 3. Consider a Poisson process $X(t)$, $-\infty < t < \infty$, with parameter λ (see Section 3.2.2). This process satisfies the following properties:

(i) $X(0) = 0$.
(ii) $X(t) - X(s)$ has a Poisson distribution with mean $\lambda(t - s)$ for $s \leq t$.
(iii) $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ are independent for $t_1 \leq t_2 \leq \cdots \leq t_n$.

We will now find the mean and covariance function of a process $X(t)$, $-\infty < t < \infty$, satisfying (i)–(iii). It follows from properties (i) and (ii) that $X(t)$ has a Poisson distribution with mean λt for $t \geq 0$ and $-X(t)$ has a Poisson distribution with mean $\lambda(-t)$ for $t < 0$. Thus

$$\mu_X(t) = \lambda t, \qquad -\infty < t < \infty.$$

Since the variance of a Poisson distribution equals its mean, we see that $X(t)$ has finite second moment and that $\operatorname{Var} X(t) = \lambda|t|$. Let $0 \leq s \leq t$. Then

$$\operatorname{cov}(X(s), X(s)) = \operatorname{Var} X(s) = \lambda s.$$

It follows from properties (i) and (iii) that $X(s)$ and $X(t) - X(s)$ are independent, and hence

$$\operatorname{cov}(X(s), X(t) - X(s)) = 0.$$

Thus

$$\begin{aligned}\operatorname{cov}(X(s), X(t)) &= \operatorname{cov}(X(s), X(s) + X(t) - X(s)) \\ &= \operatorname{cov}(X(s), X(s)) + \operatorname{cov}(X(s), X(t) - X(s)) \\ &= \lambda s.\end{aligned}$$

If $s < 0$ and $t > 0$, then by properties (i) and (iii) the random variables $X(s)$ and $X(t)$ are independent, and hence

$$\operatorname{cov}(X(s), X(t)) = 0.$$

The other cases can be handled similarly. We find in general that

$$r_X(s, t) = \begin{cases} \lambda \min(|s|, |t|), & st \geq 0, \\ 0, & st < 0. \end{cases} \tag{10}$$

The process from Example 3 is not a second order stationary process. In the next example we will consider a closely related process which is a second order stationary process.

Example 4. Let $X(t)$, $-\infty < t < \infty$, be a Poisson process with parameter λ. Set

$$Y(t) = X(t + 1) - X(t), \qquad -\infty < t < \infty.$$

Find the mean and covariance function of the $Y(t)$ process, and show that it is a second order stationary process.

Since $EX(t) = \lambda t$, it follows that

$$\begin{aligned}EY(t) &= E(X(t + 1) - X(t)) \\ &= \lambda(t + 1) - \lambda t = \lambda,\end{aligned}$$

so the random variables $Y(t)$ have common mean λ. To compute the covariance function of the $Y(t)$ process, we observe that if $|t - s| \geq 1$, then the random variables $X(s + 1) - X(s)$ and $X(t + 1) - X(t)$ are independent by property (iii). Consequently,

$$r_Y(s, t) = 0 \qquad \text{for} \qquad |t - s| \geq 1.$$

Suppose $s \leq t < s + 1$. Then

$$\begin{aligned} \operatorname{cov}(Y(s), Y(t)) &= \operatorname{cov}(X(s + 1) - X(s), X(t + 1) - X(t)) \\ &= \operatorname{cov}(X(t) - X(s) + X(s + 1) - X(t), X(s + 1) \\ &\qquad -X(t) + X(t + 1) - X(s + 1)). \end{aligned}$$

It follows from property (iii) and the assumptions on s and t that

$$\operatorname{cov}(X(t) - X(s), X(s + 1) - X(t)) = 0,$$

$$\operatorname{cov}(X(t) - X(s), X(t + 1) - X(s + 1)) = 0,$$

and

$$\operatorname{cov}(X(s + 1) - X(t), X(t + 1) - X(s + 1)) = 0.$$

By property (ii)

$$\begin{aligned} \operatorname{cov}(X(s + 1) - X(t), X(s + 1) - X(t)) &= \operatorname{Var}(X(s + 1) - X(t)) \\ &= \lambda(s + 1 - t). \end{aligned}$$

Thus

$$\operatorname{cov}(Y(s), Y(t)) = \lambda(s + 1 - t).$$

By using symmetry we find in general that

$$r_Y(s, t) = \begin{cases} \lambda(1 - |t - s|), & |t - s| < 1, \\ 0, & |t - s| \geq 1. \end{cases}$$

Thus $Y(t)$, $-\infty < t < \infty$, is a second order stationary process having mean λ and covariance function

$$r_Y(t) = \begin{cases} \lambda(1 - |t|), & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

In Figure 1 we have graphed the covariance function for three different second order stationary processes. These covariance functions are special cases of those found in Examples 1, 2, and 4 respectively. In each case $r_X(0) = 1$ and hence $r_X(t)$ is equal to the correlation between $X(0)$ and $X(t)$. In the top curve of Figure 1 we see that the correlation oscillates between -1 and 1. In the middle curve the correlation decreases exponentially fast as $|t| \to \infty$. In the bottom curve the correlation decreases linearly to zero as $|t|$ increases from 0 to 1 and remains zero for all larger values of $|t|$.

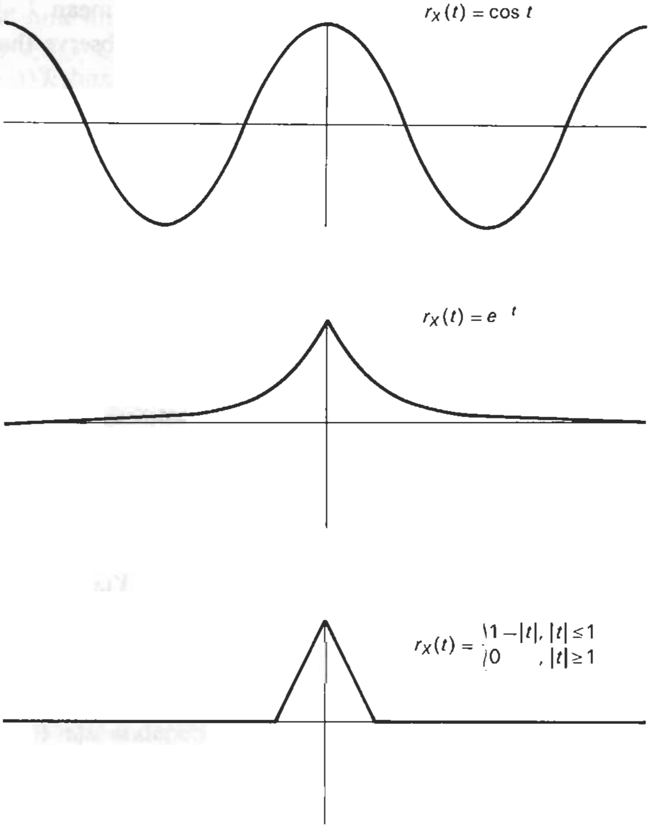

Figure 1

Consider two second order processes $X(t)$, $t \in T$, and $Y(t)$, $t \in T$. Their *cross-covariance function* is defined as

$$r_{XY}(s, t) = \operatorname{cov}(X(s), Y(t)), \qquad s, t \in T.$$

Clearly

$$r_{XY}(s, t) = r_{YX}(t, s)$$

and

$$r_{XX}(s, t) = r_X(s, t).$$

The cross-covariance function can be used to find the covariance function of the sum of two processes. Indeed,

$$\begin{aligned} r_{X+Y}(s, t) &= \operatorname{cov}(X(s) + Y(s), X(t) + Y(t)) \\ &= r_{XX}(s, t) + r_{XY}(s, t) + r_{YX}(s, t) + r_{YY}(s, t), \end{aligned}$$

which can be rewritten as

(11) $$r_{X+Y}(s, t) = r_X(s, t) + r_{XY}(s, t) + r_{YX}(s, t) + r_Y(s, t).$$

In the important case when the cross-covariance function vanishes, (11) reduces to

$$r_{X+Y}(s, t) = r_X(s, t) + r_Y(s, t). \tag{12}$$

These formulas are readily extended to sums of any finite number of processes. Consider in particular n second order stationary processes $X_1(t)$, $-\infty < t < \infty, \ldots, X_n(t)$, $-\infty < t < \infty$, whose cross-covariance functions all vanish. Then their sum

$$X(t) = X_1(t) + \cdots + X_n(t), \qquad -\infty < t < \infty,$$

is a second order stationary process such that

$$\mu_X = \sum_{k=1}^{n} \mu_{X_k} \tag{13}$$

and

$$r_X(t) = \sum_{k=1}^{n} r_{X_k}(t), \qquad -\infty < t < \infty. \tag{14}$$

Example 5. Let $Z_{11}, Z_{12}, Z_{21}, Z_{22}, \ldots, Z_{n1}, Z_{n2}$ be $2n$ independent normally distributed random variables each having mean zero and such that

$$\operatorname{Var} Z_{k1} = \operatorname{Var} Z_{k2} = \sigma_k^2, \qquad k = 1, \ldots, n.$$

Let $\lambda_1, \ldots, \lambda_n$ be real constants and set

$$X(t) = \sum_{k=1}^{n} (Z_{k1} \cos \lambda_k t + Z_{k2} \sin \lambda_k t), \qquad -\infty < t < \infty.$$

Find the mean and covariance functions of $X(t)$, $-\infty < t < \infty$.

Set

$$X_k(t) = Z_{k1} \cos \lambda_k t + Z_{k2} \sin \lambda_k t.$$

It follows from the independence of the Z's that the cross-covariance function between $X_i(t)$ and $X_j(t)$ vanishes for $i \neq j$. Thus by using (13) and (14) together with the results of Example 1, we see that $X(t)$, $-\infty < t < \infty$, is a second order stationary process having mean zero and covariance function

$$r_X(t) = \sum_{k=1}^{n} \sigma_k^2 \cos \lambda_k t, \qquad -\infty < t < \infty. \tag{15}$$

4.2. Gaussian processes

A stochastic process $X(t)$, $t \in T$, is called a *Gaussian process* if every finite linear combination of the random variables $X(t)$, $t \in T$, is normally distributed. (In this context constant random variables are regarded as normally distributed with zero variance.) Gaussian processes are also called normal processes, and normally distributed random variables are sometimes said to have a Gaussian distribution. If $X(t)$, $t \in T$, is a Gaussian process, then for each $t \in T$, $X(t)$ is normally distributed and, in particular, $EX^2(t) < \infty$. Thus a Gaussian process is necessarily a second order process. Gaussian processes have many nice theoretical properties that do not hold for second order processes in general. They are also widely used in applications, especially in engineering and in the physical sciences.

Example 6. Show that the process $X(t)$, $-\infty < t < \infty$, from Example 1 is a Gaussian process.

To verify that this is a Gaussian process, we let n be a positive integer and choose real numbers $t_1, \ldots, t_n$ and $a_1, \ldots, a_n$. Now

$$X(t) = Z_1 \cos \lambda t + Z_2 \sin \lambda t,$$

where Z_1 and Z_2 are independent and normally distributed. Thus

$$\begin{aligned} &a_1 X(t_1) + \cdots + a_n X(t_n) \\ &\quad = Z_1(a_1 \cos \lambda t_1 + \cdots + a_n \cos \lambda t_n) + Z_2(a_1 \sin \lambda t_1 + \cdots + a_n \sin \lambda t_n) \end{aligned}$$

is a linear combination of independent normally distributed random variables and therefore is itself normally distributed.

It is left as an exercise for the reader to show that the process in Example 5 is also a Gaussian process.

Two stochastic processes $X(t)$, $t \in T$, and $Y(t)$, $t \in T$, are said to have the same joint distribution functions if for every positive integer n and every choice of $t_1, \ldots, t_n$, all in T, the random variables

$$X(t_1), \ldots, X(t_n)$$

have the same joint distribution function as the random variables

$$Y(t_1), \ldots, Y(t_n).$$

One of the most useful properties of Gaussian processes is that if two such processes have the same mean and covariance functions, then they also have the same joint distribution functions. We omit the proof of this result. To see that the Gaussian assumption is necessary, observe that the process defined in Exercise 15 has the same mean and covariance functions as that from Example 1 with $\sigma^2 = 1$ but not the same joint distribution functions.

The mean and covariance functions can also be used to find the higher moments of a Gaussian process.

Example 7. Let $X(t)$, $t \in T$, be a Gaussian process having zero means. Find $EX^4(t)$ in terms of the covariance function of the process.

We recall that if X is normally distributed with mean 0 and variance σ^2, then $EX^4 = 3\sigma^4$. Since $X(t)$ is normally distributed with mean 0 and variance $r_X(t, t)$, we see that

$$EX^4(t) = 3(r_X(t, t))^2.$$

Let n be a positive integer and let $X_1, \ldots, X_n$ be random variables. They are said to have a joint *normal* (or *Gaussian*) *distribution* if

$$a_1 X_1 + \cdots + a_n X_n$$

is normally distributed for every choice of the constants $a_1, \ldots, a_n$. A stochastic process $X(t)$, $t \in T$, is a Gaussian process if and only if for every positive integer n and every choice of $t_1, \ldots, t_n$ all in T, the random variables $X(t_1), \ldots, X(t_n)$ have a joint normal distribution.

Let $X_1, \ldots, X_n$ be random variables having a joint normal distribution and a density f with respect to integration on R^n. (Such a density exists if and only if the covariance matrix of $X_1, \ldots, X_n$ has nonzero determinant.) It can be shown that f is necessarily of the form

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det \Sigma)^{1/2}} \exp\left[-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right], \tag{16}$$

where Σ is the covariance matrix

$$\Sigma = \begin{bmatrix} \operatorname{cov}(X_1, X_1) \cdots \operatorname{cov}(X_1, X_n) \\ \vdots \qquad\qquad \vdots \\ \operatorname{cov}(X_n, X_1) \cdots \operatorname{cov}(X_n, X_n) \end{bmatrix},$$

$\mathbf{x}$ and $\boldsymbol{\mu}$ are the vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

and $'$ denotes matrix transpose. In particular, if $n = 2$, then (16) can be written as

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left[-\frac{Q(x_1, x_2)}{2}\right], \tag{17}$$

where

$$Q(x_1, x_2) = \frac{1}{(1 - \rho^2)} \times \left[\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2\right].$$

Here μ_1 and σ_1^2 denote the mean and variance of X_1, μ_2 and σ_2^2 denote the mean and variance of X_2, and ρ denotes the correlation between X_1 and X_2. One can also use (16) to show that the conditional expectation of X_n given $X_1, \ldots, X_{n-1}$ is a linear function of these $n - 1$ random variables, i.e., that

$$E[X_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = a + b_1 x_1 + \cdots + b_{n-1} x_{n-1}$$

for suitable constants $a, b_1, \ldots, b_{n-1}$.

A stochastic process $X(t)$, $-\infty < t < \infty$, is said to be *strictly stationary* if for every number τ the stochastic process $Y(t)$, $-\infty < t < \infty$, defined by

$$Y(t) = X(t + \tau), \qquad -\infty < t < \infty,$$

has the same joint distribution functions as the $X(t)$ process. A strictly stationary process need not have finite second moments and hence need not be a second order process. It is clear, however, that if a strictly stationary process does have finite second moments, then it is a second order stationary process. The converse is not true in general. It is left as an exercise for the reader to demonstrate by an example that a second order stationary process need not be strictly stationary.

Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process which is also a Gaussian process. Then this process is necessarily strictly stationary. For if τ is any real number, then the $Y(t)$ process defined by $Y(t) = X(t + \tau)$, $-\infty < t < \infty$, is a Gaussian process having the same mean and covariance functions as the $X(t)$ process. It therefore has the same joint distribution functions as the $X(t)$ process.

Since the processes in Examples 1 and 5 are Gaussian and second order stationary, they are also strictly stationary. The second order stationary processes from Examples 2 and 4 are not Gaussian, but it can be shown that they too are strictly stationary.

4.3. The Wiener process

It has long been known from microscopic observations that particles suspended in a liquid are in a state of constant highly irregular motion. It gradually came to be realized that the cause of this motion is the bombardment of the particles by the smaller invisible molecules of the liquid. Such motion is called "Brownian motion," named after one of the first scientists to study it carefully.

Many mathematical models for this physical process have been proposed. We will now describe one such model. Let the location of a particle be described by a Cartesian coordinate system whose origin is the location of the particle at time $t = 0$. Then the three coordinates of the position of the particle vary independently, each according to a stochastic process $W(t)$, $-\infty < t < \infty$, satisfying the following properties:

(i) $W(0) = 0$.
(ii) $W(t) - W(s)$ has a normal distribution with mean 0 and variance $\sigma^2(t - s)$ for $s \leq t$.
(iii) $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$ are independent for $t_1 \leq t_2 \leq \cdots \leq t_n$.

Here σ^2 is some positive constant.

Property (i) follows from our choice of the coordinate system. Properties (ii) and (iii) are plausible if the motion is caused by an extremely large number of unrelated and individually negligible collisions which have no more tendency to move the particle in one direction than in the opposite direction. In particular, the central limit theorem makes it reasonable to suppose that the increments $W(t) - W(s)$ are normally distributed.

This model was initiated, in a different form, by Albert Einstein in 1905. He related the parameter σ^2 to various physical parameters including Avogadro's number. Estimation of σ^2 together with other measurements in a scientific experiment conducted shortly thereafter led to an estimate of Avogadro's number that is within 19 percent of the presently accepted value. Einstein's work and its experimental confirmation gave added evidence for the atomic basis of matter, which was still being questioned at the turn of the century.

Although the mathematical model is reasonable and fits the experimental data quite well, it has certain theoretical deficiencies that will be discussed in Section 5.3. In Chapter 6 we will discuss another mathematical model for the physical process.

A stochastic process $W(t)$, $-\infty < t < \infty$, satisfying properties (i)–(iii) is called the *Wiener process* with parameter σ^2. Mathematicians Norbert Wiener and Paul Lévy developed much of the theory, and the process is also known as the Wiener-Lévy process and as Brownian motion. The Wiener process is usually assumed to satisfy an additional property involving "continuity of the sample functions," which we will discuss in Section 5.1.2.

It follows immediately from the properties of the Wiener process that the random variables $W(t)$ all have mean 0 and that

$$(18)\qquad E(W(t_2) - W(t_1))(W(t_4) - W(t_3)) = 0, \qquad t_1 \le t_2 \le t_3 \le t_4.$$

The covariance function of the process is

$$(19)\qquad r_W(s, t) = \begin{cases} \sigma^2 \min(|s|, |t|), & st > 0, \\ 0, & st \le 0. \end{cases}$$

The proof of (19) is virtually identical to that of Formula (10) for the covariance function of the Poisson process defined in Example 3. It is left as an exercise for the reader to show that

$$(20)\qquad E(W(s) - W(a))(W(t) - W(a)) = \sigma^2 \min(s - a, t - a), \qquad s \ge a \text{ and } t \ge a.$$

The Wiener process is a Gaussian process. In other words, if $t_1 \le \cdots \le t_n$ and $b_1, \ldots, b_n$ are real constants, the random variable

$$b_1 W(t_1) + \cdots + b_n W(t_n)$$

is normally distributed. In proving this result we can assume, with no loss of generality, that one of the numbers $t_1, \ldots, t_n$, say t_k, equals zero. Then each of the random variables $W(t_1), \ldots, W(t_n)$ is a linear combination of the increments $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$. Indeed, $W(t_k) = 0$,

$$W(t_j) = (W(t_{k+1}) - W(t_k)) + \cdots + (W(t_j) - W(t_{j-1})), \qquad k < j \le n,$$

and

$$W(t_j) = (W(t_j) - W(t_{j+1})) + \cdots + (W(t_{k-1}) - W(t_k)), \qquad 1 \le j < k.$$

Thus $b_1 W(t_1) + \cdots + b_n W(t_n)$ can also be written as a linear combination of the increments $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$. Now these increments are independent and normally distributed. Thus any linear combination of them, in particular,

$$b_1 W(t_1) + \cdots + b_n W(t_n)$$

is normally distributed.

Exercises

1 Let $X(t)$, $-\infty < t < \infty$, be a second order process. Show that it is a second order stationary process if and only if $\mu_X(t)$ is independent of t and $r_X(s, t)$ depends only on the difference between s and t.

2 Let $X(t)$, $-\infty < t < \infty$, be a second order process. Show that it is a second order stationary process if and only if $EX(s)$ and $EX(s)X(s+t)$ are both independent of s.

3 Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process and set $Y(t) = X(t+1) - X(t)$, $-\infty < t < \infty$. Show that the $Y(t)$ process is a second order stationary process having zero means and covariance function

$$r_Y(t) = 2r_X(t) - r_X(t-1) - r_X(t+1).$$

4 Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process.
(a) Show that

$$\operatorname{Var}(X(s+t) - X(s)) = 2(r_X(0) - r_X(t)).$$

(b) Show that for $M > 0$

$$P(|X(s+t) - X(s)| \geq M) \leq \frac{2}{M^2}(r_X(0) - r_X(t)).$$

5 Let $X(t)$, $-\infty < t < \infty$, be a Poisson process with parameter λ and set $Y(t) = X(t) - tX(1)$, $0 \leq t \leq 1$. Find the mean and covariance functions of the $Y(t)$ process.

6 Let $U_1, \ldots, U_n$ be independent random variables, each uniformly distributed on $(0, 1)$. Let $\psi(t, x)$, $0 \leq t \leq 1$ and $0 \leq x \leq 1$, be defined by

$$\psi(t, x) = \begin{cases} 1, & x \leq t, \\ 0, & x > t. \end{cases}$$

Then

$$X(t) = \frac{1}{n} \sum_{k=1}^{n} \psi(t, U_k), \qquad 0 \leq t \leq 1,$$

is the *empirical distribution function* of $U_1, \ldots, U_n$. Compute the mean and covariance functions of the $X(t)$ process.

7 Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process having covariance function $r_X(t)$, $-\infty < t < \infty$. Set $Y(t) = X(t+1)$, $-\infty < t < \infty$. Find the cross-covariance function between the $X(t)$ process and the $Y(t)$ process.

8 Let R and Θ be independent random variables such that Θ is uniformly distributed on $[0, 2\pi)$ and R has the density

$$f_R(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-r^2/2\sigma^2}, & 0 < r < \infty, \\ 0, & r \leq 0, \end{cases}$$

where σ is a positive constant. It follows by using the change of variable formula involving Jacobians that $R \cos \Theta$ and $R \sin \Theta$ are independent random variables, each normally distributed with mean 0 and variance σ^2. Let λ be a positive constant and set

$$X(t) = R \cos (\lambda t + \Theta), \qquad -\infty < t < \infty.$$

Show that the $X(t)$ process is a second order stationary process having mean zero and covariance function

$$r_X(t) = \sigma^2 \cos \lambda t, \qquad -\infty < t < \infty.$$

9 Let $R_1, \ldots, R_n, \Theta_1, \ldots, \Theta_n$ be independent random variables such that the Θ's are uniformly distributed on $[0, 2\pi)$ and R_k has the density

$$f_{R_k}(r) = \begin{cases} \dfrac{r}{\sigma_k^2} e^{-r^2/2\sigma_k^2}, & 0 < r < \infty, \\ 0, & r \le 0, \end{cases}$$

where $\sigma_1, \ldots, \sigma_n$ are positive constants. Let $\lambda_1, \ldots, \lambda_n$ be positive constants and set

$$X(t) = \sum_{k=1}^{n} R_k \cos (\lambda_k t + \Theta_k).$$

Show that the $X(t)$ process is a second order stationary process having mean zero and covariance function

$$r_X(t) = \sum_{k=1}^{n} \sigma_k^2 \cos \lambda_k t.$$

✓**10** Show that the $X(t)$ process in Example 5 is a Gaussian process.

11 Show that the $X(t)$ process in Exercise 9 is a Gaussian process.

12 Let $X(t)$, $-\infty < t < \infty$, be a Gaussian process and let f and g be functions from $(-\infty, \infty)$ to $(-\infty, \infty)$. Show that $Y(t) = f(t)X(g(t))$, $-\infty < t < \infty$, is a Gaussian process and find its mean and covariance functions.

13 Let $X(t)$, $-\infty < t < \infty$, be a Gaussian process having mean zero and set $Y(t) = X^2(t)$, $-\infty < t < \infty$.
(a) Find the mean and covariance functions of the $Y(t)$ process.
(b) Show that if the $X(t)$ process is a second order stationary process, then so is the $Y(t)$ process.

14 Let X_1 and X_2 have the joint density given by (17).
(a) Find the conditional density of X_2 given $X_1 = x_1$.
(b) Find the conditional expectation of X_2 given $X_1 = x_1$.

15 Let Z_1 and Z_2 be independent and identically distributed random variables taking on the values -1 and 1 each with probability $1/2$. Show that $X(t) = Z_1 \cos \lambda t + Z_2 \sin \lambda t$, $-\infty < t < \infty$, is a second order stationary process which is not strictly stationary.

In the remaining problems $W(t)$, $-\infty < t < \infty$, is the Wiener process with parameter σ^2.

16 Verify Formula (20).

17 Find the distribution of $W(1) + \cdots + W(n)$ for a positive integer n. *Hint:* Use the formulas

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

and

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

18 Set

$$X(t) = \frac{W(t+\varepsilon) - W(t)}{\varepsilon}, \qquad -\infty < t < \infty,$$

where ε is a positive constant. Show that the $X(t)$ process is a stationary Gaussian process having covariance function

$$r_X(t) = \begin{cases} \dfrac{\sigma^2}{\varepsilon}\left(1 - \dfrac{|t|}{\varepsilon}\right), & |t| < \varepsilon, \\ 0, & |t| \geq \varepsilon. \end{cases}$$

✓ **19** Set

$$X(t) = e^{-\alpha t} W(e^{2\alpha t}), \qquad -\infty < t < \infty,$$

where α is a positive constant. Show that the $X(t)$ process is a stationary Gaussian process having covariance function

$$r_X(t) = \sigma^2 e^{-\alpha|t|}, \qquad -\infty < t < \infty.$$

20 Find the mean and covariance functions of the following processes:

(a) $X(t) = (W(t))^2, \quad t \geq 0;$

(b) $X(t) = tW(1/t), \quad t > 0;$

(c) $X(t) = c^{-1}W(c^2 t), \quad t \geq 0;$

(d) $X(t) = W(t) - tW(1), \quad 0 \leq t \leq 1.$