ENVIRONMENTAL SPATIOTEMPORAL MAPPING AND GROUND WATER FLOW MODELLING USING THE BME AND ST METHODS

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in the partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Environmental Sciences and Engineering, School of Public Health.

Chapel Hill
1999

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Refereed publications in scientific journals and conference proceedings produced from this work:

Bogaert, P., M.L. Serre and G. Christakos, "Efficient computational BME analysis of non-Gaussian data in terms of transformation functions", *Proceedings of 5th Annual Conference of the International Association for Mathematical Geology* , Norway, In Press, (1999)

Choi, K-M., G. Christakos, and M.L. Serre. "Recent developments in vectorial and multi-point Bayesian maximum entropy analysis." *Proceedings of the 4rth Annual Conference of the International Association for Mathematical Geology -Vol. 1,* Buccianti A., G. Nardi and R. Potenza (eds.), 91-96, De Frede Editore, Naples, Italy (1998).

Christakos, G., D.T. Hristopulos, and M.L. Serre, "BME studies of stochastic differential equations representing physical laws - Part I" *Proceedings of 5th Annual Conference of the International Association for Mathematical Geology* , Norway, In Press, (1999)

Christakos, G. and M. L. Serre. "An analysis of the Temperature-Death Rate Association in the State of North Carolina" Journal of Exposure Analysis and Environmental Epidemiology, (1999), In review.

Hristopulos, D. T., G. Christakos and M. L. Serre. "Numerical Implementation of the Space Transformation Approach for Solving the Three-Dimensional Flow Equation." SIAM Journal on Scientific Computing, **20**, 619-647 (1999).

Serre, M. L., P. Bogaert and G. Christakos. "Latest Computational Results in Spatiotemporal Prediction Using the Bayesian Maximum Entropy Method." *Proceedings of the 4rth Annual Conference of the International Association for Mathematical Geology -Vol. 1,* Buccianti A., G. Nardi and R. Potenza (eds.), 117-122, De Frede Editore, Naples, Italy (1998).

Serre, M. L. and G. Christakos. Modern geostatistics: computational BME analysis in the light of uncertain physical knowledge - the Equus Beds study, *Journal of Stochatic Environmental Research and Risk Analysis,* **13** (1), 1-26 (1999)

Serre, M. L., Christakos and D.T. Hristopulos, "BME studies of stochastic differential equations representing physical laws - Part II" *Proceedings of 5th Annual Conference of the International Association for Mathematical Geology* , Norway, In Press, (1999)

Presentations, short course and technical session chairman at national and international conferences produced from this work:

Christakos, G. and M. L. Serre. "Short Course in Spatiotemporal Mapping and Analysis." *The 1998 Annual Conference of the International Association for Mathematical Geology*, Ischia Island, Italy, October 5-8 (1998).

Serre, M. L. "The Space Transform and the Stochastic Flow Equation Problem." Superfund Student meeting at UNC-Chapel Hill, May, (1996)

Serre, M. L. "Stochastic analysis of Groundwater Flow Using the Space Transform Method." *Superfund Basic Research Program: A decade of Improving Health Through Multi-Disciplinary Research*, February 23-26, Abstract #39 (1997).

Serre, M. L. and P. Bogaert, Chairman and reviewer for the "Technical Session on New avenues in spatio-temporal estimation", *Proceedings of the Fourth Annual Conference of the International Association for Mathematical Geology*, De Frede Editore, Napoli, Chapter I (1998).

Serre, M. L., G. Christakos and D. Hristopulos. "Using the space transform method in geosciences." *Fourth SIAM Conference on Mathematical and Computational Issues in the Geosciences*, June 16-17, (1997)

ABSTRACT

Marc Laurent Serre

Environmental Spatiotemporal Mapping and Ground Water Flow Modelling Using the BME and ST Methods

(Under the direction of George Christakos)

Modelling the natural processes that shape our environment is a difficult task, both numerically and theoretically. The numerical solution of physical laws describing natural processes, such the flow and transport of pollutants in the subsurface, lead to solutions that are very expensive, even on modern day computers. Additionally natural processes exhibit high randomness in space and time and are described by a wide source of knowledge (including sparse observations, uncertain measurements, empirical laws, etc..), but classical methods of Geostatistics lack the theoretical underpinnings to accurately incorporate these types of knowledge. Therefore there is a need for novel approaches that are numerically efficient and theoretically sound. In this work two methods are implemented to respond to these challenges; the Space Transformation (ST) method used to solve efficiently the ground water flow equation, and the Bayesian Maximum Entropy (BME) method for the spatiotemporal mapping of natural processes using uncertain information. The ST approach allows to transform the complicated set of PDEs describing three-dimensional flow into a set of one-dimensional equations (i.e. Ordinary Differential Equations, or ODEs) which are much easier to solve. The method is implemented to solve the ground water flow equation for three dimensional flow domains, and numerical results show that accurate solutions are calculated and that an efficient parallel implementation is possible because the computational work is divided into independent tasks . The BME method offers a rigorous framework for spatiotemporal analysis and mapping. Due to its epistomological background and mathematical rigor, the BME approach offers a considerable flexibility in incorporating various sources of physical knowledge. In this work efficient formulations of the BME approach are derived, and the method is implemented for several mapping situations of practical interest. In particular the BME implementations presented can incorporate uncertain measurements of interval or probabilistic types. Numerical comparisons show that BME is substantially more accurate than classical kriging methods because it accounts rigorously for uncertain information. Several case studies are presented, which demonstrate the practical usefulness of BME mapping in water resources and environmental health. Finally a general framework is presented that permits to incorporate physical laws in the BME analysis. This framework unifies the goal of the ST method within the spatiotemporal mapping context, and leads to considerable gains in accuracy and flexibility

ACKNOWLEDGMENTS

[Acknowledgements to go here]

[I would like to thank my advisor, and my committee members.]

[I would like to thank the Computational Science Graduate Fellowship.]

[I would like to thank all those who supported me to make this possible, particularly my wife, my parents, and Param and Mayura.]

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