Statistics 654 Homework 1

1. (Ross) Let X be a random variable taking values in the finite interval $[0, c]$.

   (a) Show that $EX \leq c$ and $EX^2 \leq cEX$.

   (b) Use these inequalities to show that

   $$\text{Var}(X) \leq c^2[u(1-u)] \quad \text{where} \quad u = \frac{EX}{c} \in [0,1].$$

   (c) Use the result of part (b) to show that $\text{Var}(X) \leq c^2/4$.

   (d) Use the result in (c) to bound the variance of a random variable X taking values in an interval $[a, b]$ with $-\infty < a < b < \infty$.

2. Let $\Phi(x)$ and $\phi(x)$ be the cumulative distribution function and density, respectively, of the standard normal distribution. In this problem, you are asked to find an useful approximation to $1 - \Phi(x)$ when x is large. Note that for $x > 0$,

   $$1 - \Phi(x) = \Phi(-x) = \int_{-\infty}^{-x} \frac{1}{t} \cdot t\,\phi(t)\,dt$$

   (a) Integrate the last term above by parts, and show that $1 - \Phi(x) \leq \frac{1}{x}\phi(x)$ for $x > 0$.

   (b) Apply the same steps above to the integral resulting from the integration by parts in (a), and show that
   $1 - \Phi(x) \geq (\frac{1}{x} - \frac{1}{x^3})\phi(x)$.

   (c) Conclude that $(1 - \Phi(x)) \sim \phi(x)/x$. (If $a(x)$ and $b(x)$ are two functions of $x \geq 0$, $a \sim b$ means that $a(x)/b(x) \to 1$ as $x \to \infty$.)

3. Show that if f is a bounded function and $X \sim \text{Poiss}(\lambda)$ then $E[\lambda f(X+1) - Xf(X)] = 0$.

4. Let $X_1, \ldots, X_n$ be jointly distributed random variables. Show that

   $$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j).$$

What can be said if the random variables X_i are independent?

5. Let $X_1, \cdots, X_n$ be iid positive random variables such that $EX_k = a$ and $E(X_k^{-1}) = b$ are finite. For $1 \le k \le n$, let $S_k = X_1 + \cdots + X_k$ and let $V_k = X_k/S_n$.

(a) Show that $E(S_m^{-1}) < \infty$ for $m = 1, \cdots, n$. [Hint: Compare S_m^{-1} and X_1^{-1}.]

(b) Argue informally that $V_1, \cdots, V_n$ have the same distribution. Thus each V_i has the same expected value.

(c) Conclude from part (b) that $E(S_m/S_n) = m/n$ for $1 \le m \le n$. [Hint: First find an equation for the expected ratio in terms of EV_1.]

(d) Show that $E(S_n/S_m) = 1 + (n-m)\, a\, E(S_m^{-1})$ for $1 \le m \le n$.

(e) Verify the inequality $x + x^{-1} \ge 2$ for $x > 0$.

(f) Use the inequality of part (e) with $x = c(S_m/S_n)$, and an appropriate constant c, to show that $E(S_n/S_m) \ge n/m$ for $1 \le m \le n$.

6. Show that if $X \sim f$ and $g(\cdot)$ is non-negative, then $Eg(X) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$.
[Hint: Recall that $EX = \int_0^\infty P(X > t)\, dt$ if $X \ge 0$.]

7. Let X be a continuous random variable with density f_X. Find the density of $Y = |X|$ in terms of f_X.

8. [Ross]. Show that if X has a standard Cauchy distribution then X^{-1} is also Cauchy.

9. Using integration by parts, show that the gamma function $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$ satisfies the relation $\Gamma(t+1) = t\Gamma(t)$ for $t > 0$.

10. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of numbers. Show carefully that

$$\min\{a_i\} + \min\{b_i\} \;\le\; \min\{a_i + b_i\} \;\le\; \min\{a_i\} + \max\{b_i\}$$