Solutions to Hw1

1 (a) Since $0 \le X \le c$, we have $\mathbb{E}X \le \mathbb{E}c = c$
Since $0 \le X \le c$, we have $0 \le X^2 \le cX$. Hence, $\mathbb{E}X^2 \le c\mathbb{E}X$.

(b) $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 \le c\mathbb{E}X - (\mathbb{E}X)^2 = c^2[u(1-u)]$

(c) $\text{Var}(X) \le c^2[u(1-u)] \le c^2 \times \frac{1}{4} = c^2/4$ by calculus.

(d) Let Y be a random variable suth that $Y = X - a$, then $Y \in [0, b-a]$. By part (c) we have $\text{Var}(X) = \text{Var}(Y) \le (b-a)^2/4$.

2 (a)

$$\begin{aligned}
1 - \Phi(x) &= \Phi(-x) = \int_{-\infty}^{-x} \frac{1}{t} t\phi(t)dt \\
&= (-\frac{1}{\sqrt{2\pi}}) \int_{-\infty}^{-x} \frac{1}{t}(-t\exp(-\frac{t^2}{2}))dt \qquad \text{(integration by parts)} \\
&= (-\frac{1}{\sqrt{2\pi}})[-\frac{1}{x}\exp(-\frac{x^2}{2}) + \int_{-\infty}^{-x} \frac{1}{t^2}\exp(-\frac{t^2}{2})dt] \\
&= \frac{1}{x}\phi(x) - \int_{-\infty}^{-x} \frac{1}{t^2}\phi(t)dt \\
&\le \frac{1}{x}\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) = \frac{1}{x}\phi(x)
\end{aligned}$$

(b) Following from part(a),

$$\begin{aligned}
1 - \Phi(x) &= \frac{1}{x}\phi(x) - \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}t^2}\exp(-\frac{t^2}{2})dt \\
&= \frac{1}{x}\phi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} \frac{1}{t^3}(-t\exp(-\frac{t^2}{2}))dt \\
&= \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x) + 3\int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})\frac{1}{t^4}dt \\
&\ge (\frac{1}{x} - \frac{1}{x^3})\phi(x)
\end{aligned}$$

(c) From part(a) and (b), we have $1 - \frac{1}{x^2} \le \frac{1-\Phi(x)}{\phi(x)/x} \le 1$. Then, $\frac{1-\Phi(x)}{\phi(x)/x} \to 1$ as $x \to \infty$.

(3) Since f is bounded, $\mathbb{E}\lambda f(X+1)$ and $\mathbb{E}Xf(X)$ exist. Then,

$$
\begin{aligned}
& \mathbb{E}(\lambda f(X+1) - Xf(X)) \\
= & \sum_{k=0}^{\infty}\left[\frac{\lambda f(k+1)\exp(-\lambda)\lambda^k}{k!} - \frac{kf(k)\exp(-\lambda)\lambda^k}{k!}\right] \\
= & \sum_{k=0}^{\infty}\left[\frac{\lambda f(k+1)\exp(-\lambda)\lambda^k}{k!}\right] - \sum_{k=1}^{\infty}\left[\frac{kf(k)\exp(-\lambda)\lambda^k}{k!}\right] \\
= & \sum_{k=0}^{\infty}\left[\frac{\lambda f(k+1)\exp(-\lambda)\lambda^k}{k!}\right] - \sum_{k=0}^{\infty}\left[\frac{\lambda f(k+1)\exp(-\lambda)\lambda^k}{k!}\right] \\
= & 0
\end{aligned}
$$

(4)

$$
\begin{aligned}
\operatorname{Var}\left(\sum_{i=1}^n X_i\right) &= \mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 - \left(\mathbb{E}\sum_{i=1}^n X_i\right)^2 \\
&= \sum_{i=1}^n \left(\mathbb{E}X_i^2 - (\mathbb{E}X_i)^2\right) + 2\sum_{i<j}(\mathbb{E}X_iX_j - \mathbb{E}X_i\mathbb{E}X_j) \\
&= \sum_{i=1}^n \operatorname{Var}(X_i) + 2\sum_{i<j}\operatorname{Cov}(X_i, X_j)
\end{aligned}
$$

When X_i are independent, we have $\operatorname{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \operatorname{Var}(X_i)$.

(5) (a) Since $0 < X_1 \leq S_m$, $\frac{1}{S_m} \leq \frac{1}{X_1}$. Hence,

$$\mathbb{E}(S_m^{-1}) \leq \mathbb{E}(X_1^{-1}) = b < \infty$$

(b)

$$
\begin{aligned}
V_k^{-1} &= X_1/X_k + X_2/X_k + \ldots X_{k-1}/X_k + 1 \\
&+ X_{k+1}/X_k + \ldots + X_n/X_k
\end{aligned}
$$

Since $X_1, \ldots, X_n$ are iid random variables, we have

$$X_1/X_k, \ldots, X_{k-1}/X_k, X_{k+1}/X_k, \ldots, X_n/X_k$$

are (n-1) iid random variables. Hence

$$X_1/X_k + X_2/X_k + \ldots + X_{k-1}/X_k + 1 + X_{k+1}/X_k + \ldots + X_n/X_k$$

have the same distribution for each k. Therefore, $V_1, \ldots, V_n$ have the same distribution.

(c) Since $\mathbb{E}\left(\frac{S_n}{S_n}\right) = 1$, from part(b), we would conclude that

$$\mathbb{E}(V_i) = 1/n$$

Hence,

$$\mathbb{E}\left(\frac{S_m}{S_n}\right) = \mathbb{E}\left(\sum_{i=1}^{m} V_i\right) = \frac{m}{n}$$

(d) As S_m is independent of X_r with $r \geq m+1$,

$$\begin{aligned}
\mathbb{E}\left(\frac{S_n}{S_m}\right) &= 1 + \mathbb{E}\left(\frac{X_{m+1} + \cdots + X_n}{S_m}\right) \\
&= 1 + (n-m)\mathbb{E}X_i\mathbb{E}\left(\frac{1}{S_m}\right) \\
&= 1 + (n-m)a\mathbb{E}(S_m^{-1})
\end{aligned}$$

(e) Since $x + x^{-1} - 2 = (\sqrt{x} + 1/\sqrt{x})^2 \geq 0$, we have $x + x^{-1} \geq 2$.

(f) Use the inequality of part (e) with $x = c(S_m/S_n)$, we have

$$c\frac{S_m}{S_n} + \frac{1}{c}\frac{S_n}{S_m} \geq 2$$

Let $c = n/m$ and take expectation on both sides,

$$\frac{n}{m}\frac{m}{n} + \frac{1}{c}\mathbb{E}\left(\frac{S_n}{S_m}\right) \geq 2$$

Hence, we have $\mathbb{E}\left(\frac{S_n}{S_m}\right) \geq \frac{n}{m}$.

(6) Let $Y = g(X)$, then Y is a non-negative random variable. Hence, $\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > t)dt$. We have

$$\begin{aligned}
\mathbb{E}g(X) &= \mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > t)dt = \int_0^\infty \mathbb{P}(g(X) > t)dt \\
&= \int_0^\infty \mathbb{P}(X \in \{x : g(x) > t\})dt \\
&= \int_0^\infty \int_{-\infty}^\infty f(x)1\{g(x) > t\}dxdt \\
&= \int_{-\infty}^\infty f(x)\int_0^\infty 1\{g(x) > t\}dtdx \quad \text{(By Fubini theorem)} \\
&= \int_{-\infty}^\infty f(x)g(x)dx
\end{aligned}$$

(7) By CDF method ,we have

$$\mathbb{P}(Y \leq y) = \mathbb{P}(|X| \leq y) = \mathbb{P}(-y \leq x \leq y) = \int_{-y}^{y} f_X(x)dx$$

Hence,

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & (y \geq 0) \\ 0 & (y < 0) \end{cases}$$

(8) Standard Cauchy distribution is $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. Let $Y = 1/X$, then

(i) When $y \geq 0$, we have

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\frac{1}{X} \leq y) = P(X \leq 0) + P(X \geq \frac{1}{y}) = \frac{1}{2} + (1 - F_X(\frac{1}{y}))$$

Hence,

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \leq y) = \frac{1}{\pi}\frac{1}{1+1/y^2}\frac{1}{y^2} = \frac{1}{\pi}\frac{1}{1+y^2}$$

(ii) When $y < 0$, we have

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\frac{1}{X} \leq y) = \mathbb{P}(\frac{1}{y} \leq X < 0) = \frac{1}{2} - F_X(\frac{1}{y})$$

So

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \leq y) = \frac{1}{\pi}\frac{1}{1+1/y^2}\frac{1}{y^2} = \frac{1}{\pi}\frac{1}{1+y^2}$$

(9) Using integration by parts,

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}dx = t\int_0^\infty x^{t-1}e^{-x}dx = t\Gamma(t)$$

(10) (a) For any j, $\min\{a_i\} \leq a_j$, $\min\{b_i\} \leq b_j$, so we have $\min\{a_i\} + \min\{b_i\} \leq a_j + b_j$. Hence, $\min\{a_i\} + \min\{b_i\} \leq \min\{a_i + b_i\}$.

(b) $\min\{a_i + b_i\} - \max\{b_i\} = \min\{a_i + b_i\} + \min\{-b_i\} \leq \min\{a_i + b_i - b_i\} = \min\{a_i\}$. So $\min\{a_i + b_i\} \leq \min\{a_i\} + \max\{b_i\}$.

From part(a) and (b) we can conclude that,
$\min\{a_i\} + \min\{b_i\} \leq \min\{a_i + b_i\} \leq \min\{a_i\} + \max\{b_i\}$.