Statistics 654 Homework 2

1. Let X and Y be independent and identically distributed $N(0, \sigma^2)$ random variables.

   (a) Show that $X^2 + Y^2$ and $\frac{X}{\sqrt{X^2+Y^2}}$ are independent, and identify their densities.

   (b) Let $\theta = \sin^{-1} \frac{X}{\sqrt{X^2+Y^2}}$. Show that θ is uniformly distributed on $(-\pi/2, \pi/2)$.

2. Let $X \sim \Gamma(\alpha_1, 1)$ and $Y \sim \Gamma(\alpha_2, 1)$ be independent. Use the two-dimensional change of variables formula to show that $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$ are independent with $Y_1 \sim \Gamma(\alpha_1 + \alpha_2, 1)$ and $Y_2 \sim \beta(\alpha_1, \alpha_2)$.

3. Show that $1 + x \leq e^x$ for every x. Deduce that $\log x \leq x - 1$ for every $x > 0$.

4. Let X be a random variable with a finite variance and let $Y = \min(X, c)$ for some constant c. Show that the variance of Y exists and is less that or equal the variance of X. [Hint: By considering $Y - c$, show that the assertion is valid for every c if it is valid for $c = 0$. For the case $c = 0$, express X in terms of Y and $Z = \max(X, 0)$, and then consider the covariance of Y and Z.]

5. Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX = \alpha/\beta$ and $\text{Var}(X) = \alpha/\beta^2$.

6. Use Stirling's formula to establish the following inequality for the standard combinatorial coefficient n choose m:

$$\binom{n}{m} \leq \frac{1}{\sqrt{2\pi}} \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m} \left(\frac{n}{m(n-m)}\right)^{1/2}$$

(You need not show this, but note that the right hand side above becomes a lower bound when multiplied by $e^{-1/(6m)}$.)

7. Let $X_{(r)}$ be the r-th order statistic of $X_1, \cdots, X_n$ iid $\sim F$. Show that

$$P(X_{(r)} \leq u) = \sum_{t=r}^{n} \binom{n}{t} F(u)^t (1 - F(u))^{n-t}$$

and find a simple form for the density of $X_{(r)}$ (see the class notes). Evaluate the densities in the special case of the minimum ($r = 1$) and maximum ($r = n$).

8. Let f_n be the density of a t-distribution with n degrees of freedom. Show that f_n converges pointwise to the density of a standard normal distribution. You may use the fact

that $\Gamma(x) \sim \sqrt{2\pi}\, e^{-x}\, x^{x-1/2}$, *i.e.*, the ratio of $\Gamma(x)$ and the expression on the right tends to one as x tends to infinity.

9. Let X be a random variable and f, g be functions such that f is non-decreasing and g is non-increasing. Using the argument given in class for the case of two non-decreasing functions, show that

$$Ef(X)\, g(X) \;\le\; Ef(X)\, Eg(X),$$

provided that all the expectations exist.

10. Use Jensen's inequality to show that for $a, b > 0$ and $p \ge 1$,

$$(a+b)^p \le 2^{p-1}[a^p + b^p].$$

Verify this inequality in case $p = 2$ by a direct calculation.

11. Use Hölder's inequality to show that $g(x) = \log \Gamma(x)$ is convex for $x \in (0, \infty)$.

12. Show that if $f_1, \dots, f_k : I \to \mathbb{R}$ are convex, and $w_1, \dots, w_k$ are non-negative, then $f = \sum_{j=1}^k w_j\, f_j$ is convex.

13. If f_λ, $\lambda \in \Lambda$, are convex functions defined on an open interval I, show that $f = \sup_{\lambda \in \Lambda} f_\lambda$ is convex.

14. Find the moment generating function of the following distributions.

   a. Poisson(λ)

   b. Exp(λ)

   c. $\mathcal{N}(0, 1)$

Use (b) to find the expected value and variance of the Exp(λ) distribution. Use the series expansion of the MGF of the standard normal to find the moments EZ^{2k} for $Z \sim \mathcal{N}(0, 1)$.