Solutions to <u>Hw2</u>

1 (a)

- Let $Z = X^2 + Y^2$ and $W = \frac{X}{\sqrt{X^2+Y^2}}$. Then

$$\begin{cases} X &= W\sqrt{Z} \\ Y &= \pm\sqrt{(1-W^2)Z} \end{cases}$$

  By two-dimensional chage of variables formula, it follows

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(x(z,w),y(z,w))\left|\frac{\partial(x,y)}{\partial(z,w)}\right| 1\{y>0\} \\ &+ f_{X,Y}(x(z,w),y(z,w))\left|\frac{\partial(x,y)}{\partial(z,w)}\right| 1\{y\le 0\} \\ &= \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}\left(\frac{1}{2\sqrt{1-w^2}}+\frac{1}{2\sqrt{1-w^2}}\right) \\ &= \frac{1}{2\pi\sigma^2}e^{-\frac{z}{2\sigma^2}}\frac{1}{\sqrt{1-w^2}} \end{aligned}$$

  where $0 \le Z < \infty$ and $-1 \le W \le 1$.

- Let $g(z) = \frac{1}{2\pi\sigma^2}e^{-\frac{z}{2\sigma^2}}$ and $h(w) = \frac{1}{\sqrt{1-w^2}}$, then $f_{Z,W}(z,w) = g(z)h(w)$. Hence, Z and W are independent.

- By integration with respect to Z and W, we would get

$$f_Z(z) = \int_{-1}^{1} f_{Z,W}(z,w)dw = \frac{1}{2\sigma^2}e^{-\frac{z}{2\sigma^2}}$$

  where $0 \le Z < \infty$. and

$$f_W(w) = \int_0^{+\infty} f_{Z,W}(z,w)dz = \frac{1}{\pi\sqrt{1-w^2}}$$

  where $-1 \le W \le 1$.

(b) Let $\theta = \arcsin W$, then $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and is monotone. Hence, $W = \sin\theta$ and $\frac{\partial W}{\partial\theta} = \cos\theta$. Then

$$f_\theta(\theta) = f_W(W)\left|\frac{\partial W}{\partial \theta}\right| = \frac{1}{\pi\cos\theta}\cos\theta = \frac{1}{\pi}, \qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

So θ is uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(2) Since X_1 and X_2 are independent, the joint p.d.f of X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{x_1^{\alpha_1-1}x_2^{\alpha_2-1}e^{-x_1-x_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$, then

$$\begin{cases} X_1 &= Y_1Y_2 \\ X_2 &= Y_1 - Y_1Y_2 \end{cases}$$

By the two-dimensional change of variables formula, we have

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(x_1(y_1,y_2),x_2(y_1,y_2))\left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| \\ &= \frac{(y_1y_2)^{\alpha_1-1}(y_1-y_1y_2)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}e^{-y_1}y_1 \\ &= \frac{y_1^{\alpha_1+\alpha_2-1}e^{-y_1}}{\Gamma(\alpha_1+\alpha_2)}\frac{y_2^{\alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathbf{B}(\alpha_1,\alpha_2)} \\ &= \Gamma(\alpha_1+\alpha_2,1)\beta(\alpha_1,\alpha_2) \end{aligned}$$

Therefore, Y_1 and Y_2 are independent with $Y_1 \sim \Gamma(\alpha_1+\alpha_2,1)$ and $Y_2 \sim \beta(\alpha_1,\alpha_2)$.

(3) Let $f(x) = e^x - x - 1$, then $f'(x) = e^x - 1$ and $f''(x) = e^x > 0$. Let $f'(x) = 0$, we have $x = 0$. So $f(x)$ is minimized at $x = 0$. Hence $f(x) = e^x - x - 1 \geq f(0) = 0$. It follows

$$e^x \geq x + 1 \tag{1}$$

Let $y > 0$ and $y = x + 1$. Substituting y in Eqn.(1), we have

$$e^{y-1} \geq y$$

Take log on both sides of the above equation,

$$\log(y) \leq y - 1$$

for every $y > 0$.

(4)
- If the assertion is valid for $c = 0$, i.e. $\mathrm{Var}(\min(X,0)) \leq \mathrm{Var}(X)$, then for every c and $Y = \min(X,c)$,

  $$\mathrm{Var}(Y) = \mathrm{Var}(Y-c) = \mathrm{Var}(\min(X-c,0)) \leq \mathrm{Var}(X-c) = \mathrm{Var}(X)$$

  Hence it is also valid for every c.

- When $c = 0$, $Y = \min(X, 0)$ and $Z = \max(X, 0)$. Then, $X = Y + Z$.
  Since $YZ = 0$, $\mathbb{E}YZ = 0$.
  Since $Y \leq 0$, $\mathbb{E}Y \leq 0$.
  Since $Z \geq 0$, $\mathbb{E}Z \geq 0$.
  Hence, $\text{Cov}(Y, Z) = \mathbb{E}YZ - \mathbb{E}Y\mathbb{E}Z \geq 0$.

$$\begin{aligned}\text{Var}(X) &= \text{Var}(Y+Z) = \text{Var}(Y) + \text{Var}(Z) + \text{Cov}(Y, Z) \\ &\geq \text{Var}(Y) + \text{Var}(Z) \geq \text{Var}(Y)\end{aligned}$$

  The assertion is valid for $c = 0$.

(5) The p.d.f of X is given by

$$f(x) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha}}{\Gamma(\alpha)}$$

Then

$$\begin{aligned}\mathbb{E}X &= \int_0^{+\infty} xf(x)dx = \frac{\alpha}{\beta}\int_0^{+\infty}\frac{x^{(\alpha+1)-1}e^{-\beta x}\beta^{\alpha+1}}{\Gamma(\alpha+1)}dx = \frac{\alpha}{\beta} \\ \mathbb{E}X^2 &= \int_0^{+\infty} x^2 f(x)dx = \frac{\alpha(\alpha+1)}{\beta^2}\int_0^{+\infty}\frac{x^{(\alpha+2)-1}e^{-\beta x}\beta^{\alpha+2}}{\Gamma(\alpha+2)}dx = \frac{\alpha(\alpha+1)}{\beta^2}\end{aligned}$$

Hence,

$$\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\alpha^2+\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

(6) The Stirling's formula is given by:

$$n! = n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}e^{\alpha_n} \qquad \text{where } \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$$

Hence, we have

$$
\begin{aligned}
\binom{n}{m} &= \frac{n!}{m!(n-m)!} \\
&= \frac{n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}e^{\alpha_n}}{m^{m+\frac{1}{2}}e^{-m}\sqrt{2\pi}e^{\alpha_m}(n-m)^{n-m+\frac{1}{2}}e^{-(n-m)}\sqrt{2\pi}e^{\alpha_{n-m}}} \\
&= \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}m^{m+\frac{1}{2}}(n-m)^{n-m+\frac{1}{2}}}e^{\alpha_n} - e^{\alpha_m} - e^{\alpha_{n-m}} \\
&< \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}m^{m+\frac{1}{2}}(n-m)^{n-m+\frac{1}{2}}}e^{\frac{1}{12n}} - e^{\frac{1}{12m+1}} - e^{\frac{1}{12(n-m)+1}} \\
&< \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}m^{m+\frac{1}{2}}(n-m)^{n-m+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{n}{m}\right)^m\left(\frac{n}{n-m}\right)^{n-m}\left(\frac{n}{m(n-m)}\right)^{\frac{1}{2}}
\end{aligned}
$$

(7)

$$
\begin{aligned}
\mathbb{P}(X_{(r)} \leq u) &= \mathbb{P}(\text{ at least } r \text{ or more of} X_1, \ldots, X_n \\
&\quad \text{are less than or equal to } u) \\
&= \sum_{t=r}^{n}\binom{n}{t}[\mathbb{P}(X \leq u)]^t[1-\mathbb{P}(X \leq u)]^{n-t} \\
&= \sum_{t=r}^{n}\binom{n}{t}[F(u)]^t[1-F(u)]^{n-t}
\end{aligned}
$$

$$
\begin{aligned}
f_{X_{(r)}}(x) &= \frac{d}{dx}\mathbb{P}(X_{(r)} \leq x) \\
&= \sum_{t=r}^{n}\binom{n}{t}f(x)\left[t[F(x)]^{t-1}[1-F(x)]^{n-t}\right. \\
&\quad \left. -(n-t)[F(x)]^t[1-F(x)]^{n-t-1}\right] \\
&= r\binom{n}{r}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \\
&\quad + \sum_{t=r+1}^{n}\binom{n}{t}f(x)t[F(x)]^{t-1}[1-F(x)]^{n-t}
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{t=r}^{n} \binom{n}{t} f(x)[F(x)]^{t}(n-t)[1-F(x)]^{n-t-1} \\
= \; & r\binom{n}{r} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \\
& + \sum_{t=r+1}^{n} \binom{n}{t} f(x) t [F(x)]^{t-1}[1-F(x)]^{n-t} \\
& - \sum_{t=r}^{n} \binom{n}{t+1} (t+1) f(x)[F(x)]^{t}[1-F(x)]^{n-t-1} \\
= \; & r\binom{n}{r} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \\
& + \sum_{t=r+1}^{n} \binom{n}{t} f(x) t [F(x)]^{t-1}[1-F(x)]^{n-t} \\
& - \sum_{t=r+1}^{n} \binom{n}{t} f(x) t [F(x)]^{t-1}[1-F(x)]^{n-t} \\
= \; & r\binom{n}{r} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}
\end{aligned}
$$

When $r = 1$,

$$f_{X_{(1)}} = nf(x)[1-F(x)]^{n-1}$$

When $r = n$,

$$f_{X_{(n)}} = nf(x)[F(x)]^{n-1}$$

(8) The density of a t-distribution with n degrees of freedom is

$$f_n(x) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}n)} \frac{1}{(1+\frac{x^2}{n})^{\frac{(n+1)}{2}}}$$

Use facts that $\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}$ and $\lim_{n\to+\infty}(1+\frac{a_n}{n})^n = e^a$ if $a_n \to a$, we have

$$
\begin{aligned}
f_n(x) \;\; & \approx \;\; \frac{1}{\sqrt{\pi n}} \frac{\sqrt{2\pi} e^{-\frac{1}{2}(n+1)} [\frac{1}{2}(n+1)]^{\frac{1}{2}n}}{\sqrt{2\pi} e^{-\frac{1}{2}n} (\frac{1}{2}n)^{\frac{n}{2}-\frac{1}{2}}} \frac{1}{[(1+\frac{x^2}{n})^n]^{\frac{1}{2}} (1+\frac{x^2}{n})^{1/2}} \\
& = \;\; \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}} \left(1+\frac{1}{n}\right)^{\frac{n}{2}} \frac{1}{\sqrt{2}} \frac{1}{[(1+\frac{x^2}{n})^n]^{\frac{1}{2}} (1+\frac{x^2}{n})^{1/2}}
\end{aligned}
$$

Let $n \to +\infty$, we have

$$\lim_{n\to+\infty} f_n(x) = \frac{1}{\sqrt{\pi}} e^{-1/2} e^{1/2} \frac{1}{\sqrt{2}} \frac{1}{e^{\frac{x^2}{2}}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Hence, f_n converges pointwise to the density of a standard normal distribution.

(9) Let Y be a random variable such that Y is independent of X and Y and X have the same distribution. Since f is non-decreasing and g is non-increasing, we have

$$\begin{aligned} 0 &\leq [f(x) - f(y)][g(y) - g(x)] \\ &= f(x)g(y) - f(x)g(x) - f(y)g(y) + f(y)g(x) \end{aligned}$$

Take expectations on both sides, we have

$$\begin{aligned} 0 &\leq \mathbb{E}f(X)g(Y) - \mathbb{E}f(X)g(X) - \mathbb{E}f(Y)g(Y) + \mathbb{E}f(Y)g(X) \\ &= \mathbb{E}f(X)\mathbb{E}g(Y) - \mathbb{E}f(X)g(X) \qquad (X \text{ and } Y \text{ are independent}) \\ &\quad -\mathbb{E}f(X)g(X) + \mathbb{E}f(Y)\mathbb{E}g(X) \\ &= \mathbb{E}f(X)\mathbb{E}g(X) - \mathbb{E}f(X)g(X) - \mathbb{E}f(X)g(X) + \mathbb{E}f(X)\mathbb{E}g(X) \end{aligned}$$

Therefore,

$$\mathbb{E}(f(x)g(x)) \leq \mathbb{E}f(x)\mathbb{E}g(x)$$

(10) Let X be a random variable such that

$$X = \begin{cases} a & \text{w.p. } 1/2 \\ b & \text{w.p. } 1/2 \end{cases}$$

Let $g(X) = X^p$. Since $p \geq 1$, $g(X)$ is convex. By Jensen's inequality $g(\mathbb{E}X) \leq \mathbb{E}g(X)$, we have

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$$

Hence,

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

When $p = 2$,

$$(a+b)^2 = a^2 + b^2 + 2ab \leq a^2 + b^2 + a^2 + b^2 \leq 2(a^2 + b^2)$$

So the inequality holds when $p = 2$.

(11) Define random variable T such that $T \sim \mathrm{Exp}(1)$, then the p.d.f of T is given by:

$$f(t) = e^{-t} \qquad 0 \le t < \infty$$

Let X and Y be functions of T such that

$$\begin{cases} X(t) &= t^{\alpha(x-1)} \\ Y(t) &= t^{(1-\alpha)(y-1)} \end{cases}$$

Then for every $x \in (0,\infty)$, $y \in (0,\infty)$ and $\alpha \in [0,1]$,

$$\begin{aligned} \mathbb{E}XY &= \int_0^{+\infty} t^{\alpha(x-1)} t^{(1-\alpha)(y-1)} e^{-t} dt \\ &= \int_0^{+\infty} t^{\alpha x + (1-\alpha)y - 1} e^{-t} dt \\ &= \Gamma(\alpha x + (1-\alpha)y) \end{aligned}$$

Let $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, then $\frac{1}{p} + \frac{1}{q} = 1$. It follows

$$\begin{aligned} (\mathbb{E}X^p)^{\frac{1}{p}} &= \left[\int_0^{+\infty} (t^{\alpha(x-1)})^{\frac{1}{\alpha}} e^{-t} dt\right]^{\alpha} \\ &= \left[\int_0^{+\infty} t^{x-1} e^{-t} dt\right]^{\alpha} \\ &= [\Gamma(x)]^{\alpha} \end{aligned}$$

$$\begin{aligned} (\mathbb{E}X^q)^{\frac{1}{q}} &= \left[\int_0^{+\infty} (t^{(1-\alpha)(y-1)})^{\frac{1}{1-\alpha}} e^{-t} dt\right]^{1-\alpha} \\ &= \left[\int_0^{+\infty} t^{y-1} e^{-t} dt\right]^{1-\alpha} \\ &= [\Gamma(y)]^{1-\alpha} \end{aligned}$$

By Holder's inequality we have

$$\mathbb{E}XY \le (\mathbb{E}X^p)^{\frac{1}{p}} (\mathbb{E}X^q)^{\frac{1}{q}}$$

Therefore it follows

$$\Gamma(\alpha x + (1-\alpha)y) \le [\Gamma(x)]^{\alpha} [\Gamma(y)]^{1-\alpha}$$

Taking log on both sides

$$\log \Gamma(\alpha x + (1-\alpha)y) \le \alpha \log \Gamma(x) + (1-\alpha) \log \Gamma(y)$$

hence

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$$

This implies that g is convex.

(12) Since $f_i(x)$ are convex, for every $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} & f(\alpha(x) + (1-\alpha)(y)) \\ = & \sum_{i=1}^{k} w_i f_i(\alpha(x) + (1-\alpha)(y)) \\ \leq & \sum_{i=1}^{k} w_i(\alpha f_i(x) + (1-\alpha) f_i(y)) \\ = & \alpha \sum_{i=1}^{k} w_i f_i(x) + (1-\alpha) \sum_{i=1}^{k} w_i f_i(y) \\ = & \alpha f(x) + (1-\alpha) f(y) \end{aligned}$$

Hence, $f(x)$ is convex.

(13) For every $x \in I$, $y \in I$ and $\alpha \in [0,1]$, since f_λ is convex, we have

$$\begin{aligned} f_\lambda(\alpha x + (1-\alpha)y) & \leq \alpha f_\lambda(x) + (1-\alpha) f_\lambda(y) \\ & \leq \alpha \sup_{\lambda \in \Lambda} f_\lambda(x) + (1-\alpha) \sup_{\lambda \in \Lambda} f_\lambda(y) \\ & = \alpha f(x) + (1-\alpha) f(y) \end{aligned}$$

hence

$$\sup_{\lambda \in \Lambda} f_\lambda(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

so

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

This implies that f is convex.

(14) (a)

$$\begin{aligned} M_X(t) & = \mathbb{E}e^{tX} = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} e^{tk} \\ & = e^{-\lambda} e^{\lambda e^t} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k e^{-\lambda e^t}}{k!} \\ & = e^{\lambda(e^t - 1)} \end{aligned}$$

(b)

$$M_X(t) = \mathbb{E}e^{tX} = \int_0^{+\infty} \lambda e^{-\lambda x} e^{tx} dx = \frac{\lambda}{\lambda - t}, \qquad (\lambda > t)$$

(c)

$$\begin{aligned} M_X(t) &= \mathbb{E}e^{tX} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{tx} dx \\ &= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

(d) From $M_X(t) = \frac{\lambda}{\lambda - t}$ and $M_X^{(k)}(0) = \mathbb{E}X^k$, we have

$$\begin{aligned} \mathbb{E}X &= M_X'(0) = \frac{\lambda}{(\lambda - t)^2} \,|_{t=0} = \frac{1}{\lambda} \\ \mathbb{E}X^2 &= M_X''(0) = \frac{2\lambda}{(\lambda - t)^3} \,|_{t=0} = \frac{2}{\lambda^2} \end{aligned}$$

Hence,

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{\lambda^2}$$

(e)

$$M_X(t) = e^{\frac{t^2}{2}} = \sum_{k=0}^{+\infty} \frac{\left(\frac{t^2}{2}\right)^2}{k!} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}$$

Then

$$\mathbb{E}Z^{2n} = M_X^{(2n)}(0) = \frac{\mathrm{d}^{2\mathrm{n}}}{\mathrm{dt}^{2\mathrm{n}}} \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!} \bigg|_{t=0} = \frac{2k!}{2^k k!}$$