

## Statistics 164 Homework 5

1. Show directly that if  $X_n \xrightarrow{P} X$  and  $Y_n \xrightarrow{P} Y$  then  $X_n Y_n \xrightarrow{P} XY$ .
2. Let  $X, X_2, \dots, X$  be random variables defined on the same probability space such that  $X_n \rightarrow X$  with probability one. Fix  $\epsilon > 0$  and define events

$$A_m = \{|X_n - X| < \epsilon \text{ for every } n \geq m\}. \quad m \geq 1.$$

- (a) Show that  $A_m \subseteq A_{m+1}$  and that  $\{X_n \rightarrow X\} \subseteq \cup_m A_m$ .
  - (b) Conclude that  $X_n \rightarrow X$  in probability.
3. Show that if  $\mu_n \rightarrow \mu$  and  $\sigma_n^2 \rightarrow \sigma^2 > 0$ , then  $\mathcal{N}(\mu_n, \sigma_n^2) \Rightarrow \mathcal{N}(\mu, \sigma^2)$ .
  4. Suppose that  $(X_n, Y_n) \Rightarrow (X, Y)$  which, by analogy with the one-dimensional case means that  $Eg(X_n, Y_n) \rightarrow Eg(X, Y)$  for every bounded continuous function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ .
    - a. Show that  $X_n \Rightarrow X$  and  $Y_n \Rightarrow Y$ .
    - b. Show that  $f(X_n, Y_n) \Rightarrow f(X, Y)$  if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous.
  5. Suppose that  $X_n \Rightarrow X$  in law,  $Y_n \rightarrow Y$  in probability, and  $Z_n \Rightarrow z_0$  in law, where  $z_0$  is a constant. Indicate the type of convergence (if any) and limit in each of the following cases.
    - a.  $X_n^2 + Y_n$
    - b.  $X_n^2 + Z_n$
    - c.  $e^{Y_n} Z_n$
    - d.  $\sin(X_n) Z_n$

6. Suppose that  $X_n, Y_n = O_p(1)$  and  $U_n = o_p(1)$ . Establish the following relations.

- (a)  $X_n Y_n = O_p(1)$
- (b)  $X_n U_n = o_p(1)$
- (c) If  $W_n = o_p(X_n)$  then  $W_n = o_p(1)$

7. Show directly that  $X_n \xrightarrow{P} X$  implies  $X_n = O_p(1)$ .

8. The number of particles emitted by a radioactive substance in one second has a Poisson distribution with mean  $\lambda$ . Suppose that observations made in adjacent time intervals are independent. The substance is observed for two seconds, during which 5 particles are emitted. Find the probability that exactly 3 of these 5 particles were emitted during the first second.

9. (Ross) We say that  $X$  is stochastically larger than  $Y$ , written  $X \geq_{st} Y$  if  $F_X(t) \leq F_Y(t)$  for all  $t$ . Show that if  $X \geq_{st} Y$ , then  $E[X] \geq E[Y]$  when

- (a)  $X$  and  $Y$  are nonnegative random variables;

(b)  $X$  and  $Y$  are arbitrary random variables.

Hint: For part (a), recall the alternative formula for the expected value of a positive random variable given earlier in the semester. For part (b), write  $X$  as  $X = X^+ - X^-$  where  $X^+ = \max\{X, 0\}$  and  $X^- = -\min\{X, 0\}$ . Similarly, represent  $Y$  as  $Y^+ - Y^-$ . Then make use of part (a).

10. (B & D) A random variable  $X$  has a Poisson  $\mathcal{P}(\lambda)$  distribution. Given  $X = k$ ,  $Y$  has a binomial  $\mathcal{B}(k, p)$  distribution.

(a) Using the relation  $E(e^{tY}) = E(E(e^{tY}|X))$  and the uniqueness of moment generating functions show that  $Y$  has a Poiss( $\lambda p$ ) distribution.

(b) Show that  $X - Y$  has a Poiss( $\lambda(1 - p)$ ) distribution.

(c) Show that  $Y$  and  $X - Y$  are independent and find the conditional distribution of  $X$  given  $Y = y$ .

11. Show that if  $X \sim \text{Bin}(n, \lambda/n)$  and  $Y \sim \text{Poiss}(\lambda)$  for some fixed  $\lambda > 0$ , then for each  $k \geq 0$ , the ratio  $P(X = k)/P(Y = k) \rightarrow 1$  as  $n \rightarrow \infty$ .

12. Let  $X_1, X_2, \dots, X \in \mathbb{N}$  be random variables taking values in the positive integers. Let  $p_n$  and  $p$  be the p.m.f.s of  $X_n$  and  $X$ , respectively. If  $X_n \Rightarrow X$ , show that  $p_n(k) \rightarrow p(k)$  for every  $k \in \mathbb{N}$ .

13. Give an example in which a sequence of absolutely continuous random variables  $X_n$  converge in law to zero.