Statistics 164 Homework 6

1. If X has characteristic function φ, find the characteristic function of $Y = aX + b$.

2. Show that if X and $-X$ have the same distributions then the characteristic function of X is real-valued.

3. Find the characteristic functions of the following random variables.

   (a) $X \sim \text{Poisson}(\lambda)$.

   (b) $X \sim U[0,1]$

   (c) $X \sim DE(1)$

4. Use the characteristic function of the double exponential and the Inversion Theorem to find the characteristic function of the standard Cauchy distribution. Now let $X_1, \ldots, X_n$ be i.i.d. Cauchy random variables, and define $Z = n^{-1}(X_1 + \cdots + X_n)$. Show that Z has the same distribution as X_1. What does this tell you about the law of large numbers?

5. (B & D) If $X \sim N(\theta, 1)$, the random variable $Y = X^2$ is said to have a noncentral χ^2 distribution with 1 degree of freedom and noncentrality parameter θ^2.

   (a) Show that Y has density $f_Y(y) = \dfrac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}(y+\theta^2)}(e^{\theta\sqrt{y}} + e^{-\theta\sqrt{y}})$, $y > 0$.

   (b) Show that we can write $f_Y(y) = \sum_{i=0}^{\infty} P(R = i) f_{2i+1}(y)$ where $R \sim \text{Poiss}(\frac{1}{2}\theta^2)$ and f_m is the χ^2_m density. Give a probabilistic interpretation of this formula. Hint: Use the Taylor expansion for $e^{\theta\sqrt{y}}$ and $e^{-\theta\sqrt{y}}$ in powers of $\sqrt{y}$.

6. (B & D) Let S_n have a χ^2_n distribution.

   (a) Show that if n is large, $\sqrt{S_n} - \sqrt{n}$ has approximately a $N(0, \frac{1}{2})$ distribution. This is known as Fisher's approximation. Hint: Use the delta method.

   (b) Using (a), deduce that $P[S_n < x] \approx \Phi(\sqrt{2x} - \sqrt{2n})$.

7. (B&D) Let $X_1, \ldots, X_n$ be a sample from a population with mean μ and variance $\sigma^2 < \infty$. Suppose h has a second derivative $h^{(2)}$ continuous at μ and that $h^{(1)}(\mu) = 0$.

   (a) Show that $\sqrt{n}[h(\bar{X}) - h(\mu)] \to 0$ while $n[h(\bar{X}) - h(\mu)]$ is asymptotically distributed as $\frac{1}{2}h^{(2)}(\mu)\sigma^2 V$ where $V \sim \chi^2_1$.

   (b) Use part (a) to show that $n[\bar{X}(1-\bar{X}) - \mu(1-\mu)] \Rightarrow -\sigma^2 V$ with $V \sim \chi^2_1$ when $\mu = \frac{1}{2}$.

8. Let $\mathbf{X} = (X_1, \ldots, X_k)$ be a random vector such that $EX_i^2 < \infty$, $i = 1, \ldots, k$. The variance-covariance matrix of X is defined by $\mathrm{Var}(\mathbf{X}) = E[(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T]$. Show that

(a) $\mathrm{Var}(\mathbf{X}) = E(\mathbf{X}\mathbf{X}^T) - (E\mathbf{X})(E\mathbf{X})^T$.

(b) If A is an $m \times k$ constant matrix, then $\mathrm{Var}(A\mathbf{X}) = A\,\mathrm{Var}(\mathbf{X})A^T$.

(c) If $\mathbf{X}$ and $\mathbf{Y}$ are $k \times 1$ independent random vectors, then $\mathrm{Var}(\mathbf{X} + \mathbf{Y}) = \mathrm{Var}(\mathbf{X}) + \mathrm{Var}(\mathbf{Y})$.

9. Let $\mathbf{X}\,(k \times 1)$ and $\mathbf{Y}\,(l \times 1)$ be random vectors, and define the $(k \times l)$ covariance matrix $\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = E[(\mathbf{X} - E\mathbf{X})(\mathbf{Y} - E\mathbf{Y})^T]$. Show that

(a) If $\mathbf{X}$ and $\mathbf{Y}$ are independent, $\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbf{0}$.

(b) $\mathrm{Cov}(A\mathbf{X} + \mathbf{a}, B\mathbf{Y} + \mathbf{b}) = A\,\mathrm{Cov}(\mathbf{X}, \mathbf{Y})B^T$.

(c) If $k = l$ then $\mathrm{Var}(\mathbf{X} + \mathbf{Y}) = \mathrm{Var}(\mathbf{X}) + \mathrm{Cov}(\mathbf{X}, \mathbf{Y}) + \mathrm{Cov}(\mathbf{Y}, \mathbf{X}) + \mathrm{Var}(\mathbf{Y})$.

10. (MKB) Let U and V be independent $\mathcal{N}(0, 1)$ random variables. Define $Y = V$ and let

$$X = \begin{cases} U & \text{if } UV \geq 0 \\ -U & \text{if } UV < 0 \end{cases}$$

a. Show that X and Y each have a standard normal distribution, but that (X, Y) is not bivariate normal.

b. Show that X^2 and Y^2 are independent.

11. (MKB) Let (X, Y) be a random vector having a bivariate Pareto distribution with density $f(x, y) = c(x + y - 1)^{-p-2}$ with $p > 0$.

a. Show that $c = p(p + 1)$.

b. If $p > 1$, show that $EX = EY = p(p - 1)$.

c. For $p > 2$, find the covariance matrix of (X, Y).