

## Statistics 164 Homework 7 (Final Version)

1. Let  $U_1, U_2, \dots$  be iid  $\chi_1^2$  random variables, and let  $M \in \{1, 2, \dots\}$  be independent of  $\{U_i\}$  and have pmf  $p(\cdot)$ . Define  $X = \sum_{j=1}^M U_j = U_1 + \dots + U_M$ .

(a) Show that  $X$  has density  $f(x) = \sum_{j=1}^{\infty} p(j)g_j(x)$ , where  $g_j$  is the density of a central  $\chi^2$  random variable with  $j$  degree of freedom. [Hint Use the CDF method and conditioning.]

(b) Use part (a) to show that if  $N \sim \text{Poisson}(\frac{1}{2}\theta^2)$  is independent of  $\{U_i\}$  then  $X = \sum_{j=1}^{2N+1} U_j$  has density  $f(x) = \sum_{j=0}^{\infty} P(N = j)g_{2j+1}(x)$ . Conclude that  $X \stackrel{L}{=} Y^2$  where  $Y \sim N(\theta, 1)$ . (See previous homework.)

(c) Use part (b) to find the density of a non-central  $\chi^2(p, \lambda)$  distribution.

2. Show that if  $Y \sim \chi^2(p, \lambda)$  then  $EY = p + 2\lambda$ ,  $\text{Var}(Y) = 2(p + 4\lambda)$ .

3. Use the definition of the non-central  $\chi^2$  distribution to show that if  $Y_1 \sim \chi^2(p_1, \lambda_1)$  and  $Y_2 \sim \chi^2(p_2, \lambda_2)$  are independent, then  $Y_1 + Y_2 \sim \chi^2(p_1 + p_2, \lambda_1 + \lambda_2)$ .

4. Show that if  $\mathbf{X} \sim N_p(\mu, \Sigma)$  and  $\mathbf{Y} = \mathbf{X}^T \mathbf{A} \mathbf{X}$  then  $EY = \text{tr}(\mathbf{A}\Sigma) + \mu^T \mathbf{A} \mu$ .

5 (MKB). Let  $X \sim \mathcal{N}_p(\mu, \Sigma)$ . Show that  $X$  and  $GX$  have the same distribution for all orthogonal matrices  $G$  if and only if  $\mu = 0$  and  $\Sigma = \sigma^2 I$ .

6 (MKB). Let  $X \sim \mathcal{N}(0, \sigma^2 I)$  and let  $A$  be a matrix with generalized inverse  $A^-$  satisfying  $AA^-A = A$ . Show that  $Y = AX$  and  $Z = (I - A^-A)X$  are jointly multinormal and independent.

7 (MKB). Let  $X \sim \mathcal{N}_p(\mu, \Sigma)$  and suppose that  $Q$  is a  $q \times p$  matrix such that  $Q\Sigma Q^T > 0$ . Find the conditional distribution of  $X$  given  $QX = q$ . Show that the variance of the conditional distribution is given by  $\Sigma - \Sigma Q^T (Q\Sigma Q^T)^{-1} Q\Sigma$ , and that this matrix is singular, regardless of whether or not  $\Sigma$  is positive definite.

8 (MKB). Let  $X = (X_1, X_2, X_3)$  be a multinormal random vector with mean  $\mu = (\mu_1, \mu_2, \mu_3)$  and variance

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{bmatrix}$$

Find the conditional distribution of  $(X_1, X_2)$  given  $X_3 = x_3$ .

9. Show that two symmetric matrices  $A$  and  $B$  commute if they are simultaneously diagonalizable.

10. Let  $A$  be a symmetric ( $n \times n$ ) matrix. Show that the constrained maximum

$$\sup_{x: x^T x = 1} x^T A x = \lambda_n$$

where  $\lambda_n$  is the largest eigenvalue of  $A$ . Deduce from this that

$$\sup_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_n$$

as well. Find a vector for which the inequality is satisfied with equality.

11. Let  $A, B$  be ( $n \times n$ ) symmetric matrices that commute, i.e.,  $AB = BA$ , and suppose that the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$  are all distinct.

a. Let  $\Gamma$  be an orthogonal matrix that diagonalizes  $A$ , i.e.,  $\Lambda = \Gamma^T A \Gamma$  is a diagonal matrix whose non-zero elements are the eigenvalues of  $A$ . Show that  $\Lambda$  and  $B' = \Gamma^T B \Gamma$  commute.

b. Deduce from the equation  $\Lambda B' = B' \Lambda$  that  $B'$  is diagonal. Thus  $\Gamma$  simultaneously diagonalizes  $A$  and  $B$ .

12. Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be positive constants. Show that

$$\left( \prod_{k=1}^n a_k \right)^{1/n} + \left( \prod_{k=1}^n b_k \right)^{1/n} \leq \left( \prod_{k=1}^n (a_k + b_k) \right)^{1/n}$$

Hint: First divide the LHS by the RHS and then use a bound that you derived in an earlier HW assignment.