Statistics 164 Homework 8

1. Let $U_1, U_2, \ldots$ be iid χ_1^2 random variables, and let $M \in \{1, 2, \ldots\}$ be independent of $\{U_i\}$ and have pmf $p(\cdot)$. Define $X = \sum_{j=1}^{M} U_j = U_1 + \cdots + U_M$.

   (a) Show that X has density $f(x) = \sum_{j=1}^{\infty} p(j) g_j(x)$, where g_j is the density of a central χ^2 random variable with j degree of freedom. [Hint Use the CDF method and conditioning.]

   (b) Use part (a) to show that if $N \sim \text{Poisson}(\frac{1}{2}\theta^2)$ is independent of $\{U_i\}$ then $X = \sum_{j=1}^{2N+1} U_j$ has density $f(x) = \sum_{j=0}^{\infty} P(N = j)\, g_{2j+1}(x)$. Conclude that $X \overset{\mathcal{L}}{=} Y^2$ where $Y \sim N(\theta, 1)$. (See previous homework.)

   (c) Use part (b) to find the density of a non-central $\chi^2(p, \lambda)$ distribution. (Carefully reproduce the argument from the lecture).

2. Show that if $Y \sim \chi^2(p, \lambda)$ then $EY = p + 2\lambda$, $\text{Var}(Y) = 2(p + 4\lambda)$.

3. Use the definition of the non-central χ^2 distribution to show that if $Y_1 \sim \chi^2(p_1, \lambda_1)$ and $Y_2 \sim \chi^2(p_2, \lambda_2)$ are independent, then $Y_1 + Y_2 \sim \chi^2(p_1 + p_2, \lambda_1 + \lambda_2)$.

4. Show that if $\mathbf{X} \sim N_p(\mu, \Sigma)$ and $\mathbf{Y} = \mathbf{X}^T A \mathbf{X}$ then $EY = \text{tr}(A\Sigma) + \mu^T A \mu$.