STOR 831, Advanced Probability

Homework 1

1. Recall that a function $f : \mathcal{X} \to \mathbb{R}$ is lower-semicontinuous (l.s.c.) if its level sets $\{x : f(x) \leq \alpha\}$ are closed for every $\alpha \in \mathbb{R}$. Show that f is l.s.c. if and only if $x_n \to x$ implies $\liminf_n f(x_n) \geq f(x)$. Draw a picture of a function that is l.s.c., but not continuous.

2. Let $\{a_n\}$ and $\{b_n\}$ be sequences of numbers. Carefully establish relations between

$$\limsup_n (a_n + b_n) \quad \text{and} \quad \limsup_n \max\{a_n, b_n\}$$

and the quantities $\limsup_n a_n$ and $\limsup_n b_n$.

3. Show that $\{\mu_n\}$ satisifies the LDP with rate function I if and only if the following bounds hold.

(UBD'): For every $\alpha < \infty$ and every $A \in \mathcal{B}$ with $\overline{A} \subseteq \{x : I(x) > \alpha\}$

$$\limsup_n n^{-1} \log \mu_n(A) \ \leq \ -\alpha$$

(LBD'): For every $x \in D_I$ and every $A \in \mathcal{B}$ with $x \in A^o$

$$\liminf_n n^{-1} \log \mu_n(A) \ \geq \ -I(x)$$

4. Let X, μ_n and $\Lambda^*()$ be as in the setting of Cramér's theorem in $\mathbb{R}$. Show that if $EX > -\infty$ and $x \leq EX$ then

$$\mu_n((-\infty, x]) \ \leq \ \exp\{-n\Lambda^*(x)\}$$

[Hint: Follow the proof of the LD upper bound given in class.]

5. (D&Z) Establish the following.

a. If $X \sim \text{Poiss}(\lambda)$ then

$$\Lambda^*(x) \ = \ \begin{cases} \lambda - x + x\log(x/\lambda) & \text{if } x \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

b. If $X \sim \mathrm{Bern}(p)$ then

$$\Lambda^*(x) = \begin{cases} x \log\left(\frac{x}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right) & \text{if } 0 \le x \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $D_\Lambda = \mathbb{R}$ but that Λ^* is discontinuous.

c. If $X \sim \mathrm{Exp}(\lambda)$ then

$$\Lambda^*(x) = \begin{cases} \lambda x - 1 - \log(\lambda x) & \text{if } x > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

d. If $X \sim \mathcal{N}(0, \sigma^2)$ then $\Lambda^*(x) = x^2/2\sigma^2$.