Partial Solutions to Homework 2

1. Carefully depict some of the indifference curves for the following utility functions. In each case, check whether the preferences are **monotonic** and whether preferences are **convex.** Be very precise! Once you have depicted the indifference curves it is OK to use the graphs to check monotonicity and convexity, but then you need a short explanation and show how you perform the graphical test. It is of course also possible to use some algebra and the definitions of convexity and monotonicity.

   You'll have to draw these by yourself!

   (a) $u(x_1, x_2) = x_1 + \sqrt{x_2}$
   Convex and monotonic

   (b) $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$
   Convex and monotonic

   (c) $u(x_1, x_2) = 25 + \left(x_1 + \sqrt{x_2}\right)^2$
   Convex and monotonic (same prefs as $u(x_1, x_2) = x_1 + \sqrt{x_2}$*)*

   (d) $u(x_1, x_2) = k \min\{2x_1, x_2\} + 3$ where $k > 0$.
   (Weakly) convex and monotonic (I've been a bit careless in definitions. Usually, one makes a distinction between weak and strict monotonicity and convexity, where the weak version requires a weak inequality and the strict version requires a strict inequality)

   (e) $u(x_1, x_2) = \sqrt{x_1} + x_2$
   Convex and monotonic (relabeling of $u(x_1, x_2) = x_1 + \sqrt{x_2}$*, so you don't need any additional work).*

   (f) $u(x_1, x_2) = -\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2}$
   Convex, but non-monotonic (indifference curves are circles centered on (2,4).

   (g) $u(x_1, x_2) = \text{integer}(x_1) + \text{integer}(x_2)$, where integer (x) means the "integer part of x". That is

   $$\begin{aligned} \text{integer}(x) &= 0 \text{ if } 0 \leq x < 1 \\ \text{integer}(x) &= 1 \text{ if } 1 \leq x < 2 \\ \text{integer}(x) &= 2 \text{ if } 2 \leq x < 3 \end{aligned}$$

   and so on.
   Non-convex but (weakly) monotonic.

   (h) $u(x_1, x_2) = \frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2$
   Non-convex but (strictly) monotonic.

2. Let preferences be represented by $u(x_1, x_2) = x_1^a x_2^b$ and suppose that the income is m and that prices are given by p_1 and p_2

   (a) Set the consumer choice problem up as a problem in a single variable (skip if you are comfortable using Lagrangian methods).

Answer: Follow steps from lecture and write problem as

$$\max_{0\le x_1\le \frac{m}{p_1}} x_1^a \left(\frac{m-p_1x_1}{p_2}\right)^b$$

or

$$\max_{0\le x_2\le \frac{m}{p_2}} \left(\frac{m-p_2x_2}{p_1}\right)^a x_2^b$$

(b) Can you assume that $b = 1-a$? Why/why not? Why would you want to make such an assumption?

Answer: Yes, provided that $a, b > 0$. This is because we then have that $f(u) = u^{\frac{1}{a+b}}$ is strictly increasing. Hence, we can define $\widehat{a} = \frac{a}{a+b}$ and note that

$$\left(x_1^a x_2^b\right)^{\frac{1}{a+b}} = (x_1)^{\frac{a}{a+b}}(x_2)^{\frac{b}{a+b}} = x_1^{\widehat{a}} x_2^{1-\widehat{\alpha}}.$$

This shows that for every pair $a, b > 0$ we can find some $\widehat{a}$ so that $x_1^{\widehat{a}}x_2^{1-\widehat{\alpha}}$ represents the same preferences as $x_1^a x_2^b$. Consequently, we do not lose any generality by insisting on $a + b = 1$.

(c) Write down the necessary conditions for an interior solution (first order conditions) for the consumer choice problem.

Answer: in case you plugged in the budget constraint we get

$$\begin{aligned} 0 &= \frac{d}{dx_1}\left[x_1^a\left(\frac{m-p_1x_1}{p_2}\right)^{1-a}\right] \\ &= ax_1^{a-1}\left(\frac{m-p_1x_1}{p_2}\right)^{1-a} - x_1^a\left(\frac{m-p_1x_1}{p_2}\right)^{-a}\frac{p_1}{p_2} \end{aligned}$$

(d) Solve out explicitly for a bundle (x_1^*, x_2^*) that satisfies the first order conditions. Which two bundles except for the one you just derived may solve the problem? Evaluate the utility function at these two bundles and at (x_1^*, x_2^*). What is the solution to the utility maximization problem?

Answer: Proceed as in class and you'll eventually get that

$$(x_1^*, x_2^*) = \left(\frac{am}{p_1}, \frac{(1-a)m}{p_2}\right).$$

The only other bundles that can be optimal are $\left(\frac{m}{p_1}, 0\right)$ and $\left(0, \frac{m}{p_2}\right)$, but

$$\begin{aligned} u(x_1^*, x_2^*) &= u\left(\frac{am}{p_1}, \frac{(1-a)m}{p_2}\right) \\ &= \left(\frac{am}{p_1}\right)^a\left[\frac{(1-a)m}{p_2}\right]^{1-a} > 0 \\ &= u\left(\frac{m}{p_1}, 0\right) = u\left(0, \frac{m}{p_2}\right), \end{aligned}$$

so the bundle $(x_1^*, x_2^*) = \left(\frac{am}{p_1}, \frac{(1-a)m}{p_2}\right)$ does indeed solve the problem.

(e) **(Optional)** Use the Lagrangian to solve the problem and convince yourself that you get the same solution.

3. Now suppose that $u(x_1, x_2) = \ln(x_1 + x_2)$

(a) Formulate the *consumer choice problem.*
Answer: The cleanest solution is to observe that the problem is

$$\begin{aligned} & \max \ln(x_1 + x_2) \\ \text{s.t. } p_1 x_1 + p_1 x_2 \quad \leq \quad & m \end{aligned}$$

which has the same solutions as

$$\begin{aligned} & \max x_1 + x_2 \\ \text{s.t. } p_1 x_1 + p_1 x_2 \quad \leq \quad & m \end{aligned}$$

(b) Suppose that the solution is *interior.* What must be true about p_1 and p_2? If the solution is interior, how many solutions are there? Illustrate with a picture.
Answer. Since the utility function is monotonic we can plug in the constraint just like in the previous case and write the problem as

$$\max_{0 \leq x_1 \leq \frac{m}{p_1}} x_1 + \frac{m - p_1 x_1}{p_2}.$$

If the solution is interior (meaning here that $0 < x_1 < \frac{m}{p_1}$) it must be that

$$0 = \frac{d}{dx_1}\left[x_1 + \frac{m - p_1 x_1}{p_2}\right] = 1 - \frac{p_1}{p_2}.$$

Hence, an interior solution exists only if $p_1 = p_2$, in case any $x_1 \in \left[0, \frac{m}{p_1}\right]$ solves the problem.

(c) Under what condition does the consumer only consume x_1 in the solution? Under what condition does the consumer only consume x_2 in the solution? What is the most likely type of solution with this type of preferences, corner solutions or interior solutions?
Answer: It is relatively easy to see that if $p_1 < p_2$ then the consumer consumes only good 1 and if $p_1 > p_2$ the consumer consumes only good 2. One way to realize this is to draw a graph. One can also note that

$$\frac{d}{dx_1}\left[x_1 + \frac{m - p_1 x_1}{p_2}\right] = 1 - \frac{p_1}{p_2} \begin{array}{ll} < 0 & \text{if } p_1 > 0 \\ = 0 & \text{if } p_1 = 0 \\ > 0 & \text{if } p_1 < 0 \end{array}$$

4. Now suppose that preferences are represented by $u(x_1, x_2) = x_1 + \ln x_2$, the income is m and that prices are given by p_1 and p_2

(a) Carefully depict the indifference curves for these preferences. In particular, be very careful about the slopes near the x_1−axis and the x_2−axis.

(b) Formulate the consumer choice problem and write down the necessary conditions for an interior solution (first order conditions). Will the solution always be interior? If $p_1 = p_2 = 1$, what level of m is required for the solution to be interior? Can you come up with more general conditions?

Write problem as

$$\max_{0 \leq x_2 \leq \frac{m}{p_2}} \frac{m - p_2 x_2}{p_1} + \ln x_2.$$

For solution to be interior it must be that

$$-\frac{p_2}{p_1} + \frac{1}{x_2} = 0 \Leftrightarrow x_2 = \frac{p_1}{p_2}$$

is satisfied for some $x_2 \in \left(0, \frac{m}{p_2}\right)$. *This requires that* $p_1 < m$. *Now, if* $p_1 \geq m$ *we have that*

$$-\frac{p_2}{p_1} + \frac{1}{x_2} > -\frac{p_2}{p_1} + \frac{p_2}{p_1} = 0$$

for every $x_2 < \frac{p_1}{p_2}$ *and* $\frac{m}{p_2} \leq \frac{p_1}{p_2}$. *Hence, the maximand is strictly increasing over the whole range and* $(x_1, x_2) = \left(0, \frac{m}{p_2}\right)$ *solves the problem.*

5. Consider a farmer from the Eau Claire region in Wisconsin who lives a two period life and is the proud owner of an Airshyre diary cow. The cow produces 10000 pounds of milk in each period and we imagine that the farmer/consumer only consumes milk and that the milk goes bad unless it is consumed within the period. The farmer has preferences $u(c_1) + \delta u(c_2)$ where c_1 and c_2 denotes the milk consumption in each period. Assume that $u(c)$ is strictly increasing.

   (a) Suppose that there is a bank where the farmer can borrow and lend milk. Let r be the interest rate, so that if the farmer deposits s pounds of milk in the bank he gets $(1+r)s$ pounds back in the second period. Carefully derive and depict the budget set.

   Answer. This is just like in class. In the end you'll get

$$c_1 + \frac{1}{1+r} c_2 \leq 10000 + \frac{1}{1+r} 10000$$

   (b) Formulate the relevant choice problem for the farmer.

   Answer:

$$\begin{aligned} &\max_{c_1, c_2} u(c_1) + \delta u(c_2) \\ \text{s.t.} c_1 + \frac{1}{1+r} c_2 &\leq 10000 + \frac{1}{1+r} 10000 \\ c_1 &\geq 0 \\ c_2 &\geq 0 \end{aligned}$$

   (c) Derive the first order condition (the necessary condition for optimality under the assumption that the solution is not at a corner)

Write

$$\max_{0\leq c_2\leq} u\left(10000+\frac{1}{1+r}10000-\frac{1}{1+r}c_2\right)+\delta u\left(c_2\right).$$

The first order condition is

$$-u'\left(\underbrace{10000+\frac{1}{1+r}10000-\frac{1}{1+r}c_2}_{=c_1}\right)\frac{1}{1+r}+\delta u'\left(c_2\right)=0$$

Hence we can write

$$\frac{u'\left(c_1\right)}{\delta u'\left(c_2\right)}=\frac{1}{1+r}$$

(d) Assume that $u\left(c\right)=\ln c, r=0$ and $\delta=1$. Solve for the optimal consumption plan.

Answer:

$$c_1=c_2=10000$$

(e) Now, assume that there is no credit market. However, the farmer can give the cow antibiotics in period 1, which increases the milk production in period 1, but decreases it in period 2. Let x be the quantity antibiotics given to the animal and assume that $0\leq x\leq 100$. Suppose that the milk production as a function of x is $m_1\left(x\right)=10000+x$ in the first period and $m_2\left(x\right)=10000-x^2$ in the second period. Carefully depict in a graph the set of feasible consumption plans $\left(c_1,c_2\right)$ given that the farmer has this technology available, but has no access to any kind of market.

C*orrect answers have a flat segment until* $(10000, 10000)$ *and then the slope of the "budget line" will get steeper and steeper as* c_1 *increases.*