Problem Set 1

1. Suppose that Axel and Birgitta run a coffee shop as a partnership. Assume that sales depend on whether they are shirking or working hard. To make it simple, suppose that effort for agent \( J = A, B \) can take on two values, \( e^J = \{L, H\} \) and that the profit of the partnership as a function of effort choices are given by

\[
\begin{align*}
y(L, L) &= 0 \\
y(L, H) &= y(H, L) = 1 \\
y(H, H) &= 2.
\end{align*}
\]

However, effort is costly, so there is a utility cost of \( c > 0 \) to set \( e^J = H \).

In good communal spirit Axel and Birgitta have decided to split profits equally:

1. Under what assumption on \( c \) is \( e^A, e^B = (H, H) \) Pareto optimal?
2. Under what assumption on \( c \) is \( e^J = L \) a strictly dominant strategy?
3. When \( (e^A, e^B) = (H, H) \) is Pareto optimal and \( e^J = L \) a strictly dominant strategy, then what game does the “shirking game” described above correspond to?

2. Axel, Birgitta and Clint are engaged in a simultaneous three way duel (true?). Each player is given one shot and they all shoot at one of the two opponents simultaneously. The preferences of the players are simple: the higher the probability of survival, the happier is the agent. Suppose that Axel kills his target with probability \( \frac{1}{2} \), that Birgitta kills her target with probability \( \frac{3}{4} \), whereas Clint is a perfect shot whokills his target for sure. Represent the normal form of this game by noting that conditional on the decision by Clint, you can reduce it to a \( 2 \times 2 \) game between Axel and Birgitta. Hence, the normal form can be represented by making Clint pick which of the two “conditional” \( 2 \times 2 \) payoff matrices will be relevant (eg., Clint will be the “matrix player”). Derive the full set of pure strategy Nash equilibria.

3. Consider a voting game with 3 agents and two alternatives \( P = \{X, Y\} \). Suppose that each agent gets a payoff of 0 of \( X \) is implemented and that the payoff is \( \alpha_i \) if \( Y \) is implemented. Each agent can vote either \( x \) or \( y \) and \( X(Y) \) is implemnted if a majory voes \( x(y) \).

1. Suppose \( \alpha_i \neq 0 \). Is there a strictly dominant strategy for \( i \). If so, which strategy?
2. Suppose \( \alpha_i \neq 0 \). Is there a weakly dominant strategy for \( i \). If so, which strategy?
3. Suppose that \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \). Derive all pure strategy Nash equilibria.

4. Consider a duopoly model where the firms pick prices. Assume that demand for firm 1 is \( D_1(p_1, p_2) = \left( \frac{p_2}{p_1} \right)^\alpha \) and that the demand for firm 2 is \( D_2(p_1, p_2) = \left( \frac{p_1}{p_2} \right)^\beta \). Firm \( i \) has constant unit cost \( c_i \). Assuming that the firms set prices simultaneously, what is the normal form? Can you make assumptions on \( \alpha \) and \( \beta \) so that there is a dominant strategy equilibrium? If yes, calculate it and explain why these demands generate dominant strategies.
5. Consider the standard Bertrand equilibrium model between two homogenous firms, 1 and 2. Assume that the demand is \( D(p) = 1 - p \) and that both firms have constant unit costs. The firms announce prices simultaneously, and the consumer(s) go to whatever firm offers the lower price (suppose that a fair coin is flipped in case of indifference).

1. Carefully write down the normal form.
2. Which strategies are weakly dominated.
3. Derive the full set of pure strategy Nash equilibria. How does this relate to the set of weakly dominated strategies?

6. Consider three players, 1, 2, and 3, voting in a committee. There are three options, called \( x, y, z \). First, each player submits a secret vote. If any option gets a majority, then that outcome is implemented. If not, then there is a three-way tie, and we assume that player 1 picks the outcome in this case. Suppose that utilities over outcomes are:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
x & 2 & 0 & 1 \\
y & 1 & 2 & 0 \\
z & 0 & 1 & 2 \\
\end{array}
\]

1. For each player, what is the set of available strategies?
2. Solve the game by iteratively deleting weakly dominated strategies.

7. MWG 8.B.2

8. MWG 8.D.4