Assignment 11

1. Consider the optimal auction setup considered in class. Recall that the “indirect utility function” is given by

\[ U_i (\theta_i) = \theta_i E_{-i} [p_i (\theta)] - E_{-i} [t_i (\theta)] = \theta_i \rho_i (\theta_i) - \tau_i (\theta_i) \]

and that incentive compatibility can be expressed as requiring that

\[ \theta_i \rho_i (\theta_i) - \tau_i (\theta_i) \geq \theta_i \rho_i (\bar{\theta}_i) - \tau_i (\bar{\theta}_i) \]

holds for all \( i \in I \), and \( \theta_i, \bar{\theta}_i \in \Theta_i \).

1. Prove that \( U_i (\cdot) \) is convex in \( \theta_i \). What result in consumer or producer theory is this similar to?

2. Consider a more general setup with multidimensional types having payoffs \( \sum_{k=1}^{K} I_k^h \theta_k^k - t_i \). Prove that convexity extends to this setup as well.

3. In the unidimensional case, show that \((p, t)\) is incentive compatible if and only if \( \rho_i (\theta_i) = E_{-i} p_i (\theta_i) \) is increasing in \( \theta_i \) and \( U_i (\theta_i) = U_i (\bar{\theta}_i) + \int_{\theta_i}^{\bar{\theta}_i} \rho_i (y) dy \) for all \( i, \theta_i, \bar{\theta}_i \).

2. Consider a general mechanism design problem with \( n \) agents where mechanisms are on the form \((x, t)\), where

\[ x (\theta) = x (\theta_1, \ldots, \theta_n) \]

is the “social decision” given any vector of announces types \( \theta \) and

\[ t (\theta) = (t_1 (\theta_1, \ldots, \theta_n), t_2 (\theta_1, \ldots, \theta_n), \ldots, t_n (\theta_1, \ldots, \theta_n)) \],

is the vector of transfers given announces types \( \theta \). Let the utility function over the social decisions and “money” be on the form

\[ u_i (x, \theta_i) - t \]

for each agent \( i \). Also, let \( C (x) \) be the monetary costs of implementing \( x \). Then, a Groves mechanism is a mechanism \((x^G, t^G)\) where, for every \( \theta \)

\[ x^G (\theta) \ \text{solves} \ \max_x \sum_{i=1}^{n} u_i (x, \theta_i) - C (x) \]  \hspace{1cm} (1)

and, where for each \( i \)

\[ t_i^G (\theta) = C (x (\theta)) - \sum_{j \neq i} u_j (x (\theta), \theta_j) + \tau_i (\theta_{-i}) \] \hspace{1cm} (2)

and \( \tau_i (\theta_{-i}) \) may be arbitrarily chosen.
1. Consider the case with $n = 2$, call them $s$ (seller) and $b$ (for buyer). Let $x \in \{T, NT\}$ (for “trade” and “no trade”) and assume that the types take on values on the unit interval, that is $\theta_s \in [0, 1]$ and $\theta_b \in [0, 1]$. Let the utilities be given by

\[
\begin{align*}
    u_s (T, \theta_s) &= t_s \\
    u_s (NT, \theta_s) &= \theta_s + t_s \\
    u_b (T, \theta_b) &= \theta_b + t_b \\
    u_b (NT, \theta_b) &= t_b
\end{align*}
\]

Derive a Groves mechanism for this example (for simplicity let $\tau_b (\theta_s) = t_s (\theta_b) = 0$). Note, there is no cost $C(x)$ in this example.

2. Is it possible to construct a Groves mechanism for which $t_s (\theta) + t_b (\theta) = 0$ for every $\theta$?

3. Interpret the model as follows. The problem is one where there is a seller with the property rights to an indivisible good and a potential buyer, and $\theta_j$ denotes the valuation for the good for $j = s, b$. If the seller has the rights to keep the good this means that the participation constraints are that $u_s (x (\theta), \theta_s) \geq \theta_s$ for the seller and $u_b (x (\theta), \theta_b) \geq 0$ for the buyer. Is it possible for a Groves mechanism to satisfy these constraints? If so, what can you say about the surplus/deficit for the mechanism designer?

4. Now, suppose that the object is instead in the hands of a mechanism designer who assigns no value for good (participation constraints are now $u_j (x (\theta), \theta_j) \geq 0$ for both agents). Show that the Groves mechanism is equivalent to a second price auction, where each agents bids a money amount $b_i$ and the good goes to the highest bidder, but where the winner pays the price of the losing bid.