Assignment 12

1. Suppose that there are two agents, each of whom has a value for a public good drawn independently from \( \Theta = \{0, 5, 10\} \) according to probability distribution \( p = \left( \frac{1}{10}, \frac{1}{10}, \frac{8}{10} \right) \) (that is, the probability of a valuation equal to 10 is 0.8 and the probability of either 0 or 5 is 0.1). Suppose that the cost of the public good is 12.

   1. What is the ex post efficient provision rule?
   2. Show that it is impossible to design a mechanism satisfying budget balance, (interim) individual rationality and incentive compatibility that implements the efficient rule.
   3. Consider a class of mechanisms where the transfer is proportional to the valuations. What is the best possible “proportional tax mechanism” that respects incentive compatibility, individual rationality and budget balance?
   4. What is the best possible mechanism (maximizing expected surplus) satisfying these constraints?
   5. Suppose that society insist that the tax should be proportional to the benefits of the public good. What is the best mechanism within this class of mechanisms?

2. Consider the optimal auction setup considered in class: Explicitly calculate the optimal auction for the case with \( n \) bidders with valuations that are uniformly distributed on \([0, 1] \) and a seller with valuation \( \theta_0 = 0 \).

3. Again consider the optimal auction setup, but now assume that there are two bidders, \( \theta_0 = 0, \theta_1 \) is uniform over \([0, 1] \) and \( \theta_2 \) is distributed in accordance with pdf

\[
f_2(\theta_2) = \begin{cases} 
\epsilon & \text{if } \theta_2 \in \left[0, \frac{2}{3}\right] \\
3 - 2\epsilon & \text{if } \theta_2 \in \left[\frac{2}{3}, 1\right]
\end{cases}
\]

   1. Check under what conditions this setup satisfies the regularity condition that \( v_i(\theta_i) = \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} \) is strictly increasing.
   2. Under the appropriate assumptions that guarantee monotonicity of \( v_i(\theta_i) \), calculate the optimal auction. Explain what the example illustrates.

4. Prove that

\[
\lim_{\delta \to 0} \frac{\Pr(\theta_i < \theta'_i + \delta | \theta_i \geq \theta'_i)}{\delta} = \frac{f_i(\theta'_i)}{1-F_i(\theta'_i)}
\]

and explain what the interpretation of this fact is.