Econ 711
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Problem Set 5

1. Let $s_i \in S_i$ be a pure strategy and assume that S_i is compact and convex for every player $i \in I$.

   1. Let $B_i(s_{-i})$ denote the pure best responses to s_{-i} and let $B : S \rightarrow S$ where $B(s) = (B_1(s_{-1}), ..., B_n(s_{-n}))$ be the (pure) best response mapping. Show that the conditions for applying Kakutani's fixed point theorem are satisfied if $u_i(\cdot, s_{-i})$ is quasi-concave in s_i for every $i \in I$ and every $s_{-i} \in S_{-i}$.
   2. Suppose that $u_i(\cdot, s_{-i})$ is **strictly** quasi-concave in s_i for every $i \in I$ and every $s_{-i} \in S_{-i}$. Show that $B(s)$ satisfies the conditions for applying Brouwer's fixed point theorem.

2. Let G be an arbitrary finite game. Let $\sigma_i \in \Delta(S_i)$ denote a mixed strategy for player $i \in I$ and let σ and σ_{-i} denote a profile of all players' randomizations and all but is randomizations in analogy with the notation for pure strategies. Let

   $$\beta_i(\sigma) \equiv \arg \max_{\sigma \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})$$

   denote the mixed best responses for player i and let $\beta : \Delta(S) \rightarrow \Delta(S)$ given by $\beta(\sigma) = (\beta_1(\sigma_{-1}), ..., \beta_n(\sigma_{-n}))$ be the (mixed) best response correspondence.

   1. Prove that $\beta : \Delta(S) \rightarrow \Delta(S)$ satisfies all conditions needed to apply Kakutani's fixed point theorem.
   2. Replace β with $b : \Delta(S) \rightarrow \Delta(S)$ given by $b(\widehat{\sigma}) = (b_1(\widehat{\sigma}_{-1}), ..., b(\widehat{\sigma}_{-n}))$, where

      $$b_i(\widehat{\sigma}) \equiv \arg \max_{\sigma \in \Delta(S_i)} u_i(\widehat{\sigma}_i, \widehat{\sigma}_{-i}) - \|\sigma_i - \widehat{\sigma}_i\|^2,$$

      where $\|x - y\|$ denotes the Euclidean distance between vectors. Show that σ^* is a Nash equilibrium if and only if it is a fixed point of b.
   3. Show that b satisfies all conditions for applying Brouwer's fixed point theorem.