1 Games on Normal Form

1.1 Examples of Normal Form Games

A Game in Normal (sometimes strategic) form consists of:

1. A set of players I, which we assume is a finite set $I = \{1, ..., n\}$

2. A strategy space $S = \times_{i=1}^{n} S_i$, where S_i is the strategy space for agent i. We say that $s_i \in S_i$ is a **strategy** chosen by player i, and $s = (s_1, ..., s_n) \in S$ is referred to as a **strategy profile**.

3. A payoff function $u : S \to R^n$, where $u_i(s)$ is the utility for agent i given strategy profile $s \in S$.

In terms of interpretations, the strategy space for an agent should be thought of as anything the player could conceivably do. Hence, the set of all strategy profiles describes everything that can possibly happen and the payoff functions describe how the agents evaluate all possible outcomes.

Example 1 (Prisoners Dilemma) *Suppose that* $n = 2$ *and that* $S_i = \{D, C\}$ *for* $i = 1, 2$. *Also, suppose that the payoff functions are given by,*

$$\begin{array}{ccc} u_1(D,D) = -2 & & u_2(D,D) = -2 \\ u_1(D,C) = -10 & \text{and} & u_2(D,C) = -1 \\ u_1(C,D) = -1 & & u_2(C,D) = -10 \\ u_1(C,C) = -5 & & u_2(C,C) = -5 \end{array}.$$

A more standard (and efficient) way to represent this game is in a payoff matrix as in Figure 1. This is a version of the famous Prisoner's Dilemma.

Example 2 (Second Price Auction) *Again, let* $n = 2$ *and suppose that* $S_1 = S_2 = \mathbb{R}_+$.

Player 2

		D	C
Player 1	D	-2,-2	-10,-1
	C	-1,-10	-5,-5

Figure 1: A Prisoners Dilemma GAme

Moreover, for every pair $b_1, b_2 \in S_1 \times S_2$*, let*

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_2) & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_1 & \text{if } b_1 < b_2 \\ \frac{1}{2}(v_2 - b_1) & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 > b_2 \end{cases}.$$

These payoffs may be interpreted as follows: v_i *is the willingness to pay for the object or the reservation price and the rules of the game are that the highest bidder wins the object and pays the bid of the second highest bidder.*

Example 3 (Public Goods Game) *Again, let* $S_i = R_+$ *for each* $i \in I$ *and suppose that payoffs are given by*

$$u_i\left(e_i - y_i, \sum_{j=1}^{n} y_j\right),$$

where $u : \mathbb{R}_+^2 \rightarrow R$ *is to be thought of as a utility function over two Here, we may interpret* e_i *as the endowment (in terms of a private good),* y_i *as a voluntary contribution towards a public good and* $\sum_{j=1}^{n} y_j$ *as the level of the public good (=sum of contributions).*

Example 4 (Cournot Duopoly) *Let* $S_1 = R_+$ *and consider two firms producing/selling a homogenous good. For simplicity, let the inverse demand be given by* $P(Q) = \max\{1 - Q, 0\}$*, where* Q *is interpreted as the total quantity of the good on the market. Also, assume that the* *firms produce the good at constant unit cost $c \geq 0$. The payoff function for a single firm is then naturally taken to be the profit as a function of the strategic variables,*

$$u_i(q_i, q_j) = P(q_1 + q_2) q_i - cq_i = [1 - q_1 - q_2 - c] q_i.$$

1.2 Dominant and Dominated Strategies

Example 5 (Prisoners Dilemma) *Consider the Prisoner's Dilemma Game. In this game there is really only one plausible outcome. For each player the strategy C ("confess") is better than D ("don't confess") regardless of what the other player does. That is, if the other player plays "confess" we see that "confess" is optimal as $-1 > -2$ and if the other player plays "don't confess" we see that "confess" is optimal as $-5 > -10$. This is what is known as a strictly dominant strategy.*

Player 2

		D	C
Player 1	D	-2,-2	-10,-1
	C	-1,-10	-5,-5

Figure 2: The Dominant Strategy Equilibrium is (C,C)

Notation:

1. $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i} = \times_{j \neq i} S_j$
2. $(s_i', s_{-i}) = (s_1, ..., s_{i-1}, s_i', s_{i+1}, ..., s_n) \in S$

Definition 1 *s_i' strictly dominates s_i'' if $u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$.*

- This is non-standard because dominance is usually defined over the set of mixed strategies. We will consider randomizations in a few weeks.

- Language: s_i' strictly dominates s_i'' and s_i'' is strictly dominates by s_i' is used interchangeably.

Next, we say that a strategy is strictly dominant if it strictly dominates all other strategies.

Definition 2 *s_i' is strictly dominant if $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_i \in S_i \backslash \{s_i'\}$ and all $s_{-i} \in S_{-i}$.*

- strictly dominant strategy must be unique.
- is optimal regardless of strategy used by others, so if s_i' is a strictly dominant strategy, then

$$\{s_i'\} = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

Definition 3 *s_i' weakly dominates s_i'' if $u_i(s_i', s_{-i}) \geq u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i})$ for some $s_{-i} \in S_{-i}$*

Naturally, we say that a strategy is weakly dominant if it weakly dominates all other strategies:

Definition 4 *s_i' is weakly dominant if for every $s_i \in S_i$ we have that $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some $s_{-i} \in S_{-i}$.*

Example 6 *Consider the second price auction. Recall that*

$$u_i(b_i, b_j) = \begin{cases} v_i - b_j & \textit{if } b_i > b_j \\ \frac{1}{2}(v_i - b_j) & \textit{if } b_i = b_j \\ 0 & \textit{if } b_i < b_j \end{cases}.$$

In this game we can get a unique prediction by using weak dominance:

CLAIM: *Bidding $b_i = v_i$ is the unique dominant strategy in the second price auction.*

Proof. *Suppose that $b_j < v_i$. Then*

$$u_i(v_i, b_j) = v_i - b_j \geq u_i(b_i, b_j)$$

for all b_i*. Moreover, if* $b_j > 0$ *and* $b_i < b_j$ *we have that* $u_i(b_i, b_j) = 0 < v_i - b_j = u_i(v_i, b_j)$*.*

Suppose that $b_j \geq v_i$*. Then,* $u_i(v_i, b_j) = 0 \geq u_i(b_i, b_j)$ *for all* b_i *and* $u_i(v_i, b_j) = 0 > u_i(b_i, b_j)$ *if* $b_i > b_j > v_i$*. Hence, bidding the true valuation is a dominant strategy.*

(Uniqueness) Suppose that $b_i < v_i$ *and consider* b_j *such that* $b_i < b_j < v_i$*. Then*

$$u_i(b_i, b_j) = 0 < v_i - b_j = u_i(v_i, b_j),$$

so there is no dominant strategy where $b_i < v_i$*. Suppose instead that* $b_i > v_i$*, and consider* b_j *so that* $b_i > b_j > v_i$*. Then,*

$$u_i(b_i, b_j) = v_i - b_j < 0 = u_i(v_i, b_j),$$

so there is no dominant strategy where $b_i > v_i$*. It follows that bidding* $b_i = v_i$ *is the unique dominant strategy.* ■

Example 7 *Consider the* n *player 2nd price auction. Consider agent* 1*. Then*

$$u_1(b) = \begin{array}{cc} v_1 - \max\{b_2, ..., b_n\} & \text{if } b_1 > \max\{b_2, ..., b_n\} \\ \frac{v_1 - \max\{b_2, ..., b_n\}}{|\arg\max_{i \in I} b_i|} & \text{if } b_1 = \max\{b_2, ..., b_n\} \\ 0 & \text{if } b_1 < \max\{b_2, ..., b_n\} \end{array}.$$

It is immediate that we may replace b_j *with* $\max\{b_2, ..., b_n\}$ *and observe that in case of a tie it doesn't matter what the probability of getting the object is. We can therefore extend the result*

Example 8 (Groves-Clarke Mechanism in a Simple Public Good Setup) *Consider the following environment:*

- n *agents*
- *agent* i *has valuation* v_i
- *binary public good which costs* $C > 0$ *if provided.*
- *utility of agent* i *given by* $v_i - t_i$ *if good is provided and* $-t_i$ *if it isn't. The efficient decision is therefore to provide iff* $\sum_{i=1}^{n} v_i \geq C$*.*

If we'd simply ask for the valuations and trusted what people would say we would create incentives to mis-report true preferences. However, consider the following game. Suppose that $v_i \geq 0$ for all agents and let all agents report some type $\widehat{v}_i \in S_i = \mathbb{R}$ (negative valuations are OK). Moreover,

1. *provide the public good if and only if* $\sum_i \widehat{v}_i \geq C$

2. *If good is not provided, don't transfer anything.*

3. *If good is provided, pay (or tax if negative)* $\sum_{j \neq i} \widehat{v}_j - C$

In this setup we have that:

CLAIM: *The unique weakly dominant strategy is for every player i to report his/her true valuation.*

Proof. *Write the payoff for agent i as*

$$u_i\left(v_i, \widehat{v}_1, ..., \widehat{v}_n\right) = \begin{cases} v_i + \sum_{j \neq i} \widehat{v}_j - C & \text{if } \sum_{j=1}^n \widehat{v}_j \geq C \\ 0 & \text{if } \sum_{j=1}^n \widehat{v}_j < C \end{cases}.$$

The crucial property of this construction is that the report changes the payoff only when the social decision is changed. Sometimes called "pivot mechanism".

Suppose first that $v_i + \sum_{j \neq i} \widehat{v}_j - C > 0$. *Then:*

1. *announcing* $\widehat{v}_i = v_i$ *gives payoff* $v_i + \sum_{j \neq i} \widehat{v}_j - C > 0$

2. *announcing some* $\widehat{v}_i \neq v_i$ *will give a payoff of either* $v_i + \sum_{j \neq i} \widehat{v}_j - C$ *or* 0.

Hence, announcing $\widehat{v}_i = v_i$ *is optimal.*

Suppose instead that $v_i + \sum_{j \neq i} \widehat{v}_j - C < 0$. *Then:*

1. *announcing* $\widehat{v}_i = v_i$ *gives payoff* 0

2. *announcing some* $\widehat{v}_i \neq v_i$ *will give a payoff of either* $v_i + \sum_{j \neq i} \widehat{v}_j - C < 0$ *or* 0.

If $v_i + \sum_{j \neq i} \widehat{v}_j - C = 0$ *the agent is indifferent. We conclude that truth-telling is weakly dominant.*

(Uniqueness) Consider $\widehat{v}_i < v_i$ *and let* $\sum_{j \neq i} \widehat{v}_j$ *be such that* $v_i + \sum_{j \neq i} \widehat{v}_j - C > 0$ *and* $\widehat{v}_i + \sum_{j \neq i} \widehat{v}_j - C < 0$*, or*

$$C - v_i < \sum_{j \neq i} \widehat{v}_j < C - \widehat{v}_i.$$

We conclude that any $\widehat{v}_i < v_i$ *is weakly dominated by truth-telling. Suppose that* $\widehat{v}_i > v_i$ *and pick* $\sum_{j \neq i} \widehat{v}_j$ *be such that* $\widehat{v}_i + \sum_{j \neq i} \widehat{v}_j - C > 0$ *and* $v_i + \sum_{j \neq i} \widehat{v}_j - C < 0$*, or*

$$C - \widehat{v}_i < \sum_{j \neq i} \widehat{v}_j < C - v_i.$$

We conclude that any $\widehat{v}_i > v_i$ *is weakly dominated by truth-telling.* ■

Example 9 (Groves Mechanism in More General Model.) *Now suppose that,*

- n *agents*
- *Society must chose a "social decision"* $x \in X$*.*
- *Decision* x *costs* $C(x)$
- *Agents* i *gets payoff* $u_i(v_i, x) - t_i$

Because of quasi-linearity/transferable utility, the efficient decision given true preferences $v = (v_1, ..., v_n)$ *is*

$$x^*(v) \in \arg\max_{x \in X} \sum_{i=1}^{n} u_i(v_i, x) - C(x)$$

Consider the (Groves) mechanism where again agents are asked to submit reports and where:

1. *The social decision is given by* $x^*(\widehat{v})$
2. *Agent* i *is paid (pays if negative)* $t(\widehat{v}) = \sum_{j \neq i} u_j(\widehat{v}_j, x^*(\widehat{v})) - C(x^*(\widehat{v}))$

The payoff for i as a function of the reports is then

$$\begin{aligned} & u_i\left(v_i, x^*\left(\widehat{v}_i, \widehat{v}_{-i}\right)\right)+t\left(\widehat{v}\right) \\ = \; & u_i\left(v_i, x^*\left(\widehat{v}_i, \widehat{v}_{-i}\right)\right)+\sum_{j \neq i} u_j\left(\widehat{v}_j, x^*\left(\widehat{v}\right)\right)-C\left(x^*\left(\widehat{v}\right)\right)\end{aligned}$$

By definition

$$x^*\left(v_i, \widehat{v}_{-i}\right) \in \arg \max_{x e X} u_i\left(v_i, x\right)+\sum_{j \neq i} u_j\left(\widehat{v}_j, x\right)-C\left(x\right),$$

implying that truth-telling is a dominant strategy.

The generalization has the advantage that it is more clear that payments are set up so as to internalize the externalities in a way so that each agent behaves so as to maximize social surplus.

1.3 Iterated Elimination of Strictly Dominated Strategies

The reader should be warned that the standard approach to dominance as well as iterations of dominance relies on randomized strategies. We'll deal with that later.

However, if a strategy is strictly dominated it is not a best response to anything the opponent(s) can do. Rationality would therefore rule out play of strictly dominated strategies. But, assuming that alla agents understand the structure of the game, then other agents should understand that strictly dominated strategies will not be played, so one should be able to eliminate these. Hence, we can ask again whether there are additional strategies that are strictly dominated given the smaller set of surviving strategies.

Example 10 *Consider the game*

	L	M	R
T	$1,0$	$1,2$	$0,1$
B	$0,3$	$0,1$	$2,0$

STEP 1: R strictly dominated by M

STEP 2: With only L, M left, B dominated by T

STEP 3: Against T, M is optimal. Hence (T, M) is the unique outcome from iterative deletion of strictly dominated strategies.

Example 11 *Consider the Cournot duopoly with linear demand and $c = 0$. We note that problem*

$$\max_{q_1} \left[1 - q_1 - q_2\right] q_1$$

is solved where

$$1 - 2q_1 - q_2 = 0$$

or

$$q_1 = \frac{1 - q_2}{2}.$$

Since there is a unique maximizer that is strictly decreasing in the quantity of player 2 it is clear that there is no dominant strategy.

However:

1. *we note that $q_1 \in \left[0, \frac{1}{2}\right]$ and $q_2 \in \left[0, \frac{1}{2}\right]$ since no firm would want to produce more than the monopoly output.*

2. *but then*

$$\begin{aligned} q_1 &\geq \frac{1 - \frac{1}{2}}{2} = \frac{1}{4} \\ q_2 &\geq \frac{1 - \frac{1}{2}}{2} = \frac{1}{4} \end{aligned}$$

   so $q_i \in \left[\frac{1}{4}, \frac{1}{2}\right]$ for $i = 1, 2$. Remark: we may think of this iteration as imposing one iteration of "I know that you are rational"

3. *but then,*

$$q_i \leq \frac{1 - \frac{1}{4}}{2} = \frac{3}{8},$$

   so $q_i \in \left[\frac{1}{4}, \frac{3}{8}\right]$ for $i = 1, 2$. Remark: to get to this point we also assume that "I know that you know that I am rational"

4. but then

$$q_i \geq \frac{1-\frac{3}{8}}{2} = \frac{5}{16},$$

so $q_i \in \left[\frac{5}{16}, \frac{3}{8}\right]$ *, where we assume that "I know that you know that I know that you are rational".*

And so on. Continuing [DRAW] we see from a picture that the unique prediction is $\left(\frac{1}{3}, \frac{1}{3}\right)$ *. One way to see this algebraically, is to note that each recursion gives us an interval* $[a_n, b_n]$ *where*

$$\begin{aligned} a_n &= \frac{1-b_{n-1}}{2} \\ b_n &= \frac{1-a_{n-1}}{2}. \end{aligned}$$

Hence, the set of iteratively weakly undominated strategies must satisfy

$$\begin{aligned} a_\infty &= \frac{1-\frac{[1-a_\infty]}{2}}{2} = \frac{1+\alpha_\infty}{4} \Leftrightarrow \alpha_\infty = \frac{1}{3} \\ b_\infty &= \frac{1-\frac{[1-b_\infty]}{2}}{2} = \frac{1+b_\infty}{4} \Leftrightarrow b_\infty = \frac{1}{3}. \end{aligned}$$

Hence, we get a unique prediction from common knowledge of rationality.

1.4 Iterated Elimination of Weakly Dominated Strategies

Suppose instead that one would iteratively eliminate weakly dominated strategies. When, using iterated elimination of strictly dominated strategies one can prove (we'll get back to this when discussion rationalizability) that the order of elimination doesn't matter. However, consider Figure order.

- Suppose that we eliminate D for player 1 first. Then L becomes dominated for player 2 and the solution is (U,R)

- If instead we eliminate L first, then we cannot eliminate anything else, so the prediction becomes $\{(\text{U,R}), (\text{D,R})\}$.

Player 2

		L	R
Player 1	U	1,0	0,1
	D	0,0	0,2

Figure 3: Example illustrating that the order matters when iteratively deleting weakly dominated strategies

2 Nash Equilibrium

Birgitta

		Left	Center	Right
Axel	Top	0,4	4,0	5,3
	Middle	4,0	0,4	5,3
	Bottom	3,5	3,5	6,6

Figure 4: A Game with no Dominant Strategy

Now consider the game in Figure 4. Observe that:

- If Birgitta plays “Left”, then “Middle” is the best option for Axel
- If Birgitta plays “Center”, then “Top” is the best option for Axel
- If Birgitta plays “Right”, then “Bottom” is the best option for Axel
- We conclude that there is no dominant strategy for Axel and you can verify that the same is true for Birgitta.

Hence, even iterated elimination of weakly dominated strategies would be completely toothless in the game described in Figure 4

However, a way out is to say that an " equilibrium" is a situation where each agent behaves optimally given the strategy chosen by the other(s), which is the concept of *Nash equilibrium.* I.e.,

Definition 5 *$s^* \in S$ is a (Pure strategy) Nash equilibrium if $u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$ for all $i \in I$ and all $s_i \in S_i$.*

Nash equilibria are relatively simple objects in terms of formalism. However, it is not obvious how to justify Nash equilibrium play. Just like dominance, Nash equilibrium assumes that agents are rational utility maximizers. However, unlike iterated elimination of dominant strategies we require more than rationality and common knowledge of rationality. In addition, there is also some *assumed* coordination of beliefs built into the notion of Nash equilibrium. Simply put, all players have rational expectations of equilibrium play.

There are many informal as well as more formal arguments trying to justify the Nash equilibrium concept. Justifications include:

- Preplay agreements
- "Mass action/norm" interpretation
- Any non-Nash prediction problematic in that somebody would like to change behavior. That is, an economist predicting non-Nash (point) prediction would have to understand the structure of the game better than the players being modeled.
- Evolutionary arguments.

Recall that:

Definition 6 *A correspondence $F : X \to Y$ is a mapping that assigns a set $F(x) \subseteq Y$ to any point $x \in X$ (alternatively, a function from X to the power set of Y).*

Definition 7 *Given game $\Gamma = (I, S, u)$ the best reply correspondence for player i, denoted $\beta_i : S_{-i} \twoheadrightarrow S_i$, is defined as*

$$\beta_i\left(s_{-i}\right) = \arg\max_{s_i \in S_i} u_i\left(s_i, s_{-i}\right)$$

Definition 8 *The best reply correspondence for game* $\Gamma = (I, S, u)$*, denoted* $\beta : S \twoheadrightarrow S$ *is defined by*

$$\beta(s) = (\beta_1(s_{-1}), ..., \beta_i(s_{-i}), ..., \beta_n(s_{-n})).$$

Claim s^* *is a Nash equilibrium if an only if* $s^* \in \beta(s^*)$.

Proof. $(\Rightarrow)$ Suppose s^* is a Nash equilibrium, but that $s^* \notin \beta(s^*)$. Then, there exists i such that

$$\begin{aligned} s_i^* &\notin \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*) = \beta_i(s_{-i}^*) \\ &\Rightarrow \text{ exists } s_i' \text{ such that} \\ u_i(s_i', s_{-i}^*) &> u_i(s_i^*, s_{-i}^*), \end{aligned}$$

which contradicts assumption that s^* is Nash

$(\Longleftarrow)$ $s^* \in \beta(s^*) \Rightarrow$

$$s_i^* \in \beta_i(s_{-i}^*) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \text{ for every } i \in I,$$

which is equivalent with

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for every } i \in I \text{ and } s_i \in S_i.$$

Example 12

Birgitta

Axel		Left	Center	Right
	Top	$0, \underline{4}$	$\underline{4}, 0$	5,3
	Middle	$\underline{4}, 0$	$0, \underline{4}$	5,3
	Bottom	3,5	3,5	$\underline{6}, \underline{6}$

Figure 5: "Bottom", "Right" is the Unique Nash Equilibrium

Again consider the game from Figure 4. In a matrix form game the characterization of Nash equilibria as fixed points of the best reply correspondence gives us a foolproof algoritm for finding all Pure strategy Nash equilibria. First, assume that Birgitta plays "Left". Then *we check the first column to find the best replies for Axel, which is "Middle" in the particular example. To keep track, underline the relevant entry in the payoff matrix (corresponding with (Middle,Left)). Next, assume Birgitta plays "Center", in which case the unique best reply is "Top". Underline the appropriate entry in the payoff matrix. In the final column we see that "Bottom" is the best response to "Right".*

Next, reverse the roles and find/underline the best replies for Birgitta to Axel's strategies. Once all pure best replies have been found, any cell in the payoff matrix where both payoffs are underlined corresponds to strategies that are mutual best responses and therefore a Nash equilibrium. In this example, the only Nash equilibrium is (Bottom, Right).

Example 13 *(Cournot duopoly). The payoff of firm i is $(1 - q_1 - q_2 - c)\, q_i$, so the unique best reply is*

$$\beta_i\,(q_j) = \frac{1 - q_j - c}{2}.$$

A Nash equilibrium will thus satisfy

$$\begin{aligned} q_1^* &= \beta_1\,(q_2^*) \\ q_2^* &= \beta_2\,(q_1^*)\,. \end{aligned}$$

Calculating we find $q_1^* = q_2^* = \frac{1-c}{3}$.