

15 Optimal Auctions with Private Values and Independent Types

15.1 Set-up

A seller seeks to sell a single indivisible good. There are n potential buyers (aka “bidders”). We denote the set of bidders $I = \{1, \dots, n\}$. Bidder i is equipped with a von Neumann Morgenstern utility function and receives utility

$$\begin{aligned} \theta_i - t_i & \quad \text{if } i \text{ gets the object and transfers } t_i \text{ units to seller} \\ -t_i & \quad \text{if agent } i \text{ does not get object and transfers } t_i \text{ units to seller} \end{aligned}$$

Here θ_i , which is to be interpreted as the willingness to pay or valuation for the object, is a random variable. We will assume that θ_i is known only to the agent and refer to it as the “type” of the agent. For technical reasons we will assume θ_i is distributed in accordance with a nice probability distribution. Specifically, we assume that:

1. θ_i has support $[\underline{\theta}_i, \bar{\theta}_i]$, where $0 \leq \underline{\theta}_i$.
2. θ_i follows cumulative density F_i with density f_i , which is assumed to be uniformly bounded away from 0 on its support.
3. a qualitatively crucial assumption that is already built into this formulation is that θ_i and θ_j are independent for every $(i, j) \in I$. We will use notation

$$\begin{aligned} f(\theta) &= \times_{i=1}^n f_i(\theta_i) \\ F(\theta) &= \times_{i=1}^n F_i(\theta_i) \end{aligned}$$

for joint densities. Similarly, we will sometimes use notation $f_{-i}(\theta_{-i}) = \times_{j \neq i} f_j(\theta_j)$.

We let θ_0 be the seller’s valuation for the object. For now, we will assume that this is common knowledge. Again we assume payoffs that are linear in the transfer, so the utility function is taken to be,

$$\begin{aligned} \theta_0 + \sum_{i=1}^n t_i & \quad \text{if seller keeps the object} \\ \sum_{i=1}^n t_i & \quad \text{if object is sold to a bidder} \end{aligned}$$

Appealing to the revelation principle we restrict attention to truthful equilibria in direct revelation mechanisms. One way to write down such a mechanism is as a pair (p, x)

$$\begin{aligned} p & : \Theta \rightarrow \Delta^{n+1} \\ t & : \Theta \rightarrow R^n, \end{aligned}$$

where $\Theta = \times_{i=1}^n [\underline{\theta}_i, \bar{\theta}_i]$, $p(\theta) = (p_0(\theta), p_1(\theta), \dots, p_n(\theta))$ is interpreted as the probability distribution over who gets the “prize” and $t(\theta) = (t_1(\theta), \dots, t_n(\theta))$ are the transfers given announcements θ . Note that we allow the seller to keep the object, and $p_0(\theta)$ is the probability that this happens.

15.1.1 Using the “Indirect Utility Function”

Since types are independent we may write E_{-i} for the expectations operator $\Theta_{-i} = \times_{j \neq i} [\underline{\theta}_j, \bar{\theta}_j]$. Let $U_i : [\underline{\theta}_i, \bar{\theta}_i] \rightarrow \mathbb{R}$ be defined as

$$\begin{aligned} U_i(\theta_i) & = \int_{\Theta_{-i}} (\theta_i p_i(\theta) - t_i(\theta)) dF_{-i}(\theta), = E_{-i}[\theta_i p_i(\theta) - t_i(\theta)] \\ & = \theta_i E_{-i}[p_i(\theta)] - E_{-i}[t_i(\theta)] \end{aligned} \tag{1}$$

where $dF_{-i}(\theta)$ is shorthand notation for $\times_{j \neq i} dF_j(\theta_j) = \times_{j \neq i} f_j(\theta_j) d\theta_j$.

Remark 1 *The expression in (1) obviously depends on (p, t) , which are choice variables for the seller. Hence, it would be more kosher to call the object $U_i(\theta_i, p, t)$ to stress this endogeneity. However, I have chosen (which is rather standard) to suppress this in the interest of brevity.*

Define

$$\begin{aligned} \rho_i(\theta_i) & = E_{-i}[p_i(\theta)] \\ \tau_i(\theta_i) & = E_{-i}[t_i(\theta)], \end{aligned}$$

which have clear interpretations as the perceived probability of getting the object and the expected transfer for type θ_i (both conditional on all others telling the truth, which is OK as we will make sure that everyone will have incentives to do so). Using the revelation principle we have that there is no loss generality in considering direct mechanisms in which truth-telling for all agents and types is a Bayesian Nash equilibrium in the incomplete information game induced by (p, t) .

The condition that truth-telling is a Bayes Nash equilibrium in the game induced by (p, t) will henceforth be referred to as incentive compatibility, which can be expressed as

$$\begin{aligned} U_i(\theta_i) &= \theta_i \mathbb{E}_{-i}[p_i(\theta)] - \mathbb{E}_{-i}[t_i(\theta)] \\ &= \theta_i \rho_i(\theta_i) - \tau_i(\theta_i) \geq \theta_i \rho_i(\widehat{\theta}_i) - \tau_i(\widehat{\theta}_i), \end{aligned} \quad (2)$$

for all $i \in I$, and $\theta_i, \widehat{\theta}_i \in \Theta_i$.

It is useful to observe that $U_i(\theta_i) = \theta_i \rho_i(\theta_i) - \tau_i(\theta_i)$ is the “indirect utility function” for agent i . Also, it is almost immediate from (2) that $U_i(\theta_i)$ is convex. This follows for the same reason as convexity of the minimized cost function in old fashioned theory of the firm. A possibility is that $\rho_i(\theta_i)$ and $\tau_i(\theta_i)$ are constant in θ_i , in which case $U_i(\theta_i)$ is linear. If $\rho_i(\theta_i)$ and $\tau_i(\theta_i)$ are not constant in θ_i we may think of $U_i(\theta_i)$ as the upper envelope of a family of linear function \Rightarrow convexity.

Lemma 1 $U_i(\theta_i)$ is convex.

Proof. Consider θ'_i and θ''_i and let $\theta_i^\lambda \equiv (1 - \lambda)\theta'_i + \lambda\theta''_i$ for some $\lambda \in [0, 1]$. As

$$\begin{aligned} U_i(\theta'_i) &= \theta'_i \rho_i(\theta'_i) - \tau_i(\theta'_i) \geq \theta'_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) \\ U_i(\theta''_i) &= \theta''_i \rho_i(\theta''_i) - \tau_i(\theta''_i) \geq \theta''_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) \end{aligned}$$

Multiply first inequality by $(1 - \lambda)$ and second by λ and add and we get

$$\begin{aligned} &(1 - \lambda)U_i(\theta'_i) + \lambda U_i(\theta''_i) \\ &\geq (1 - \lambda) \left[\theta'_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) \right] + \lambda \left[\theta''_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) \right] \\ &= \left[(1 - \lambda)\theta'_i + \lambda\theta''_i \right] \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) = U_i(\theta_i^\lambda). \end{aligned}$$

■

Remark 2 Lemma 1 is one of the rare properties that actually generalizes to the multidimensional case. That is, suppose that the payoff function would be

$$\sum_{k=1}^K I_i^k \theta_i^k - t_i,$$

where $I_i^k \in \{0, 1\}$ is a dummy indicating whether i gets the k th object. Also, let $\theta_i = (\theta_i^1, \dots, \theta_i^K)$, $\theta = (\theta_1, \dots, \theta_n)$, $\rho_i(\theta_i) = (\rho_i^1(\theta_i), \dots, \rho_i^K(\theta_i))$, with each $\rho_i^k(\theta_i) = \mathbb{E}_{-i}[p_i^k(\theta)]$. In this case we

have that

$$\begin{aligned} U_i(\theta'_i) &= \theta'_i \rho_i(\theta'_i) - \tau_i(\theta'_i) \geq \theta'_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda) \\ U_i(\theta''_i) &= \theta''_i \rho_i(\theta''_i) - \tau_i(\theta''_i) \geq \theta''_i \rho_i(\theta_i^\lambda) - \tau_i(\theta_i^\lambda). \end{aligned}$$

The argument obviously generalizes.

Next, we use convexity and the fact that types are unidimensional to get a very useful characterization of incentive compatibility.

Lemma 2 (p, t) is incentive compatible if and only if $\rho_i(\theta_i) = E_{-i} p_i(\theta_i)$ is increasing in θ_i and

$$U_i(\theta_i) = U_i(\widehat{\theta}_i) + \int_{\widehat{\theta}_i}^{\theta_i} \rho_i(y) dy \text{ for all } i, \theta_i, \widehat{\theta}_i \quad (3)$$

Remark 3 The convention in (3) is that if $\theta_i < \widehat{\theta}_i$, then $\int_{\widehat{\theta}_i}^{\theta_i} \rho_i(y) dy = - \int_{\theta_i}^{\widehat{\theta}_i} \rho_i(y) dy$.

Remark 4 $U_i(\theta_i) = \theta_i \rho_i(\theta_i) - \tau_i(\theta_i)$ is the maximized utility for agent i of the type θ_i , so $U'(\theta_i) = \rho_i(\theta_i)$ could be thought of as a consequence of the envelope theorem (recall Shepherd, Hotelling, Roy etc...). The expression in (3) is thus simply a combination of the envelope result and the fundamental theorem of calculus.

For completeness we'll provide a direct proof of Lemma 2 which does not rely on the "envelope theorem".

Proof. (Necessity) $[\rho_i(\theta_i)$ increasing] Let $\theta_i < \theta'_i$. Pair wise incentive compatibility implies that

$$\begin{aligned} U_i(\theta_i) &\geq \theta_i \rho_i(\theta'_i) - \tau_i(\theta'_i) = U_i(\theta'_i) - (\theta'_i - \theta_i) \rho_i(\theta'_i) \\ U_i(\theta'_i) &\geq \theta'_i \rho_i(\theta_i) - \tau_i(\theta_i) = U_i(\theta_i) + (\theta'_i - \theta_i) \rho_i(\theta_i), \end{aligned}$$

where $(\theta'_i - \theta_i) > 0$. Hence,

$$\rho_i(\theta_i) \leq \frac{U_i(\theta'_i) - U_i(\theta_i)}{\theta'_i - \theta_i} \leq \rho_i(\theta'_i),$$

which shows that $\rho_i(\cdot)$ must be increasing.

[(3)] Convex functions are differentiable almost everywhere. Pick any θ_i in the interior of $[\underline{\theta}_i, \bar{\theta}_i]$ for which U_i is differentiable and let $\bar{\epsilon} > 0$ be small enough so that an open $\bar{\epsilon}$ -ball $B_{\bar{\epsilon}}$ around θ_i is contained in $[\underline{\theta}_i, \bar{\theta}_i]$. Pick any $\epsilon \in (0, \bar{\epsilon})$ and use incentive compatibility twice to get

$$\begin{aligned} U_i(\theta_i + \epsilon) &\geq (\theta_i + \epsilon) \rho_i(\theta_i) - \tau_i(\theta_i) = U_i(\theta_i) + \epsilon \rho_i(\theta_i) \\ U_i(\theta_i - \epsilon) &\geq (\theta_i - \epsilon) \rho_i(\theta_i) - \tau_i(\theta_i) = U_i(\theta_i) - \epsilon \rho_i(\theta_i) \end{aligned}$$

Hence

$$\frac{U_i(\theta_i) - U_i(\theta_i - \varepsilon)}{\varepsilon} \leq \rho_i(\theta_i) \leq \frac{U_i(\theta_i + \varepsilon) - U_i(\theta_i)}{\varepsilon}.$$

As U_i is differentiable at θ_i it follows that

$$U'(\theta_i) = \lim_{\varepsilon \rightarrow 0} \frac{U_i(\theta_i) - U_i(\theta_i - \varepsilon)}{\varepsilon} \leq \rho_i(\theta_i) \leq \lim_{\varepsilon \rightarrow 0} \frac{U_i(\theta_i + \varepsilon) - U_i(\theta_i)}{\varepsilon} = U'(\theta_i),$$

which shows that $U'(\theta_i) = \rho_i(\theta_i)$. Hence,

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} U'(y) dy = \int_{\hat{\theta}_i}^{\theta_i} \rho_i(y) dy.$$

(Sufficiency) Suppose that (3) holds and $\rho_i(\cdot)$ is weakly increasing. Consider $\theta_i > \theta'_i$. Then, since $\rho_i(y) \geq \rho_i(\hat{\theta}_i)$ for all $y \geq \hat{\theta}_i$

$$\begin{aligned} U_i(\theta_i) &= U_i(\theta'_i) + \int_{\theta'_i}^{\theta_i} \rho_i(y) dy \\ &\geq U_i(\theta'_i) + \int_{\theta'_i}^{\theta_i} \rho_i(\theta'_i) d\theta \\ &= U_i(\theta'_i) + (\theta_i - \theta'_i)\rho_i(\theta'_i) \\ &= \theta'_i \rho_i(\theta'_i) - \tau_i(\theta'_i) + (\theta_i - \theta'_i)\rho_i(\theta'_i) \\ &= \theta_i \rho_i(\theta'_i) - \tau_i(\theta'_i) \end{aligned}$$

Hence, θ_i has no incentive to pretend to be θ'_i . A symmetric calculation works for the case where $\theta_i < \theta'_i$ ■

15.2 Revenue Equivalence

A remarkable property of single-unit auctions with continuous type spaces is that the expected revenue from the auction is fully determined by the allocation rule.

Lemma 3 *Suppose that (p, t) is incentive compatible, then*

$$Et_i(\theta) = \int_{\Theta} \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) p_i(\theta) f(\theta) d\theta - U_i(\underline{\theta}_i)$$

This result says that the **expected** transfer is fully determined by:

1. $U_i(\underline{\theta}_i)$, the utility of the lowest type.
2. The allocation rule p_i

In other words, up to a constant/lump sum, the allocation rule nails the expected profit for the seller. For example, for any two auction formats that guarantee that the highest bidder wins the object and the lowest type is indifferent between participation and participation, the expected revenue is the same. First and second price auctions are examples of such equivalent auctions. The implication for the seller is that the design of the optimal mechanism boils down to:

1. How much utility should the seller give to the lowest type (obvious answer is as little as possible)?
2. The allocation rule.

Proof. Using the definition of $U_i(\theta_i)$ and Lemma 3

$$\begin{aligned} U_i(\theta_i) &= \theta_i \rho_i(\theta_i) - \tau_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy \Leftrightarrow \\ \tau_i(\theta_i) &= \int_{\Theta_{-i}} t_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i} = \theta_i \rho_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy - U_i(\underline{\theta}_i) \end{aligned} \quad (4)$$

$\tau_i(\theta_i) \equiv E_{-i} t_i(\theta_i)$, implying that $E t_i(\theta) = E_{\theta_i} [E_{-i} t_i(\theta_i)] = \int_{\underline{\theta}_i}^{\theta_i} \tau_i(\theta_i) f_i(\theta_i) d\theta_i$. Integrating (4) over $[\underline{\theta}_i, \bar{\theta}_i]$ therefore yields

$$E t_i(\theta) = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\theta_i \rho_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy \right] f_i(\theta_i) d\theta_i - U_i(\underline{\theta}_i) \quad (5)$$

Remark 5 *The revenue equivalence theorem is actually seen directly from (5). The right hand side contains terms determined by the allocation rule and $U_i(\underline{\theta}_i)$ only!*

Define $H(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy$. By an integration by parts

$$\begin{aligned} \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy \right] f_i(\theta_i) d\theta_i &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} H(\theta_i) f_i(\theta_i) d\theta_i \\ &= [H(\theta_i) (F_i(\theta_i) - 1)]_{\underline{\theta}_i}^{\bar{\theta}_i} - \int_{\underline{\theta}_i}^{\bar{\theta}_i} \frac{dH(\theta_i)}{d\theta_i} (F_i(\theta_i) - 1) d\theta_i \\ \left/ \begin{array}{l} \frac{dH(\theta_i)}{d\theta_i} = \rho_i(\theta_i) \\ H(\underline{\theta}_i) = 0 \\ F_i(\bar{\theta}_i) - 1 = 0 \end{array} \right/ &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \rho_i(\theta_i) (1 - F_i(\theta_i)) d\theta_i = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \rho_i(\theta_i) \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} f_i(\theta_i) d\theta_i \end{aligned}$$

Hence

$$\int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\theta_i \rho_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy \right] f_i(\theta_i) d\theta_i = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right] \rho_i(\theta_i) f_i(\theta_i) d\theta_i \quad (7)$$

$$\begin{aligned} &= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right] \left[\int_{\Theta_{-i}} p_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i} \right] f_i(\theta_i) d\theta_i \\ &= \int_{\Theta} \left[\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right] p_i(\theta) f(\theta) d\theta \end{aligned} \quad (8)$$

combined with (5) this gives the result. ■

15.3 Optimal Auctions (for the Seller)

We assume that the seller wants to maximize expected revenue subject to incentive and **participation constraints**. Requiring that nobody can be forced to participate in the auction means that $U_i(\theta_i) \geq 0$, which is implied by $U_i(\underline{\theta}_i) \geq 0$.

The participation constraint for the lowest type must bind because otherwise the allocation rule can be kept intact, but the transfer increased by a constant for all types of the agent.

After noting that $p_0(\theta) = 1 - \sum_{i=1}^n p_i(\theta)$ we can write the relevant optimization problem as.

$$\begin{aligned} &\max \int_{\Theta} \sum_{i=1}^n \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} - \theta_0 \right) p_i(\theta) f(\theta) d\theta \quad (9) \\ &\text{s.t. } \int_{\Theta_{-i}} p_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i} \text{ is weakly increasing} \\ &\quad p_i(\theta) \geq 0 \text{ for all } \theta \in \Theta \\ &\quad \sum_{i=1}^n p_i(\theta) \leq 1 \text{ for all } \theta \in \Theta \end{aligned}$$

15.4 Solving the Regular Case

Definition 1 *The cumulative distribution F_i is said to be regular if the function*

$$v_i(\theta_i) \equiv \theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)}$$

is strictly increasing (we refer to $v_i(\theta_i)$ as the “virtual valuation” for reasons to be discussed below).

Most parametric distributions that are used by economists are regular. The most well known sufficient condition is the monotone hazard rate conditions (see Bagnoli and Bergstrom [?]).

Now, if we ignore the constraint that $\int_{\Theta_{-i}} p_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i}$ must be weakly increasing the problem is

$$\begin{aligned} & \max \int_{\Theta} \sum_{i=1}^n (v_i(\theta_i) - \theta_0) p_i(\theta) f(\theta) d\theta \\ \text{s.t. } & p_i(\theta) \geq 0 \text{ and } \sum_{i=1}^n p_i(\theta) \leq 1 \text{ for all } \theta \in \Theta \end{aligned}$$

We note that this is a problem that can be solved by pointwise optimization, which may be solved by solving

$$\begin{aligned} & \max_{p_i(\theta) \in R^n} \sum_{i=1}^n (v_i(\theta_i) - \theta_0) p_i(\theta) \\ \text{s.t. } & p_i(\theta) \geq 0 \text{ and } \sum_{i=1}^n p_i(\theta) \leq 1 \end{aligned}$$

for every $\theta \in \Theta$. The solution to this problem is more or less obvious. Pick the agent with the highest “virtual valuation” and give the object to that agent unless this virtual valuation is below the seller’s actual valuation. That is,

$$p_i(\theta) = \begin{cases} 1 & \text{if } v_i(\theta_i) > \max\{\theta_0, v_1(\theta_1), \dots, v_{i-1}(\theta_{i-1}), v_{i+1}(\theta_{i+1}), \dots, v_n(\theta_n)\} \\ 0 & \text{if } v_i(\theta_i) < \max\{\theta_0, v_1(\theta_1), \dots, v_{i-1}(\theta_{i-1}), v_{i+1}(\theta_{i+1}), \dots, v_n(\theta_n)\} \end{cases} \quad (10)$$

If there are ties, then the seller may flip a coin, give it to the agent with the lowest index among the “winners” or apply any other tie-breaking rule one may think of—it doesn’t matter for the expected payoff calculations for the bidders since ties happen with probability zero.

We notice that since (by the regularity assumption) $\theta'_i > \theta_i \Rightarrow v_i(\theta'_i) > v_i(\theta_i)$. Hence $p_i(\theta'_i, \theta_{-i}) \geq p_i(\theta_i, \theta_{-i})$ for every θ_{-i} , which implies that $\int_{\Theta_{-i}} p_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i}$ is weakly increasing in θ_i . We thus conclude:

Proposition 1 *If $v_i(\theta_i)$ is strictly increasing for all θ_i , then the allocation rule $p : \Theta \rightarrow \Delta^{n+1}$ where $p_i(\theta)$ is given by (10) for each θ such that there is a “unique winner” (which may be the seller) and where ties are broken arbitrarily when they occur and where $p_0(\theta) = 1 - \sum_{i=1}^n p_i(\theta)$ together with any transfer rule that*

$$\int_{\Theta_{-i}} t_i(\theta) f_{-i}(\theta_{-i}) d\theta_{-i} = \tau_i(\theta_i) = \theta_i \rho_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \rho_i(y) dy \quad (11)$$

constitutes an optimal auction for the seller.

15.5 Transfer Schemes that Support the Optimal Allocation Rule

We notice that only *expected* transfers matter from the point of view of the bidders, so there is huge multiplicity in terms of the choice of transfer scheme. A natural way to “nail” transfers is to impose that

$$t_i(\theta) = \theta_i p_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} p_i(\theta_{-i}, y) dy \quad (12)$$

holds for every θ_{-i} . The left hand side of (11) is the expectation over $t_i(\theta)$ with respect to θ_{-i} and the right hand side is the expectation of $\theta_i p_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} p_i(\theta_{-i}, y) dy$ with respect to θ_{-i} , so this is obviously one way to determine the transfers. In the regular case we assume that each v_i is strictly increasing and therefore invertible. Define,

$$z_i(\theta_{-i}) = \begin{cases} \bar{\theta}_i & \text{if } v_i(\bar{\theta}_i) < \max\{\theta_0, \max_{j \neq i} v_j(\theta_j)\} \\ v_i^{-1}(\max\{\theta_0, \max_{j \neq i} v_j(\theta_j)\}) & v_i(\underline{\theta}_i) \leq \max\{\theta_0, \max_{j \neq i} v_j(\theta_j)\} \leq v_i(\bar{\theta}_i) \\ \underline{\theta}_i & v_i(\underline{\theta}_i) > \max\{\theta_0, \max_{j \neq i} v_j(\theta_j)\} \end{cases}$$

The point is that this notation allows us to express (10) briefly as $p_i(\theta) = 1$ if $\theta_i > z_i(\theta_{-i})$ and $p_i(\theta) = 0$ if $\theta_i < z_i(\theta_{-i})$. Hence

$$\int_{\underline{\theta}_i}^{\theta_i} p_i(\theta_{-i}, y) dy = \begin{cases} \int_{z_i(\theta_{-i})}^{\theta_i} 1 dy = \theta_i - z_i(\theta_{-i}) & \text{if } \theta_i > z_i(\theta_{-i}) \\ 0 & \text{if } \theta_i \leq z_i(\theta_{-i}) \end{cases},$$

which substituted back into (12) implies that

$$t_i(\theta) = \begin{cases} z_i(\theta_{-i}) & \text{if } \theta_i > z_i(\theta_{-i}) \Leftrightarrow p_i(\theta) = 1 \\ 0 & \text{if } \theta_i \leq z_i(\theta_{-i}) \Leftrightarrow p_i(\theta) = 0 \end{cases}. \quad (13)$$

This is similar to a second price auction. A winning pays the price (determined by the **virtual** valuations of the other bidders) that is the minimal valuation that still keeps her among the set of “winning bidders”.

15.6 Some Special Cases

15.6.1 Symmetric Bidders

Suppose that $[\underline{\theta}_i, \bar{\theta}_i] = [\underline{\theta}, \bar{\theta}]$ and $f_i(y) = h(y)$ for all bidders. Since the virtual valuation function for all agents are identical and given by

$$v(\theta_i) = \theta_i - \frac{(1 - H(\theta_i))}{h(\theta_i)}$$

Then (ruling out trivial cases where θ_0 is so high so that the seller never wants to sell we have that

$$v(\theta_i) \leq \max \left\{ \theta_0, \max_{j \neq i} v(\theta_j) \right\} \leq v(\bar{\theta}_i)$$

is always satisfied, and

$$z_i(\theta_{-i}) = v^{-1} \left(\max \left\{ \theta_0, \max_{j \neq i} v(\theta_j) \right\} \right) = \max \left\{ v^{-1}(\theta_0), \max_{j \neq i} \theta_j \right\}$$

We have thus concluded that a second price auction with reservation price $v^{-1}(\theta_0)$ is a seller optimal mechanism. The second price auction allocates the object to the bidder with the highest valuation above the reservation, and by the revenue equivalence result any other mechanism that does the same will also be optimal. A first price auction with the same reservation price will be one example of a mechanism that works.

15.6.2 Discriminating Optimal Auction with Asymmetric Bidders

The optimality of second price and first price auctions is nice as these are commonly used auction formats. However, these formats are no longer optimal if there are **known** differences between the bidders. Then the optimal auction will in general involve asymmetric treatment of agents. An example is:

- Two bidders
- $\theta_0 = 0$
- θ_1 uniform over $[0, 1]$
- θ_2 distributed in accordance with pdf

$$f_2(\theta_2) = \begin{cases} \epsilon & \text{if } \theta_2 \in [0, \frac{2}{3}] \\ 3 - 2\epsilon & \text{if } \theta_2 \in (\frac{2}{3}, 1] \end{cases}$$

This distribution satisfies the regularity assumption given that $\epsilon \leq 1$, which we assume.

The virtual valuations are

$$\begin{aligned} v_1(\theta_1) &= \theta_1 - (1 - \theta_1) \text{ for all } \theta_1 \in [0, 1] \\ v_2(\theta_2) &= \begin{cases} \theta_2 - \frac{(1-\epsilon)\theta_2}{\epsilon} & \text{if } \theta_2 \in [0, \frac{2}{3}] \\ \theta_2 - \frac{1-\epsilon\frac{2}{3}-(3-2\epsilon)(\theta_2-\frac{2}{3})}{(3-2\epsilon)} & \text{if } \theta_2 \in (\frac{2}{3}, 1] \end{cases} \end{aligned}$$

We note that

$$v_2\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{(1 - \frac{\epsilon}{2})}{\epsilon} = \frac{2(\epsilon - 1)}{2\epsilon} < 0 \text{ if } \epsilon < 1$$

whereas $v_1\left(\frac{1}{2}\right) = 0$. Both virtual valuations are continuous in $\left[0, \frac{2}{3}\right]$, so if $\epsilon < 1$ there is a range $\left[\frac{1}{2}, \delta\right]$ where $v_1(x) > 0$ and $v_2(x) < 0$ for all $(\theta_1, \theta_2) \in \left[\frac{1}{2}, \delta\right] \times \left[\frac{1}{2}, \delta\right]$.

Intuitively, the gain from discriminating in favor of the “weak” bidder is that it allows the seller to extract more surplus from agent two agents with higher types. This makes sense because the probability of a valuation in the range $\left[\frac{2}{3}, 1\right]$ is higher for agent 2 than for agent 1.

15.7 A Single “Bidder”

In this case we find that

$$t_i(\theta_0) = z_i(\theta_{-1}) = v^{-1}(\theta_0),$$

which is just monopoly pricing. To see this, look at the problem

$$\max_p (1 - F(p))(p - \theta_0),$$

which is the standard monopoly pricing problem. The first order condition is

$$\begin{aligned} 1 - F(p) - f(p)(p - \theta_0) &= 0 \Leftrightarrow \\ v(p) &= p - \frac{1 - F(p)}{f(p)} = \theta_0 \Leftrightarrow \\ p &= v^{-1}(\theta_0) \end{aligned}$$

15.8 Optional Reading: Illustration of the “General” Case (Riley-Zeckhausers’ result)

Drop the i subscript and write θ for the valuation of the single customer. A mechanism is now a pair

$$p : \Theta \rightarrow [0, 1] \text{ (probability of trade)}$$

$$t : \Theta \rightarrow [0, 1] \text{ (pricing function)}$$

The optimal auction problem (9) simplifies to (the monopoly pricing problem)

$$\begin{aligned} \max_{p: \Theta \rightarrow [0, 1]} \int p(\theta) \left[\theta - \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta \\ \text{s.t. } p \text{ is increasing in } \theta \end{aligned}$$

Let

$$\begin{aligned} h(q) &= F^{-1}(q) - \frac{1 - q}{f(F^{-1}(q))} \\ H(q) &= \int_0^q h(r) dr \end{aligned}$$

We notice that $H'(q) = h(q)$. Hence:

1. H is convex if and only if h is increasing in q
2. h is increasing in q iff $h(F(\theta))$ is increasing, which requires that $\theta - \frac{1 - F(\theta)}{f(\theta)}$ is increasing. Hence
3. For a regular problem, $H : [0, 1] \rightarrow R$ is a convex function

Let $G : [0, 1] \rightarrow R$ be given by the convex hull of the function H

$$\begin{aligned} G(q) &= \min_{\lambda, r, s \in [0, 1]} \lambda H(r) + (1 - \lambda) H(s) \\ \text{s.t. } \lambda r + (1 - \lambda) s &= q \end{aligned}$$

The geometric interpretation is that one first take the convex hull of the epigraph to H and let G be the lower bound of that function. Let

$$g(q) = G'(q)$$

where the derivative is defined (almost everywhere) and extend to $[0, 1]$ by right-continuity. Finally, let

$$c(\theta) = g(F(\theta))$$

Consider the problem

$$\max_{p: \Theta \rightarrow [0,1]} \int p(\theta) g(F(\theta)) f(\theta) d\theta \quad (14)$$

The result is

Proposition 2 *The solution to (14),*

$$p^*(\theta) = \begin{cases} 1 & \text{if } g(F(\theta)) \geq 0 \\ 0 & \text{if } g(F(\theta)) < 0 \end{cases},$$

is an optimal selling mechanism.

Proof. $H(q) \geq G(q)$ for all q and x is increasing, so x differentiable almost everywhere)

$$\int H(F(\theta)) - G(F(\theta)) \frac{dx(\theta)}{d\theta} d\theta \geq 0$$

Integrate by parts,

$$\begin{aligned} \int H(F(\theta)) \frac{dx(\theta)}{d\theta} d\theta &= [H(F(\theta)) x(\theta)]_{\underline{\theta}}^{\bar{\theta}} - \int h(F(\theta)) f(\theta) x(\theta) d\theta \\ &= H(1)x(1) - H(0)x(0) - \int x(\theta) h(F(\theta)) f(\theta) d\theta \\ \int G(F(\theta)) \frac{dx(\theta)}{d\theta} d\theta &= G(1)x(1) - G(0)x(0) - \int x(\theta) g(F(\theta)) f(\theta) d\theta \end{aligned}$$

But $H(1) = G(1)$ and $H(0) = G(0)$. Hence we conclude that

$$\int x(\theta) h(F(\theta)) f(\theta) d\theta \leq \int x(\theta) g(F(\theta)) f(\theta) d\theta$$

holds for any increasing function $x(\theta)$. Since p^* maximizes the right hand side we have that for every other increasing function x

$$\begin{aligned} \int x^*(\theta) g(F(\theta)) f(\theta) d\theta &\geq \int x(\theta) h(F(\theta)) f(\theta) d\theta \\ &= \int x(\theta) \left[\theta - \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta, \end{aligned}$$

which completes the proof. ■

16 Public Goods

Consider a simple public good economy with a binary public good, a set of agents $I = \{1, \dots, n\}$, type spaces $[\underline{\theta}_i, \bar{\theta}_i]$, and maintain the smoothness and independence assumptions on distributions from the analysis of optimal auctions. As we did there, we let $f_i(\theta_i)$ and $F_i(\theta_i)$ denote the probability density and cumulative density respectively. The individual payoffs are taken to be

$$\begin{aligned} \theta_i - t_i & \quad \text{if } i \text{ consumes the good and transfers } t_i \text{ units to provider} \\ -t_i & \quad \text{if agent } i \text{ does not consume the good and transfers } t_i \text{ units to provider} \end{aligned},$$

which is identical to the formulation in the optimal auction setup. The only difference is that agent i consumes the good and receives utility gain θ_i if and only if all other agents consume the good. Intuitively, this creates a positive externality (as opposed to a negative externality in the case of private goods).

We assume that the cost of provision is some $C > 0$.

16.1 The Efficient Provision rule, the Groves-Clarke Mechanism, and Bayesian Implementation

A direct mechanism can be taken to be a pair $\langle p, t \rangle$, where $p : \Theta \rightarrow [0, 1]$ is the probability of provision as a function of announced types and $t : \Theta \rightarrow \mathbb{R}^n$ are the transfers. The expected surplus corresponding with mechanism $\langle p, t \rangle$ is

$$\mathbb{E} \left[\left(\sum_{i=1}^n \theta_i - C \right) p(\theta) \right],$$

which and we see directly that the ex ante Pareto efficient rule is to set $p^*(\theta) = 1$ if $\sum_i \theta_i \geq C$ and 0 otherwise.¹ Consider the following *Groves-Clarke mechanism* (where we make the distinction between announced types $\hat{\theta}$ and true types θ in order to be able to distinguish dominant strategies),

$$p^*(\hat{\theta}) = \begin{cases} 1 & \text{if } \sum_i \hat{\theta}_i \geq C \\ 0 & \text{otherwise} \end{cases},$$

$$t_i(\hat{\theta}) = \begin{cases} C - \sum_{j \neq i} \hat{\theta}_j & \text{if } \sum_i \hat{\theta}_i \geq C \\ 0 & \text{otherwise} \end{cases}$$

The payoff for player i is .

$$u_i(\theta_i, \hat{\theta}) = p^*(\hat{\theta}) (\theta_i - t_i(\hat{\theta})) = \begin{cases} \theta_i + \sum_{j \neq i} \hat{\theta}_j - C & \text{if } \sum_{j=1}^n \hat{\theta}_j \geq C \\ 0 & \text{otherwise} \end{cases}$$

and it is quite easy to see that truthful reporting is a dominant strategy for every player.

¹In these notes we deal exclusively with quasi-linear/transferrable utility models. Ex ante Pareto efficiency is then the same as surplus maximization. However, one can think of examples where it is reasonable to assign a higher welfare weight on certain types than others, which corresponds to interim Pareto efficiency. The set of interim Pareto efficient allocation rules is clearly larger. See Ledyard and Palfrey [?] for a characterization.

Remark 6 *One explanation of the dominant strategy result is that each agent is bribed with or pays what would be the social surplus for the rest of the economy if reports are truthful. This implies that the agent behaves as if maximizing social surplus in an economy where the reports of others are the true valuations. Since we consider private values, truthful reporting solves the problem for each agent and the ex post efficient rule is thus implementable in dominant strategies.*

16.2 More General Groves-Clarke Mechanisms

Groves-Clarke mechanisms are applicable to environments with transferable utility and private values, so the payoff function for agent i will be taken to be on the form

$$u_i(a, \theta_i) - t_i.$$

We could build it into the payoff functions, but we will assume that $a \in A$ costs $C(a)$ as this is a more intuitive way of modelling the costs a mechanism $\langle y, t \rangle$ is a Groves mechanism if

$$\begin{aligned} y(\theta) &\in \arg \max_{a \in A} \sum_{i=1}^n u_i(a, \theta_i) - C(a) \\ t_i^G(\theta) &= C(y(\theta)) - \sum_{j \neq i} u_j(y(\theta), \theta_j) + h_i(\theta_{-i}) \end{aligned}$$

By defining $S(\theta) = \max_{a \in A} \sum_{i=1}^n u_i(a, \theta_i) - C(a)$ as the efficient social surplus given type profile θ we can write the Groves transfers as

$$t_i^G(\theta) = -S(\theta) + u_i(y(\theta), \theta_i) + h_i(\theta_{-i}).$$

This formulation should make it clear that agents are bribed with the social surplus, less the own utility, which makes all agents fully internalize the effects from their own choices on other agents.

16.3 Failure of Ex Post Budget Balance

We know that (subject to some unstated conditions)

Theorem 1 *Suppose that each Θ_i is convex. Then, $\langle y, t \rangle$ is a dominant strategy incentive compatible mechanism that implements the ex post efficient outcome if and only if it is a Groves-Clarke mechanism.*

In addition we know that Groves-Clarke mechanisms are rarely budget balancing.

Theorem 2 *In general, there exists no Groves-Clarke mechanism that satisfies ex post budget balance.*

The response to this general impossibility of satisfying dominant strategy incentive compatibility and ex post budget balance was to weaken the notion of implementation to Bayesian implementation.

16.4 Example: The Consequences of Imposing Budget Balance and Individual Rationality/Veto Power

Consider a simple version of Mailath and Postlewaite [?] with two agents, a and b , whose valuations are i.i.d. draws from $\Theta = \{0, 1\}$ and the cost of providing the good is C . The probability of a high valuation ($\theta_i = 1$) is given by α . For each $j, k \in \{0, 1\}^2$, write ρ_{jk} for the probability of provision when agent a announces valuation $\theta_a = j$ and agent 2 announces $\theta_b = k$ and let t_{jk}^i be the taxes paid for $i = a, b$. The incentive compatibility constraints are then

$$\begin{aligned}
(1 - \alpha)\rho_{10}(1 - t_{10}^a) + \alpha\rho_{11}(1 - t_{11}^a) &\geq (1 - \alpha)\rho_{00}(1 - t_{00}^a) + \alpha\rho_{01}(1 - t_{01}^a) & \text{(IC)} \\
-(1 - \alpha)\rho_{00}t_{00}^a - \alpha\rho_{01}t_{01}^a &\geq -(1 - \alpha)\rho_{10}t_{10}^a - \alpha\rho_{11}t_{11}^a \\
(1 - \alpha)\rho_{01}(1 - t_{01}^b) + \alpha\rho_{11}(1 - t_{11}^b) &\geq (1 - \alpha)\rho_{00}(1 - t_{00}^b) + \alpha\rho_{10}(1 - t_{10}^b) \\
-(1 - \alpha)\rho_{00}t_{00}^b - \alpha\rho_{10}t_{10}^b &\geq -(1 - \alpha)\rho_{01}t_{01}^b - \alpha\rho_{11}t_{11}^b
\end{aligned}$$

for player a and the individual rationality constraints reads

$$\begin{aligned}
(1 - \alpha)\rho_{10}(1 - t_{10}^a) + \alpha\rho_{11}(1 - t_{11}^a) &\geq 0 & \text{(IR)} \\
-(1 - \alpha)\rho_{00}t_{00}^a - \alpha\rho_{01}t_{01}^a &\geq 0 \\
(1 - \alpha)\rho_{01}(1 - t_{01}^b) + \alpha\rho_{11}(1 - t_{11}^b) &\geq 0 \\
-(1 - \alpha)\rho_{00}t_{00}^b - \alpha\rho_{10}t_{10}^b &\geq 0
\end{aligned}$$

and the relevant constraints for b are analogous. Budget balance implies that

$$t_{jk}^a + t_{jk}^b \geq C \quad \text{(BB)}$$

for each $j, k \in \{0, 1\}^2$. The mechanism design problem is thus to solve

$$\max \alpha^2 \rho_{11} (2 - C) + \alpha (1 - \alpha) \rho_{10} (1 - C) + \alpha (1 - \alpha) \rho_{01} (1 - C) - (1 - \alpha)^2 \rho_{00} C$$

subject to the constraints in s.t. (15),(15) and (BB). Suppose that $0 < C < 1$. Clearly this means that the *ex post* efficient rule is to set

$$\begin{aligned}\rho_{11}^* &= \rho_{10}^* = \rho_{01}^* = 1 \\ \rho_{00}^* &= 0\end{aligned}$$

However, to satisfy the (IR) constraints it is necessary that $t_{01}^{a*} = t_{10}^{b*} = 0$, which using budget balance means that $t_{01}^{b*} = t_{10}^{a*} = C$. Plugging into the (IC) constraints for the high type we get

$$(1 - \alpha)(1 - C) + \alpha(1 - t_{11}^{j*}) \geq \alpha$$

for $j = a, b$. Observe that $t_{11}^{a*} + t_{11}^{b*} \geq C$. The left hand side is monotonic (decreasing), which means that only if

$$\begin{aligned}(1 - \alpha)(1 - C) + \alpha\left(1 - \frac{C}{2}\right) &\geq \alpha \\ &\Downarrow \\ \alpha &\leq \frac{2(1 - C)}{2 - C}\end{aligned}$$

is it possible to satisfy both (IC) constraints for the high types. For example, with $C = 1/2$, this tells us that the *ex post* efficient rule is only implementable if $\alpha \leq 2/3$ and that as $C \rightarrow 1$, the *ex post* efficient rule is implementable only when α is arbitrarily small.

17 Correlated Types: an Example

Consider a discretized version of the optimal auction setup with two bidders and valuations $\theta_i \in \{l, h\}$ with $l < h$. We know that a second price auction is dominant strategy incentive compatible (regardless of how ties are broken). Hence, we may get economic efficiency by implementing, say, a symmetric second price auction. Write t_{jk} for transfer when the player in question has valuation j and the other player has valuation k . These payments are

$$\begin{aligned}t_{hh} &= \frac{1}{2}h \\ t_{hl} &= l \\ t_{lh} &= 0 \\ t_{ll} &= \frac{1}{2}l\end{aligned}$$

and the associated efficient allocation rule is

$$\begin{aligned}
p_{hh} &= \frac{1}{2} \\
p_{hl} &= 1 \\
p_{lh} &= 0 \\
p_{ll} &= \frac{1}{2}
\end{aligned}$$

Assume that the probability distribution is symmetric and given by

player 1 \ player 2	h	l	
h	α	γ	(15)
l	γ	β	

where $\gamma = \frac{1-\alpha-\beta}{2}$.

We now want to adjust the transfers in the second price auction so that the **surplus gets extracted from the bidders** and transferred to the seller of the object **without reducing the total surplus generated**. If this works, then the seller is in a position analogous to a perfectly price discriminating monopolist.

We can always add a component to the transfer that depends on the private information of the other agent only without changing incentives. So, let $g_{-i}(h)$ denote a term added to the transfer if the other agent is of type $\theta_j = h$ and $g_{-i}(l)$ be a term paid if the other agent is of type l . If we can find $g_{-i}(h)$ and $g_{-i}(l)$ so that

$$\begin{aligned}
\Pr[\theta_{-i} = h | \theta_i = h] [-g_{-i}(h)] + \Pr[\theta_{-i} = l | \theta_i = h] [h - l - g_{-i}(l)] &= 0 \\
\Pr[\theta_{-i} = h | \theta_i = l] [-g_{-i}(h)] + \Pr[\theta_{-i} = l | \theta_i = l] [-g_{-i}(l)] &= 0
\end{aligned}$$

is satisfied it means that the interim expected utility of each bidder is driven down to the participation level. Expressing the conditional probabilities in terms of the entries in the matrix in (15) we rewrite these equations as

$$\begin{aligned}
-\frac{\alpha}{\alpha + \gamma} g_{-i}(h) + \frac{\gamma}{\alpha + \gamma} [h - l - g_{-i}(l)] &= 0 \\
\frac{\gamma}{\gamma + \beta} g_{-i}(h) + \frac{\beta}{\gamma + \beta} g_{-i}(l) &= 0.
\end{aligned}$$

The second equation implies that $g_{-i}(h) = -\frac{\beta}{\gamma}g_{-i}(l)$, and substituting into the first equation we have that

$$\frac{\alpha\beta}{\gamma}g_{-i}(l) + \gamma[h - l - g_{-i}(l)] = 0.$$

We first note that if $\alpha\beta = \gamma^2$ there is no way that the equation can be solved. In this case we can solve for the relation between α and β to get $\sqrt{\beta} = 1 - \sqrt{\alpha}$, so

player 1 \ player 2	h	l
h	$\sqrt{\alpha}\sqrt{\alpha}$	$\sqrt{\alpha}(1 - \sqrt{\alpha})$,
l	$\sqrt{\alpha}(1 - \sqrt{\alpha})$	$(1 - \sqrt{\alpha})^2$

which is the case with independent valuations. In any other case we can solve for

$$g_{-i}(l) = \frac{\gamma^2(h - l)}{\gamma^2 - \alpha\beta}$$

$$g_{-i}(h) = -\frac{\beta\gamma(h - l)}{\gamma^2 - \alpha\beta},$$

meaning that it is indeed possible for the seller to extract the full surplus of the transaction regardless of how tiny the correlation between the types is. One may however note that very large transfers are required when the correlation is small, which can be seen by observing that $|g_{-i}(l)| \rightarrow \infty$ as $\gamma^2 \rightarrow \alpha\beta$.