

## 2 Extensive Form Games

**Definition 1** A finite extensive form game is an object

$$K = \{I, (T, \prec), P, A, H, u, \rho\},$$

where

- $I = \{0, 1, \dots, n\}$  is the set of agents (0 denotes “nature”)
- $(T, \prec)$  is the game tree
- $P$  is the player partitioning
- $A$  is the set of actions
- $H$  is the informational partitioning
- $u : Z \rightarrow \mathbb{R}^n$  is the payoff function, where  $Z \subset T$  is the set of endnodes
- $\rho$  is the probability distribution over moves by nature.

Next, we will fill in the details:

**Definition 2** A (finite) game tree  $(T, \prec)$  consists of a finite set of nodes  $T$  and a binary relation  $\prec$  on  $T$  (with interpretation “comes before”) satisfying:

1. *Asymmetry:* there exists no  $t, t'$  such that  $t \prec t'$  and  $t' \prec t$ .
2. *Transitivity:* if  $t \prec t'$  and  $t' \prec t''$ , then  $t \prec t''$ .
3. *Each node is complete description of the history:* for all  $t, t', t'' \in T$ , if  $t \prec t''$  and  $t' \prec t''$ , then either  $t \prec t'$  or  $t' \prec t$ .

Together, these assumption implies that there is a unique path to every node.

Let

$$Z = \{t \in T \mid \nexists t' \in T \text{ s.t. } t \prec t'\}$$

**Definition 3**  $P = \{P_j\}_{j=0}^n$  is a player partitioning if it is a partition of  $T \setminus Z$ .

Can generate such a partition from function (each node belongs to single player)

$$i : T \setminus Z \rightarrow \{0, 1, \dots, n\}$$

We would then have that

$$P_j = i^{-1}(j) = \{t \in T \setminus Z \text{ such that } i(t) = j\}.$$

**Definition 4** For every  $t \in T \setminus Z$  let  $S(t)$  denote the immediate successors,

$$S(t) = \{t' \in T \mid t \prec t' \text{ and } \nexists t'' \in T \text{ s.t. } t \prec t'' \prec t'\}$$

**Definition 5** Let  $A = \{A(t)\}_{t \in T \setminus Z}$ , where  $A(t)$  is a finite set and  $\alpha = \{\alpha(t)\}_{t \in T \setminus Z}$  be a vector of one-to-one functions

$$\alpha(t) : A(t) \rightarrow S(t),$$

then  $A(t)$  describes the actions at node  $t \in T \setminus Z$  and  $\alpha(t)$  describes how the actions lead to successor nodes.

To see that  $\alpha(t)$  must be one-to-one we note that if we would have several actions mapping into the same successor, then the consequence of the actions are identical, as we would just specify multiple paths in between two nodes. This would clearly be redundant.

**Definition 6**  $H$  is an information partition if it is a list  $\{H_j\}_{j=1}^n$  where each  $H_j$  is a partition of  $P_j$  (the set of nodes where  $j$  acts) and satisfies:

1. For every  $t \in P_j$  there exists  $h_j \in H_j$  such that  $t \in h_j$  (all nodes belonging to a player is in some information set)
2. For  $h_j \in H_j$  and every  $t, t' \in h_j$  we have that  $t \not\prec t'$  and  $t' \not\prec t$
3. For  $h_j \in H_j$  and every  $t, t' \in h_j$  we have that  $A(t) = A(t') \equiv A(h_j)$

Notive that assuming that  $H_j$  is a partition of  $P_j$  implies that  $h_j \cap h'_j = \emptyset$  for every  $h_j, h'_j \in H_j$ .

**Definition 7** An extensive form payoff function is a map  $u_i : Z \rightarrow R$

Assuming that if nature moves, then it is only at the “root” of the game (this is without loss of generality) we can define the probability distribution over moves by nature as follows.

**Definition 8** Let  $t_0$  be the unique node with such that there exists no  $t \prec t_0$  (eg.  $t_0$  is the root) and suppose that  $P_0 = \{t_0\}$ . Then the distribution over moves by nature is a probability distribution over the immediate succesors to  $t_0$ ,  $\rho \in \Delta(S(t_0))$ .

Usually it is assumed that players do not forget anything they have learnt or done (EXAMPLE-did I wash my hair?):

**Definition 9** An extensive form game  $K$  is said to be of perfect recall if for every  $h'_j \in H_j$ , every  $(t', t'') \in h'_j$  and every  $t \prec t'$  where player  $j$  moves there exists a node  $\hat{t}$  (possibly  $t$ ) such that:

1. there is some information set  $h_j \in H_j$  such that  $(t, \hat{t}) \in h_j$
2.  $\hat{t} \prec t''$

Intuitively, when  $t'$  and  $t''$  are in the same information set it means that the nodes are distinguished by information the player doesn't have. Fixing a predecessor  $t$ , the existence of a predecessor  $\hat{t}$  in the same infomration set says that the information distinguishing  $t'$  and  $t''$  is not known when the agent is in node  $t$ . As this must be true for all predecessors, the definition rules out any imperfect recall.

**Definition 10** A (pure) strategy for player  $i$  is a map  $s_i : H_i \rightarrow \cup_{h_i \in H_i} A(h_i)$  satisfying  $s_i(h_i) \in A(h_i)$ .

That is, a strategy is a fully specified contingent plan of action.

We note that:

- If there are no chance moves, a pure strategy profile  $s$  specifies a unique set of nodes that takes us from the root of the game to a terminal node.
- Call the nodes the **outcome path**
- Call the terminal node reached by  $s$  the **outcome**.
- If nature has non-trivial moves, let the outcome be the **distribution over  $Z$  given a particular strategy profile**.

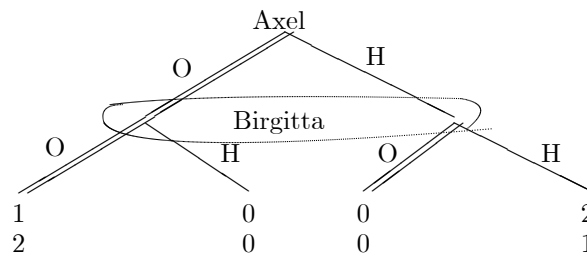


Figure 1: The Battle of the Sexes in Extensive Form

**Example 1** Consider Figure 1, which depicts a version of the battle of the sexes in extensive form ( $O$  is for Opera and  $H$  is for Hockey).. Notice that the example should illustrate why each node in the same information set must have the same available actions. If the set of actions following opera would be different from the set of actions following hockey Birgitta would understand whether opera or hockey would be played. Also note that, since there is nothing to condition on, the notion of a strategy is no different from that in the strategic form and that the extensive and the strategic form representations clearly contain the same information.

**Example 2** In Figure 2 we consider a variant of the battle of the sexes with Axel moving first. A strategy for Axel is then

$$S_{AXEL} = \{O, H\},$$

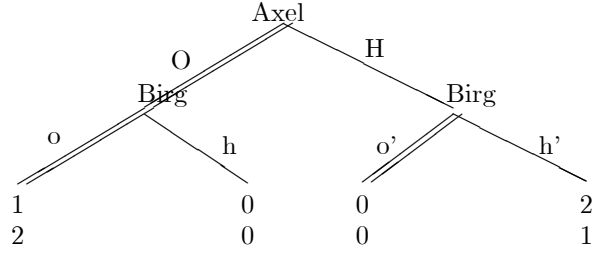


Figure 2: The Battle of the Sexes with Axel Moving First

whereas Birgitta now actually can condition play on Axels actions,

$$S_{BIRG} = \{oo', oh', ho', hh'\}.$$

Notice that we can represent this as a normal form game

	$oo'$	$oh'$	$ho'$	$hh'$
$O$	1, 2	1, 2	0, 0	0, 0
$H$	0, 0	2, 1	0, 0	2, 1

In general:

- If there are no moves by nature, then extensive form payoffs can be translated into normal form payoffs by letting

$$u_i(s) = u_i(z),$$

where  $z$  is the (unique) endnode implied by strategy profile  $s$

- If there are moves by nature, let  $p \in \Delta(Z)$  be the probability distribution induced by  $s$  and let

$$u_i(s) = \sum_z p(z) u_i(z)$$

**Example 3** Return to the battle of the sexes in Figure 2. Notice that we have three Nash equilibria,  $(O, oo')$  and  $(H, oh')$  and  $(H, hh')$ . Note:

1. In the extensive form we only need to check nodes on the equilibrium path. Off equilibrium path play still needed to check payoff from deviations.

2. In the normal form we check the best responses

	$oo'$	$oh'$	$ho'$	$hh'$
$O$	$\underline{1}, \underline{2}$	$1, \underline{2}$	$\underline{0}, 0$	$0, 0$
$H$	$0, 0$	$\underline{2}, \underline{1}$	$\underline{0}, 0$	$\underline{2}, \underline{1}$

Obviously the set of Nash equilibria is the same in the extensive and normal forms.

3. Note that  $oh'$  weakly dominates all the other strategies. Hence,  $(H, oh')$  is the unique prediction from iterative elimination of weakly dominated strategies.

4. In this class of games, iterative elimination of weakly dominated strategies has a different interpretation. Suppose that we require play to be sequentially rational in that each player must optimize in every information set. In this example, that tells us that Birgitta must play  $oh'$ , implying that the (backwards induction) outcome is  $(H, oh')$ . Intuitively, the Nash equilibrium  $(O, oo')$  is implausible as it relies on the non-credible threat to carry out an action that would be suboptimal should Axel pick  $H$ .

## 2.1 Backwards Induction

**Definition 11** An extensive form game  $K$  is said to have perfect information if  $P_j = H_j$ , implying that every information set is a singleton.

The battle of the sexes with first mover advantage just considered is a simple example of such a game. Consider the following procedure:

1. First consider the set of nodes  $T^1$  that are immediate precursor to terminal nodes.

That is

$$T^1 = \{t \in T \mid \nexists t' \in T \text{ s.t. } t \prec t' \prec z\}.$$

Now, for every  $i \in I$  and  $t \in P_i \cap T^1$  we observe that an action is the same as picking a terminal node. Hence, rationality at node  $t$  requires that agent  $i$  solves

$$\max_{z \in Z \mid t \prec z} u_i(z).$$

For every  $t \in T^1$ , let  $z(t) \in \{z \in Z \mid t \prec z\}$  be an optimal solution, which exists because it is a finite maximization problem.

2. Next, let  $T^2 \subset T \setminus \{T^1 \cup Z\}$  that are immediate precursor to the nodes in  $T^1$ ,

$$T^2 = \{t \in T \mid \exists t' \in T^1 \text{ s.t. } t \prec t' \text{ and } \nexists t'' \text{ s.t. } t \prec t'' \prec t'\}.$$

Now, an action is just picking a node in  $t' \in T^1$ , which corresponds to a terminal node  $z(t')$  so rationality requires that the agent solves

$$\max_{t' \in T^1} u_i(z(t')).$$

Again, we may simply denote a solution by  $z(t)$ , which exists because we are maximizing over a finite set of alternatives.

3. Continuing inductively it is clear that:

- If each node in  $T^K$  corresponds to a unique terminal node (after arbitrary tie-breaking), then there exists optimal actions given the (sequentially rational) continuation play in every node in  $T^{K+1}$ .
- If  $T$  is finite then the process stops in a finite number of recursions.
- In the final step,  $T^K$  is a singleton. Player controlling that node faces a simple decision problem (solution exists due to finiteness in every step)

We have just described the process of backwards induction. Backwards induction equilibria are:

- Applicable in games of perfect information.
- “Credible”. Requires optimal behavior in all nodes. Nash equilibrium requires optimal behavior in all nodes along the equilibrium path  $\Rightarrow$  backwards induction equilibria are Nash, but the converse is not true.

We have thus proved:

**Theorem 1 (Zermelo-Kuhn)** *Any finite game of perfect information has at least one backwards induction equilibrium. Hence, a Nash equilibrium in pure strategies exists for this class of games.*

**2.2 The Reduced Normal Form**

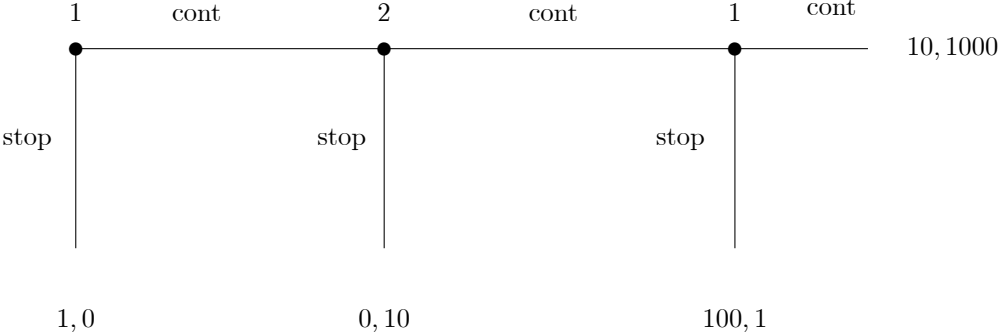


Figure 3: The Centipede Game

**Example 4** *Consider the version of Rosenthal’s centipede game depicted in Figure 3. To derive the normal form, let  $s$  ( $c$ ) denote stop (continue) by player at the first node, and  $s'$  ( $c'$ ) denote stop (continue) by player at the third node. We then have*

	$S$	$C$
$ss'$	1, 0	1, 0
$sc'$	1, 0	1, 0
$cs'$	0, 10	100, 1
$cc'$	0, 10	10, 1000

*Obviously,  $ss'$  and  $sc'$  are equivalent strategies in the sense that the outcome is the same against either  $S$  or  $C$  (for the simple reason that  $s$  is a move that makes the choice between*

$s'$  and  $c'$  irrelevant for the outcome). Deleting one of the equivalent strategies we obtain the **reduced normal form**

	$S$	$C$
$s$	1, 0	1, 0
$cs'$	0, 10	100, 1
$cc'$	0, 10	10, 1000

In the example, we deleted a strategy because  $s$  meant that the choice between  $s'$  and  $c'$  irrelevant. In general two strategies are strategically equivalent if different in information sets that the player herself exclude from being reached by an earlier move.

**Remark 1** *There is a little literature on the (lack of) logical foundations for backwards induction reasoning. In the context of the centipede game, the analysis says that:*

1. *Player 1 will stop if the third node is reached ( $100 > 10$ )*
2. *understanding that player 1 will stop if play reaches the third node, player 2 stops if the second node is reached ( $10 > 1$ )*
3. *in the first node player 1 should therefore stop too ( $1 > 0$ )*

The “problem” is that if both players understand this analysis, then the second node should never be reached as this is inconsistent with backwards induction. Hence, a seemingly fair question at node 2 would be: why on earth did player 1 continue? While the answer to this question is unclear, it is obvious that reaching the node implies that player 1 failed to backwards induct. The key question is then: should we assume that player 1 will be able to backwards induct at the third node when he/she failed so miserably in the first node?

Notice that:

1.  $cc'$  is weakly dominated by  $cs'$  (because  $s'$  is optimal after C), which leaves

	$S$	$C$
$s$	1, 0	1, 0
$cs'$	0, 10	100, 1

2. After eliminating  $cc''$ ,  $C$  is weakly dominated by  $S$ , which leaves

$$\begin{array}{r} S \\ s \quad 1, 0 \\ cs' \quad 0, 10 \end{array}$$

3. What remains is a decision problem, where  $cs'$  is dominated by  $s$ .

In the example it is clear that backwards induction is the same thing as iterative elimination of weakly dominated strategies. This is generally true for games of perfect information (implying that iterative elimination of weakly dominated strategies has the same logical difficulties as backwards induction).