6 The Principle of Optimality

Definition 1 *A $T-$shot deviation from a strategy s_i is a strategy $\widehat{s}_i$ such that there exists T such that*

$$\widehat{s}_i\left(h_t\right) = s_i\left(h_t\right)$$

for all $h_t \in H$ with $t \geq T$.

Definition 2 *A one-shot deviation from a strategy s_i is a strategy $\widehat{s}_i$ such that there exists some unique t and $\widetilde{h}_t \in H_t$ such that*

$$\widehat{s}_i\left(\widetilde{h}_t\right) \neq s_i\left(\widetilde{h}_t\right).$$

In other words, $s_i\left(h_t\right) = \widehat{s}_i\left(h_t\right)$ for every $h_t \neq \widetilde{h}_t$.

Proposition 1 *Suppose that there exists some $M < \infty$ such that $-M \leq u_i\left(s\right) \leq M$ for all $s \in S$. Then, a necessary and sufficient condition for $s = \left(s_1, .., s_n\right)$ to be subgame perfect in the infinitely repeated game $G\left(\delta, \infty\right)$ is that there exists no profitable one-shot deviation after any history $h_t \in H$.*

For proving this:

1. Necessity is trivial.
2. We first use continuity to show that, if there is a profitable deviation, then there is a profitable finite period deviation.
3. The second step proceeds inductively to locate a profitable one-shot deviation for every finite period deviation.

Proof. Suppose that s is not subgame perfect. We want to show that there is a one-shot deviation.

By definition of subgame perfection there is some h_τ and $i \in I$ such that $s_i|h_\tau$ is not a best reply against $s_{-i}|h_\tau$. Thus, there exists $\widehat{s}_i$ such that the continuation payoff from sticking to s_i is below that of deviating to $\widehat{s}_i$. We write this as

$$U_i\left(s_i, s_{-i}|h_\tau\right) < U_i\left(\widehat{s}_i, s_{-i}|h_\tau\right)$$

Define

$$\varepsilon = U_i\left(\widehat{s}_i, s_{-i}|h_\tau\right) - U_i\left(s_i, s_{-i}|h_\tau\right) > 0$$

and let T be such that

$$\delta^{T-\tau} M < \frac{\varepsilon}{3}.$$

Now, consider a $T-\tau$ shot deviation $\overline{s}_i$ where

$$\overline{s}_i\left(h_t\right) = \begin{cases} \widehat{s}_i\left(h_y\right) & \text{if } t \leq T \\ s_i\left(h_t\right) & \text{if } t > T \end{cases}.$$

Now, let $\widehat{h}_T$ be the history implied by playing continuation strategies $\left(\widehat{s}_i, s_{-i}\right)$ up to time T after history h_τ. Then, we may write

$$\begin{aligned}
& U_i\left(\overline{s}_i, s_{-i}|h_\tau\right) - U_i\left(s_i, s_{-i}|h_\tau\right) \\
= & \sum_{t=\tau}^{\infty} \delta^t u_i\left(a^t\left(\overline{s}_i, s_{-i}|h_\tau\right)\right) - U_i\left(s_i, s_{-i}|h_\tau\right) \\
= & \underbrace{\sum_{t=\tau}^{\infty} \delta^t u_i\left(a^t\left(\overline{s}_i, s_{-i}|h_\tau\right)\right)}_{=\sum_{t=\tau}^{T} \delta^t u_i\left(a^t\left(\widehat{s}_i, s_{-i}|h_\tau\right)\right) + \sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(s|\widehat{h}_T\right)\right)} - U_i\left(s_i, s_{-i}|h_\tau\right) \\
& + \sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(\widehat{s}_i, s_{-i}|h_\tau\right)\right) - \sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(\widehat{s}_i, s_{-i}|h_\tau\right)\right) \\
= & U_i\left(\widehat{s}_i, s_{-i}|h_\tau\right) - U_i\left(s_i, s_{-i}|h_\tau\right) + \sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(s|\widehat{h}_T\right)\right) - \sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(\widehat{s}_i, s_{-i}|h_\tau\right)\right) \\
= & \varepsilon + \underbrace{\sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(s|\widehat{h}_T\right)\right)}_{\geq -\delta^{T-\tau} M > -\frac{\varepsilon}{3}} - \underbrace{\sum_{t=T+1}^{\infty} \delta^t u_i\left(a^t\left(\widehat{s}_i, s_{-i}|h_\tau\right)\right)}_{\leq \delta^{T-\tau} M < \frac{\varepsilon}{3}} \geq \varepsilon - 2\delta^{T-\tau} M < \frac{\varepsilon}{3}.
\end{aligned}$$

Hence, there is a finite shot deviation.

Now,

- Let $\overline{s}_i$ be a T-shot deviation from s_i.
- Without loss of generality, let the deviation start at $t = 1$.
- For each $t = 1, ..., T$ let $\overline{h}_t$ be the history induced by play of $(\overline{s}_i, s_{-i})$

Now, consider the one-shot deviation s_i^T where

$$s_i^T(h_t) = \begin{cases} \overline{s}_i(h_t) & \text{if } h_t = \overline{h}_T \\ s_i(h_t) & \text{if } h_t \neq \overline{h}_T \end{cases} .$$

There are now two possibilities:

1. Either s_i^T is a profitable one show deviation from s_i,
2. or playing
$$\overline{s}_i \begin{cases} \overline{s}_i(h_t) & \text{if } t \leq T-1 \\ s_i(h_t) & \text{if } t > T-1 \end{cases}$$
is a profitable $T-1$-shot deviation. Proceeding inductively it follows that there must be a profitable one-shot deviation. ■

6.1 Supporting Inefficient Play in Infinitely Repeated Games

The focus in many applications of repeated games is on supporting good/efficient outcomes. However, the same ideas can also be used to support bad/inefficient outcomes. Consider the stage game

	L	C	R
T	$5, 5$	$4, 4$	$0, 0$
M	$4, 4$	$3, 3$	$0, 0$
B	$0, 0$	$0, 0$	$-2, -2$

Clearly (T,L) is the unique stage game Nash equilibrium as well as the unique Pareto efficient outcome. Consider the infinite repetition and let

$$s_1(h_0) = \text{M and } s_1(h_t) = \begin{cases} \text{M} & \text{if } (a_{t-2}, a_{t-1}) = (\text{MC,MC}) \text{ or } a_{t-2} \neq \text{MC and } a_{t-1} = \text{BR} \\ \text{B} & \text{otherwise} \end{cases}$$

$$s_2(h_0) = \text{C and } s_2(h_t) = \begin{cases} \text{C} & \text{if } (a_{t-2}, a_{t-1}) = (\text{MC,MC}) \text{ or } a_{t-2} \neq \text{MC and } a_{t-1} = \text{BR} \\ \text{R} & \text{otherwise} \end{cases}$$

If the players follow this strategy the outcome is (MC,MC,....) giving each player a payoff of

$$\frac{3}{1-\delta} = 3 + \delta 3 + \sum_{t=2}^{\infty} \delta^t 3$$

A one shot deviation to T (for player 1) or L (for player 2) would give payoff

$$4 - \delta + \sum_{t=2}^{\infty} \delta^t 3$$

Hence (this is the best on the equilibrium path deviation), the specified strategy is a Nash equilibrium if

$$\begin{aligned} \frac{3}{1-\delta} &= 3 + \delta 3 + \sum_{t=2}^{\infty} \delta^t 3 \geq 4 - \delta + \sum_{t=2}^{\infty} \delta^t 3 \\ &\iff \\ \delta &\geq \frac{1}{4} \end{aligned}$$

To check for subgame perfection, we now only need to rule out profitable deviations from the punishment phase. Using the Principle of optimality, we only need to check one-shot deviations, that is whether

$$\begin{aligned} -2 + 3\delta + \sum_{t=2}^{\infty} \delta^t 3 &\geq 0 - 2\delta + \sum_{t=2}^{\infty} \delta^t 3 \\ &\iff \\ \delta &\geq \frac{2}{5}. \end{aligned}$$

Hence, the strategy profile is subgame perfect.

7 Folk Theorems

7.1 A Simple Folk Theorem

Now consider an arbitrary stage game $G = (n, A, u)$. Let $NE(G)$ denote the set of Nash equilibria to the stage game. Now:

Definition 3 *A strategy profile s^* is called a Nash reversion profile if there exists some stage game Nash equilibrium $a^* \in NE(G)$ and a sequence $\{a^t\}_1^\infty \in A^\infty$ such that*

$$s_i^*(h_t) = \begin{cases} a_i^t & \text{if } h_t = (a^1, ..., a^{t-1}) \\ a_i^* & \text{if } h_t \neq (a^1, ..., a^{t-1}) \end{cases}$$

Lemma 1 *A Nash reversion strategy that calls for playing path $\{a^t\}_1^\infty$ with Nash reversion threat a^* is subgame perfect if and only if*

$$U_i\left(s^* | h_t = \left(a^1, ..., a^{t-1}\right)\right) = \sum_{\tau=t}^{\infty} u_i(a^\tau) \geq u_i\left(a_i, a_{-i}^t\right) + \frac{\delta}{1-\delta} u_i(a^*)$$

for every $a_i \in A_i$, t and i.

Proof. Obvious as playing a static Nash in every period is an equilibrium of subgames following a deviation and the condition stated in the Lemma says that deviating from the sequence is worse than triggering Nash reversion, implying that following the sequence is a Nash equilibrium in every subgame on the equilibrium path as well. ■

It is more or less immediate that:

Proposition 2 *Let $a \in A$ be a stage game action profile such that $u_i(a) > u_i(a^*)$ for all $i \in I$. Then, there exists some $\underline{\delta} < 1$ such that playing $a \in A$ in every period is supportable as a subgame perfect equilibrium of the infinite repetition of G.*

Proof. By Lemma 1 we have that we can support playing $a \in A$ in every period as a subgame perfect equilibrium by a threat of Nash reversion if and only if

$$\begin{aligned} \frac{u_i(a)}{1-\delta} &\geq u_i\left(a_i, a_{-i}^t\right) + \frac{\delta}{1-\delta} u_i(a^*) \\ &\Leftrightarrow \\ u_i(a) - u_i(a^*) &\geq (1-\delta) u_i\left(a_i, a_{-i}^t\right) \end{aligned}$$

Since $(1-\delta) u_i\left(a_i, a_{-i}^t\right) \rightarrow 0$ as $\delta \rightarrow 1$ and $u_i(a)-u_i\left(a^*\right)>0$ it follows that there is $\underline{\delta}<1$ such that $u_i(a)-u_i\left(a^*\right) \geq(1-\delta) u_i\left(a_i, a_{-i}^t\right)$ whenever $\delta \geq \underline{\delta}$. ■

7.2 More Sophisticated Folk Theorems

Nash reversion can be used to support also non-stationary paths of play, which can be used to convexify the set of supportable stage game payoffs:

Definition 4 *The set of feasible (average) payoffs in an infinitely repeated game is*

$$F = CONV\left\{x \in R^n \mid \text{there exists } a \in A \text{ s.t. } u(a) = x\right\}$$

Proposition 3 *For any $v \in F$ such that there exists some stage game Nash equilibrium a^* such that $v_i > u_i\left(a^*\right)$ there exists some $\underline{\delta} < 1$ such that there is a subgame perfect equilibrium of the infinite repetition of G where the payoff is $\frac{1}{1-\delta} v_i$ for each player $i \in I$.*

IDEA:

1. If $x \in F$ there exists some $\left\{a^1, a^2, .., a^k\right\}$ and some $\lambda \in \Delta^k$ such that

$$x_i = \sum \lambda^j a_i^j$$

2. Can approximate this by strategies where a^j is played approximately λ^j percent of the time. If δ is close enough to 1 the exact timing isn't important (but technical work is needed to show this).

3. Use same idea as in the simple Folk theorem to use Nash reversion to support the play.

Next, we note that it is not necessary to use Nash reversion as punishments. More severe punishments do exist.

Definition 5 *The minmax value in stage game is*

$$\underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i\left(a_i, a_{-i}\right)$$

Definition 6 *The set of individually rational payoffs is*

$$I = \{x \in R^n | v_i \geq \underline{v}_i \text{ for every } i \in I\}$$

Proposition 4 *For every $v \in F \cap I$ there exists some $\underline{\delta} < 1$ such that there is a subgame perfect equilibrium of the infinite repetition of G where the payoff is $\frac{1}{1-\delta} v_i$ for each player $i \in I$.*

Proof somewhat difficult, but idea similar to construction of equilibrium with inefficient play.

Remark 1 *Issues:*

1. *Minmax doesn't always exist.*

2. *Cheat with boundaries...must restrict to interior or rely on public randomizations.*

7.3 Renegotiation Proofness

- Subgame perfection without "bite" in many repeated game contexts.

- One idea: suppose that layers can "renegotiate" in every period. Equilibria such as trigger strategy equilibria are then somewhat suspect as players may try to strike a deal upon reaching punishment phase. "Let bygones be bygones and return to the jolly equilibrium path"...which would potentially destabilize the equilibrium.

- This idea is formalized as *renegotiation proofness*, which may be viewed as importing some cooperative ideas into non-cooperative game theory.

Consider the finite case first. Let G^T be a $T-$fold repetition of stage game G.

Definition 7 *Let the set of Nash equilibria of G^T that are not Pareto dominated be called the efficient equilibria of G^T, denoted $P\left(G^T\right)$. That is,*

$$P\left(G^T\right) = \left\{s \in NE\left(G^T\right) | \text{there exists no } s' \in NE\left(G^T\right) \text{ s.t. } u_i\left(s'\right) > u_i\left(s\right) \text{ for all } i \in I\right\}$$

Definition 8 *s^* is renegotiation proof if $s^*|h_t$ is an efficient equilibrium of the continuation game after any history h_t. That is $s^*|h_T \in P(G)$ for every history leading to the final stage, $s^*|h_{T-1} \in P(G^2)$ for every history leading to the penultimate stage and $s^*|h_t \in P\left(G^{T-t+1}\right)$ for every history of length t.*

Example 1 *Consider the case with G given by the following mix of a coordination game and a prisoners dilemma*

	D_A	D_B	C
D_A	$\underline{1}, \underline{1}$	$0, 0$	$\underline{4\frac{1}{4}}, 0$
D_B	$0, 0$	$\underline{2}, \underline{2}$	$\underline{4\frac{1}{4}}, 0$
C	$0, \underline{4\frac{1}{4}}$	$0, \underline{4\frac{1}{4}}$	$3, 3$

We have two stage game Nash equilibria. (D_A, D_A) and (D_B, D_B). First, consider G^3 and let4

$$\begin{aligned} s_i^1 &= C \\ s_i^2(h_2) &= s_i^3(h_3) = \begin{cases} D_B & \text{if } a^1 = CC \\ D_A & \text{if } a^1 \neq CC \end{cases} \end{aligned}$$

Since any history leads to repeated play of a stage game Nash in period 2 and 3 we have that play is Nash in all subgames starting at times 2 and 3. Hence we need to check whether there is a profitable one shot deviation in the first period, that is if

$$\begin{aligned} 3 + \delta 2 + \delta^2 2 &\geq 4\frac{1}{4} + \delta + \delta^2 \\ \delta(1+\delta) &\geq \frac{5}{4} \\ \left(\delta + \frac{1}{2}\right)^2 &\geq 1 \end{aligned}$$

which clearly holds for $\delta \geq \frac{1}{2}$. *Next, let* $T > 3$ *and consider the following strategies*

$$\begin{aligned} s_i^1 &= C \\ s_i^t &= \begin{cases} C & \text{if } t \leq T-1 \text{ and } h_t = (CC, ..., CC) \\ D_A & \text{if } t \leq T-1 \text{ and } h_t \neq (CC, ..., CC) \end{cases} \\ s_i^T(h_2) &= s_i^{T-1}(h_3) = \begin{cases} D_B & \text{if } h_{T-1} = (CC, ..., CC) \\ D_A & \text{if } h_{T-1} \neq (CC, ..., CC) \end{cases} \end{aligned}$$

It is easy to check that these strategies are subgame perfect given that $\delta \geq \frac{1}{2}$. *However, this is not renegotiation proof:*

1. *In the last period we see that* (D_B, D_B) *is the only efficient equilibrium in the stage game.*

2. *It follows that* (D_B, D_B) *in the first period and* (D_B, D_B) *in the second period is the only efficient equilibrium in* G^2.

3. *By induction, there is a unique renegotiation proof subgame perfect equilibrium which is to play* (D_B, D_B) *after any history of play.*

Example 2 *Consider the case with* G *given by the following mix of a coordination game and a battle of the sexes*

	F	O	C
F	$\underline{3}, \underline{1}$	$0, 0$	$\underline{5}, 0$
O	$0, 0$	$\underline{1}, \underline{3}$	$\underline{5}, 0$
C	$0, \underline{5}$	$0, \underline{5}$	$4, 4$

Again, we have two stage game Nash equilibria. (F, F) *and* (O, O). *First, consider* G^3 *and* *let*

$$\begin{aligned} s_i^1 &= C \\ s_i^2(h_2) &= \begin{cases} F & \text{if } a_1^1 = C \text{ or } a_2^1 \neq C \\ O & \text{if } a_1^1 \neq C \text{ and } a_2^1 = C \end{cases} \\ s_i^3(h_3) &= \begin{cases} O & \text{if } a_2^1 = C \text{ or } a_1^1 \neq C \\ F & \text{if } a_2^1 \neq C \text{ and } a_1^1 = C \end{cases} \end{aligned}$$

The outcome path is CC, FF, OO. With payoffs

$$4 + \delta 3 + \delta^2 \text{ for player 1}$$

$$4 + \delta + \delta^3 3 \text{ for player 2}$$

It is clear that the play in the subgames starting at time 2 and 3 is subgame perfect. Need to check for one shot deviation at time 1. If player 2 doesn't have a deviation, then there is no deviation for player 1 either (since 1 get rewarded earlier). The profile is immune towards deviations from 2 if

$$\begin{aligned} 4 + \delta + \delta^2 3 &\geq 5 + \delta + \delta^2 \\ 2\delta^2 &\geq 1, \end{aligned}$$

which is true for every $\delta \geq \sqrt{\frac{1}{2}}$.

In this case renegotiation proofness doesn't have any bite at all as there is no Pareto dominated equilibrium in stage game.

Proposition 5 *In a finitely repeated game, there is either a unique renegotiation proof equilibrium, or every renegotiation proof equilibrium is near Pareto efficient if δ is near unity.*

7.4 Renegotiation in infinitely repeated games

There are variants. Here is one:

Definition 9 *Consider a two player game. A strategy profile s in $G(\infty)$ is internally consistent if for all h, h'*

$$u_i(s|h) > u_i(s|h') \Rightarrow u_j(s|h) \leq u_j(s|h')$$

for $i, j = 1, 2$.

IDEA: Rule out punishments where both players are made worse off for the remainder of the game. Such a punishment would be vulnerable to a "deal"-let's go back to the equilibrium path!

Definition 10 *s^* is negotiation proof if it is internally consistent and subgame perfect.*

7.4.1 Example

	D	C
D	$1,1$	$4,0$
C	$0,4$	$3,3$

Grim-Trigger is not negotiation proof as

$$\begin{pmatrix} \frac{3}{1-\delta} \\ \frac{3}{1-\delta} \end{pmatrix} > \begin{pmatrix} \frac{1}{1-\delta} \\ \frac{1}{1-\delta} \end{pmatrix}.$$

Need strategies that:

1. Punishes the deviator

2. Rewards the other player

Consider

$$\begin{aligned} s_i^1 &= C \\ s_i^t &= \begin{cases} C & \text{if } a_j^{t-1} = C \text{ or if } a^{t-2}, a^{t-1} \in \{(CD, DC), (DC, CD)\,(CC, DD)\} \\ D & \text{otherwise} \end{cases} \end{aligned}$$

Outcome path (CC) in every period. Nash if

$$\begin{aligned} \frac{3}{1-\delta} &\geq 4+0+\delta^2\frac{3}{1-\delta} \\ \frac{3}{1-\delta}\left(1-\delta^2\right) &= \frac{3}{1-\delta}\left(1-\delta\right)\left(1+\delta\right)=3\left(1+\delta\right)\geq 4 \\ \delta &\geq \frac{1}{3} \end{aligned}$$

Deviation from punishment not profitable if

$$\begin{aligned} 0+\delta\frac{3}{1-\delta} &\geq 1+0+\delta^2\frac{2}{1-\delta} \\ \delta\left(1-\delta\right)\frac{3}{1-\delta} &= 3\delta\geq 1 \\ \delta &\geq \frac{1}{3}. \end{aligned}$$

(obvious that the player that is rewarded has no profitable deviation). Continuation payoffs are

$$\begin{pmatrix} \frac{3}{1-\delta} \\ \frac{3}{1-\delta} \end{pmatrix} \text{ in “cooperative phase" (which includes after DD)}$$

$$\begin{pmatrix} 4+\frac{\delta 3}{1-\delta} \\ 0+\frac{\delta 3}{1-\delta} \end{pmatrix} \text{ after deviation by player 2 only}$$

$$\begin{pmatrix} 0+\frac{\delta 3}{1-\delta} \\ 4+\frac{\delta 3}{1-\delta} \end{pmatrix} \text{ after deviation by player 1 only}$$

These are the only continuation payoffs to consider, and we see that

$$4+\frac{\delta 3}{1-\delta}>\frac{3}{1-\delta}>0+\frac{\delta 3}{1-\delta},$$

so the strategy is negotiation proof.